Erhaltung der Masse

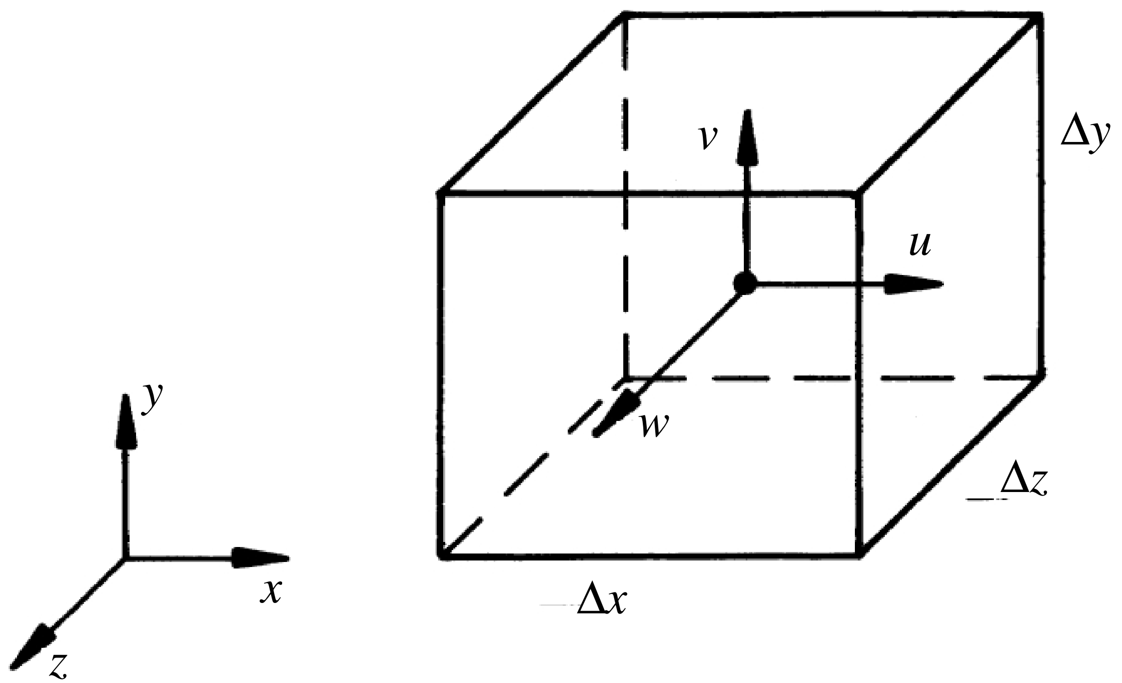
Die Masse des Systems bleibt bei Bewegung durch das Strömungsfeld konstant

$B = mb$, für $b = 1$

$$\boxed{\frac{dm_{sys}}{dt} = \frac{\partial}{\partial t}\int_{KV} \rho \, dV + \int_{KF} \rho \, \vec{v} \cdot \vec{n} \, dA = 0}$$

integrale Form

differentielle Form über Gaußschen Satz oder am Element

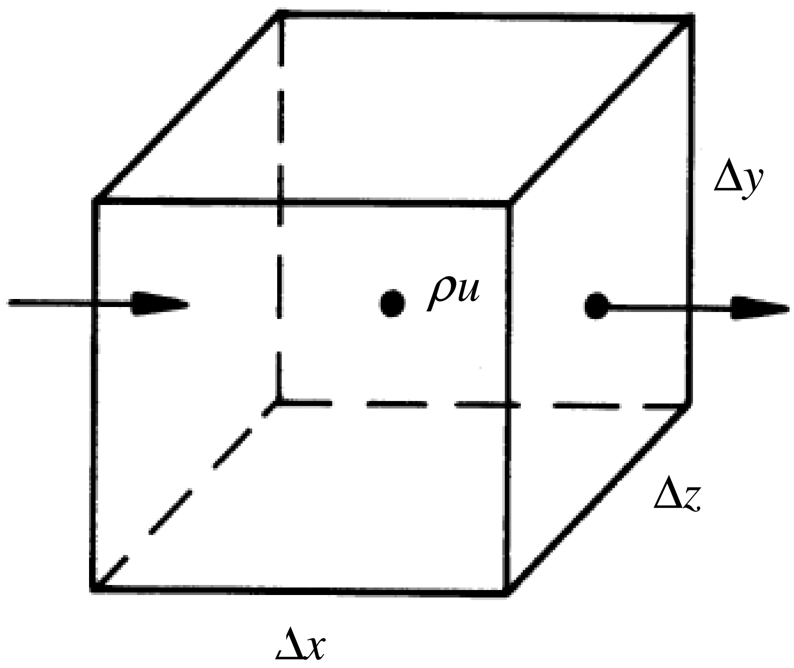


Zeitliche lokale Massenänderung :

$$\frac{\partial \rho}{\partial t}\ \Delta x\ \Delta y\ \Delta z$$

Massenfluss über die Oberfläche des Elements (in x-Richtung) :

$$\left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z$$



$$\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z$$

Nettomassenfluss in x-Richtung

$$\left(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z - \left(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z = \frac{\partial(\rho u)}{\partial x}\Delta x \Delta y \Delta z$$

In y- und z-Richtung erhält man

$$\frac{\partial(\rho v)}{\partial y}\Delta x \Delta y \Delta z \quad , \quad \frac{\partial(\rho w)}{\partial z}\Delta x \Delta y \Delta z$$

$\Longrightarrow$ differentielle Form der Kontinuitätsgleichung

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\ u)}{\partial x} + \frac{\partial(\rho\ v)}{\partial y} + \frac{\partial(\rho\ w)}{\partial z} = 0$$

bzw.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\ \vec{v}) = 0$$

mit

$$\vec{\nabla}(\bullet) = \frac{\partial(\bullet)}{\partial x}\ \vec{i} + \frac{\partial(\bullet)}{\partial y}\ \vec{j} + \frac{\partial(\bullet)}{\partial z}\ \vec{k}$$

Mit lok. konvek.

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}$$

$$= \frac{\partial\rho}{\partial t} + \vec{v}\cdot\nabla\rho$$

folgt :

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

bzw.

$$\frac{d\rho}{dt} + \rho\vec{\nabla}\cdot\vec{v} = \frac{d\rho}{dt} + \rho\, div\,\vec{v} = 0$$

Sonderfälle :

- stat. Strömung $\frac{\partial}{\partial t} \to 0$

$$\vec{\nabla} \cdot (\rho\ \vec{v}) = \frac{\partial(\rho\ u)}{\partial x} + \frac{\partial(\rho\ v)}{\partial y} + \frac{\partial(\rho\ w)}{\partial z} = 0$$

- Inkompressibles Fluid ρ = konstant

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

Erhaltung des Impulses

$\vec{F}$: Auf die Fluidmasse wirkende resultierende Kraft

$\vec{I} = \int_{Sys} \vec{v}\, dm$: Impuls

Anwendung auf ein differentielles Massesystem:

$$\Delta\vec{F} = \frac{d(\vec{v}\,\Delta m)}{dt}$$

$\Delta m_{sys} =$ konstant

$$\Delta\vec{F} = \Delta m \frac{d\vec{v}}{dt} = \Delta m \vec{a}$$

$\Delta\vec{F}$: Oberflächen- und Volumenkräfte

Volumenkraft : Gewichtskraft maßgeblich

in kartesischen Koordinaten:

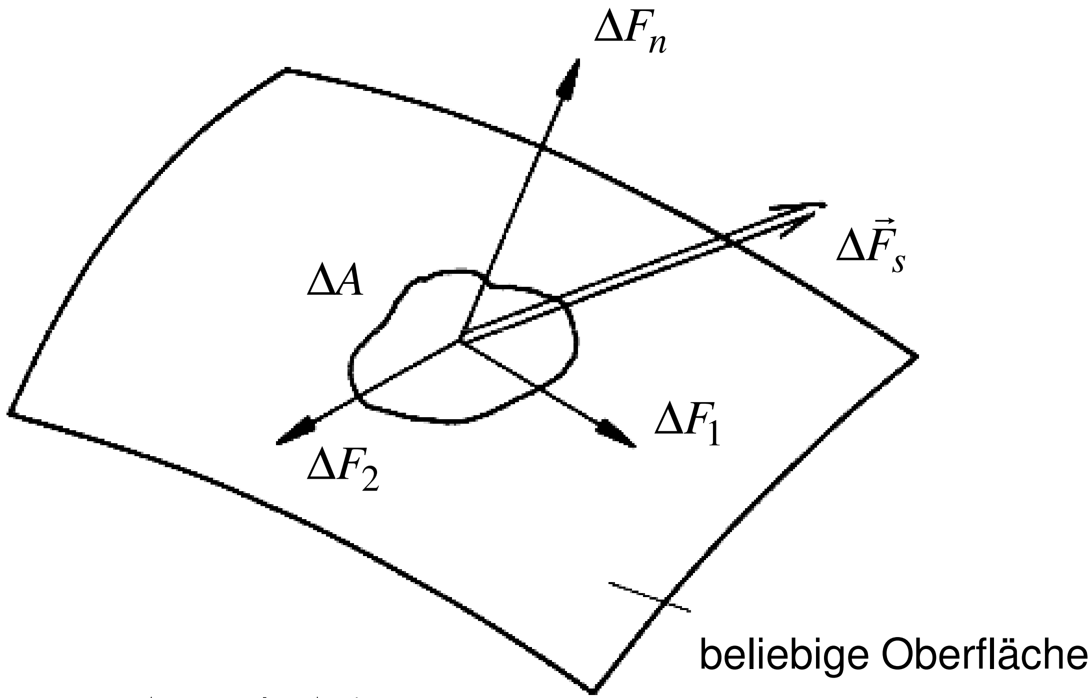
$$\Delta F_{bx} = \Delta m g_x$$

$$\Delta F_{by} = \Delta m g_y$$

$$\Delta F_{bz} = \Delta m g_z$$

Oberflächenkraft:

Element ↔ Umgebung



$$\Delta F_1 \perp \Delta F_2 \quad , \quad \Delta F_n \perp \Delta A$$

Normalspannung:

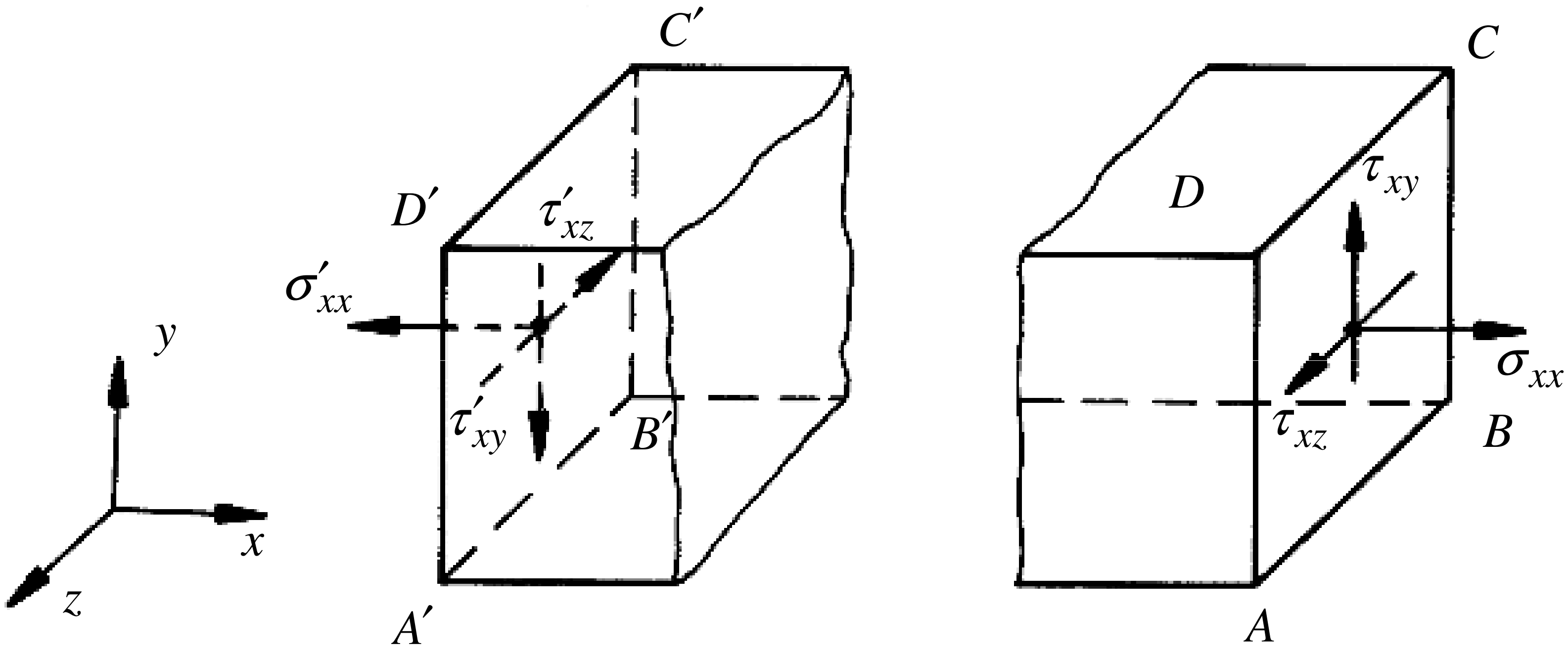
$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A}$$

Schubspannung:

$$\tau_1 = \lim_{\Delta A \to 0} \frac{\Delta F_1}{\Delta A}$$

$$\tau_2 = \lim_{\Delta A \to 0} \frac{\Delta F_2}{\Delta A}$$

die übliche Zeichenkonvektion gilt



Bedeutung der Indizes :

S_{ij} : i Richtung der Normalen der Ebene

j Richtung der Spannung

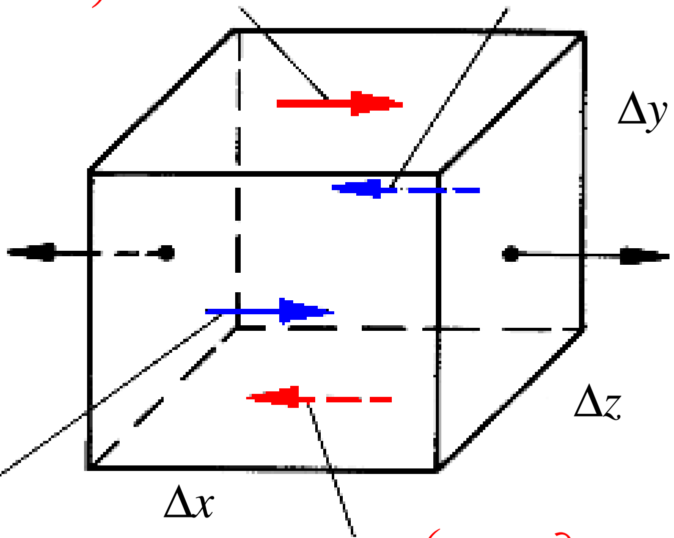
Spannungen auf 3 orthogonalen, durch einen Punkt gehende Ebenen definieren den Spannungszustand eindeutig

Oberflächenkräfte (nicht vollständig):

$$\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{\Delta y}{2}\right)\Delta x \Delta z$$

$$\left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y$$

$$\left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z$$



$$\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{\Delta x}{2}\right)\Delta y \Delta z$$

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\Delta z}{2}\right)\Delta x \Delta y$$

$$\left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{\Delta y}{2}\right)\Delta x \Delta z$$

Summation liefert die Komponenten der resultierenden Oberflächenkraft:

$$\Delta\vec{F}_S = \Delta F_{Sx}\,\vec{i} + \Delta F_{Sy}\,\vec{j} + \Delta F_{Sz}\,\vec{k}$$

Kräfte in x- / y- / z-Richtung:

$$\Delta F_{Sx} = \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}\right)\Delta x\Delta y\Delta z$$

$$\Delta F_{Sy} = \left(\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z}\right)\Delta x\Delta y\Delta z$$

$$\Delta F_{Sz} = \left(\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}\right)\Delta x\Delta y\Delta z$$

Einsetzen in $\Delta\vec{F} = \Delta m\vec{a}$

mit $\Delta\vec{F} = \Delta\vec{F}_s + \Delta\vec{F}_b$

$$\Delta m = \rho\Delta x\Delta y\Delta z$$

$$\vec{a} = \frac{\partial\vec{v}}{\partial t} + \underline{\underline{u\frac{\partial\vec{v}}{\partial x} + v\frac{\partial\vec{v}}{\partial y} + w\frac{\partial\vec{v}}{\partial z}}}$$

ergibt $\vec{v} = u\vec{i} + v\vec{j} + w\vec{k}$

Differentielle Form der Impulserhaltung

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x+\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z} \qquad \text{X}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z} \qquad \text{Y}$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z} \qquad \text{Z}$$

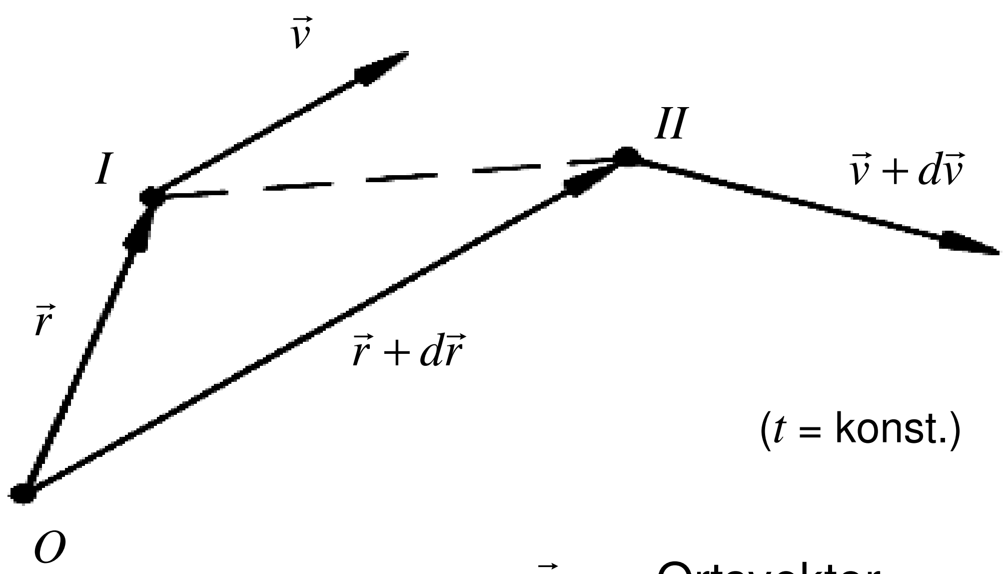
Kinematik des Fluidelements

Aufgabe : Spannungen durch Geschwindigkeitskomponenten ausdrücken

Bewegung eines Elements

$\Longrightarrow$ Änderung seiner Lage und seiner Form

Bestimmung der Verformung mittels der relativen Bewegung zwischen 2 Punkten I und II



(t = konst.)

$\vec{r}$: Ortsvektor

$d\vec{v}$: Geschwindigkeitsänderung

$d\vec{v}$ in Komponentenschreibweise

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz$$

$$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$$

Um den Zusammenhang mit Translation, Rotation, Dehnung und Scherung zu erkennen, wird $d\vec{v}$ umgeschrieben

$$du = \dot{\varepsilon}_x dx + \dot{\varepsilon}_{xy} dy + \dot{\varepsilon}_{xz} dz + \omega_{zx} dz - \omega_{xy} dy$$

$$dv = \dot{\varepsilon}_{yx} dx + \dot{\varepsilon}_y dy + \dot{\varepsilon}_{yz} dz + \omega_{xy} dx - \omega_{yz} dz$$

$$dw = \dot{\varepsilon}_{zx} dx + \dot{\varepsilon}_{zy} dy + \dot{\varepsilon}_z dz + \omega_{yz} dy - \omega_{zx} dx$$

Der Vergleich beider Gleichungssysteme ergibt folgende Definitionen für $\dot{\varepsilon}_i$ und $\dot{\varepsilon}_{ij}$ bzw. ω_{ij}

$$\dot{d}_{ij} = \begin{pmatrix} \dot{\varepsilon}_x & \dot{\varepsilon}_{xy} & \dot{\varepsilon}_{xz} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_y & \dot{\varepsilon}_{yz} \\ \dot{\varepsilon}_{zx} & \dot{\varepsilon}_{zy} & \dot{\varepsilon}_z \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$\omega_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

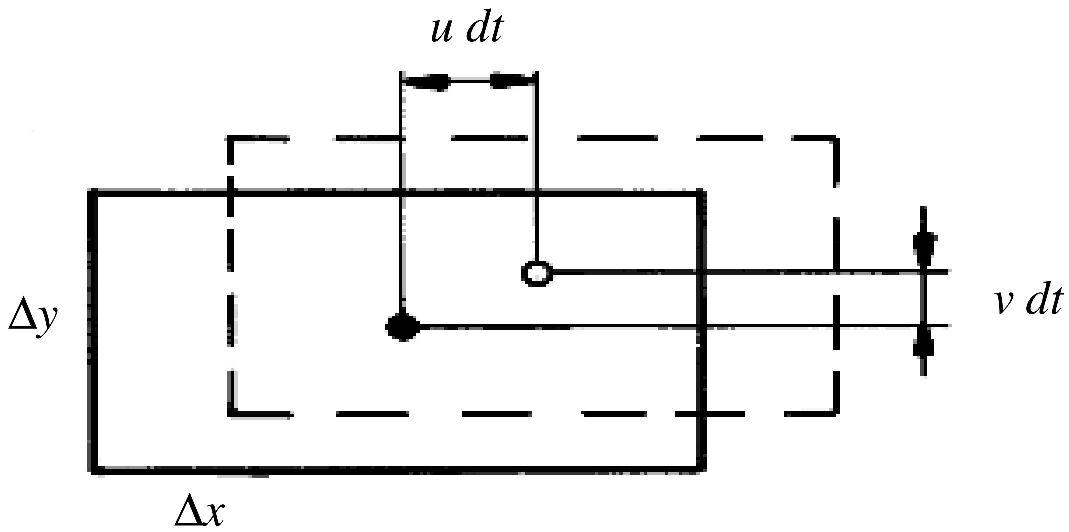
$$\omega_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

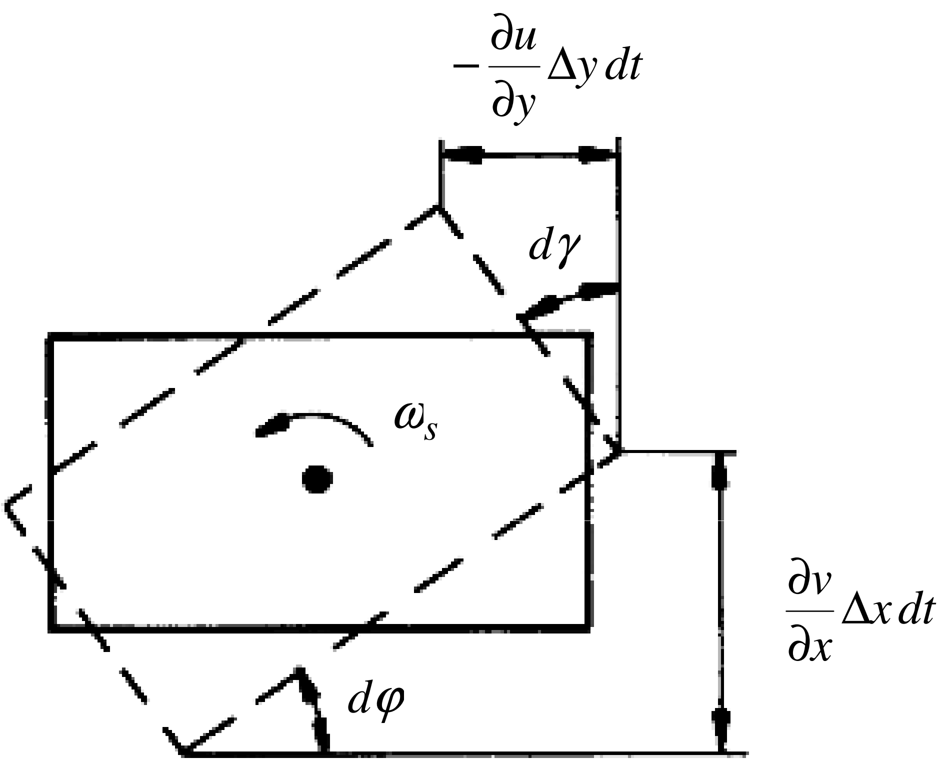
Bedeutung von $\dot{\varepsilon}_i$, $\dot{\varepsilon}_{ij}$, ω_{ij} ?

Unverformtes Element bewegt sich in der Strömung

Translation:



Rotation:



$\tan(d\varphi) \approx d\varphi$

$$d\varphi = \frac{\partial v}{\partial x} dt = -\frac{\partial u}{\partial y} dt$$

$\Rightarrow$ Zeitliche Winkeländerung $\dot{\varphi} = \frac{d\varphi}{dt}$ aus dem arithmetischen Mittel

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \omega_{xy}$$

$\Rightarrow$ Drehung um z-Achse

In 3D erhält man :

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \omega_{yz}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \omega_{zx}$$

$\Rightarrow$ Dreh- oder Wirbelvektor $\vec{\omega}$

$$\vec{\omega} = \omega_x \vec{i} + \omega_y \vec{j} + \omega_z \vec{k}$$

$\Rightarrow$ ω_{ij} - Terme : Rotation des unverformten Elements

Relative Volumenänderung (Volumendilatation) $\frac{1}{\Delta V}\frac{d(\Delta V)}{dt}$

$$\frac{1}{\Delta V}\frac{d(\Delta V)}{dt}=\frac{1}{\Delta V}\left\{\left[\Delta x+\frac{\partial u}{\partial x}\Delta x\,dt\right]\cdot\left[\Delta y+\frac{\partial v}{\partial y}\Delta y\,dt\right]\cdot\left[\Delta z+\frac{\partial w}{\partial z}\Delta z\,dt\right]-\Delta x\Delta y\Delta z\right\}\frac{1}{dt}$$

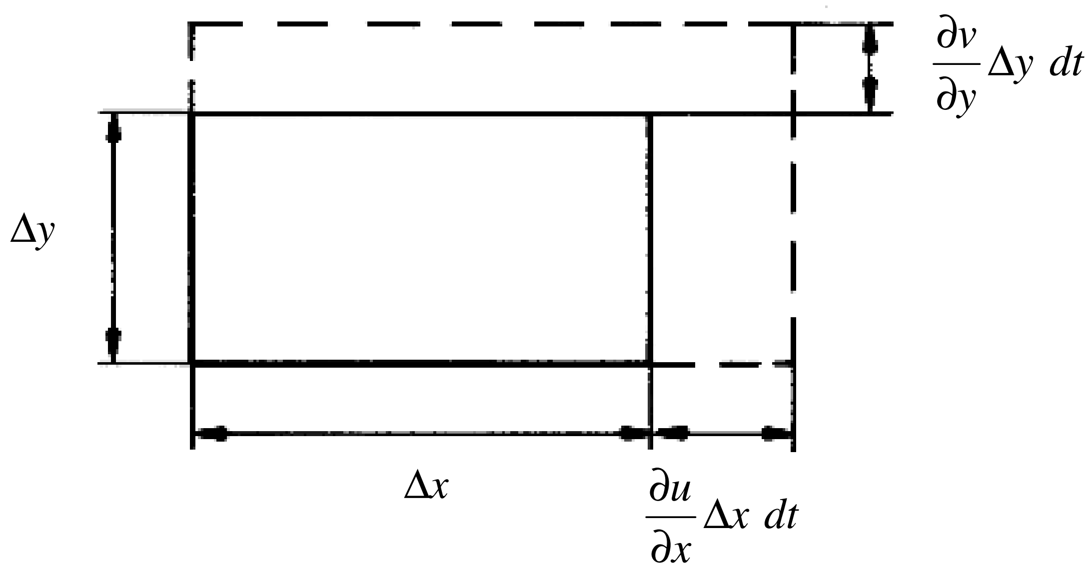
$$\frac{1}{\Delta V}\frac{d(\Delta V)}{dt}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\vec{\nabla}\cdot\vec{v}=div\,\vec{v}$$

$\longrightarrow$ Kontinuitätsgleichung für inkompr. Fluide

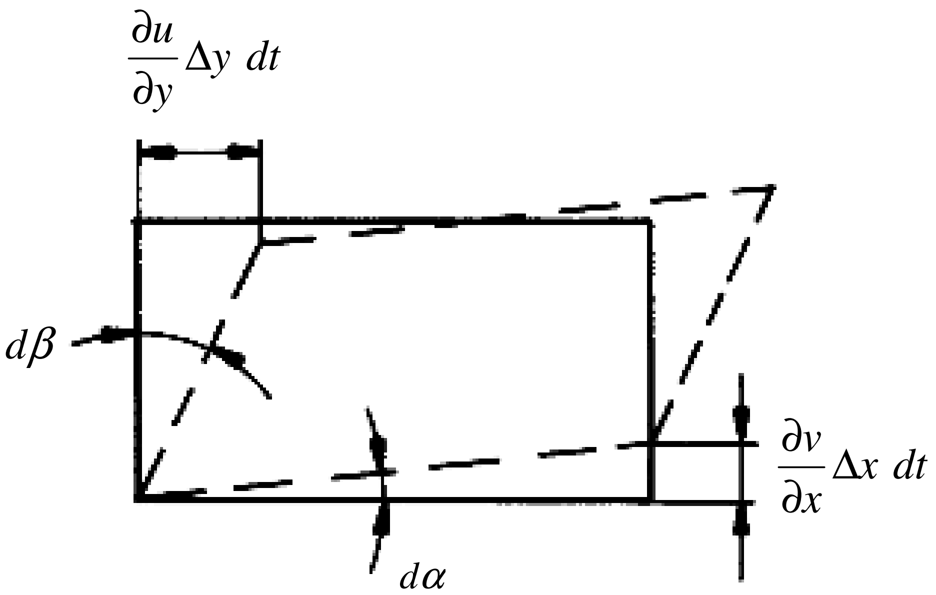
Rechteck $\xrightarrow{\textbf{Scherung}}$ Parallelepiped

$$d\gamma = d\alpha + d\beta$$

Verformung des Elements



Dehnung



Scherung

Dehnungsgeschwindigkeit entspricht der zeitlichen Dehnungsänderung pro Kantenlänge

$$\dot{\varepsilon}_x = \frac{\partial u}{\partial x} \quad , \quad \dot{\varepsilon}_y = \frac{\partial v}{\partial y} \quad , \quad \dot{\varepsilon}_z = \frac{\partial w}{\partial z}$$

entspricht den Hauptdiagonalen der Matrix d_{ij}

$$d\alpha = \frac{\partial v}{\partial x} dt$$

$$d\beta = \frac{\partial u}{\partial y} dt$$

$$\frac{d\gamma}{dt} : \left(\frac{d\alpha}{dt}\right); \left(\frac{d\beta}{dt}\right)$$

gesamte Winkeländerung pro Zeit

Schergeschwindigkeit $\dot{\gamma}_{ij}$ über arithmetische Mittelung

$$\dot{\gamma}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\dot{\gamma}_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\dot{\gamma}_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial z}\right)$$

Vergleiche mit $d_{ij} - Matrix$:

$\Longrightarrow$ $\dot{\varepsilon}_{ij} \mathrel{\hat{=}} \dot{\gamma}_{ij}$ – Terme

$\Longrightarrow$ d_{ij} – Matrix : enthält **Dehnung** und **Scherung**

$\Longrightarrow$ Tensor der Deformation

Spannungstensor τ_{ij} in der Impulserhaltung

Zusammenhang zwischen τ_{ij} und $\dot{d}_{ij}$ durch Newtonschen Reibungsansatz :

Tangentiale Spannung ~ Schergeschwindigkeit

und mittels Isotropie des Elements

$\dot{\varepsilon}_x, \dot{\varepsilon}_y, \dot{\varepsilon}_z$ und $div\vec{v}$ $\longrightarrow$ viskositätsbedingte Normalspannungen

Ansatz:

$$\sigma_{xx} = -p + 2\eta\dot{\varepsilon}_x + \lambda\, div\,\vec{v} \qquad \tau_{xy} = \tau_{yx} = 2\eta\,\dot{\gamma}_{xy}$$

$$\sigma_{yy} = -p + 2\eta\dot{\varepsilon}_y + \lambda\, div\,\vec{v} \qquad \tau_{yz} = \tau_{zy} = 2\eta\,\dot{\gamma}_{yz}$$

$$\sigma_{zz} = -p + 2\eta\dot{\varepsilon}_z + \lambda div\,\vec{v} \qquad \tau_{zx} = \tau_{xz} = 2\eta\,\dot{\gamma}_{zx}$$

Die Größen η und λ sind Proportionalitätsfaktoren

Die Normalspannungen werden i. a. umformuliert

$$\sigma_{yy} = -p + \eta\left(2\dot{\varepsilon}_y - \frac{2}{3} div\,\vec{v}\right) + \hat{\eta}\, div\,\vec{v}$$

$$\sigma_{xx} = -p + \eta\left(2\dot{\varepsilon}_x - \frac{2}{3} div\,\vec{v}\right) + \hat{\eta}\, div\,\vec{v}$$

$$\sigma_{zz} = -p + \eta\left(2\dot{\varepsilon}_z - \frac{2}{3} div\,\vec{v}\right) + \hat{\eta}\, div\,\vec{v}$$

η : dynamische Viskosität

$\hat{\eta} = \lambda + \frac{2}{3}\eta$: Volumenviskosität

Stokes ‘sche Hypothese : $\hat{\eta} = \lambda + \frac{2}{3}\eta = 0$

inkompressible Strömungen : $div\,\vec{v} = 0$

$\Longrightarrow$ mittlere Normalspannung σ :

$$\overline{\sigma} = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) = -p + \hat{\eta}\, div\,\vec{v}$$

Einsetzen der Normal- und Tangentialspannung

$\Rightarrow$ **Navier- Stokes Gleichungen**

$$\rho\frac{du}{dt}=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\eta\left(2\frac{\partial u}{\partial x}-\frac{2}{3}div\,\vec{v}\right)\right]+\frac{\partial}{\partial y}\left[\eta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\eta\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]$$

$$\rho\frac{dv}{dt}=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial y}\left[\eta\left(2\frac{\partial v}{\partial y}-\frac{2}{3}div\,\vec{v}\right)\right]+\frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial x}\left[\eta\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]$$

$$\rho\frac{dw}{dt}=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial z}\left[\eta\left(2\frac{\partial w}{\partial z}-\frac{2}{3}div\,\vec{v}\right)\right]+\frac{\partial}{\partial x}\left[\eta\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+\frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]$$

inkompressible Strömung, η = konst

$$\longrightarrow \quad div\,\vec{v} = 0$$

Vereinfachung der 2. Ableitungen

$$\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{\partial}{\partial x}\left[\eta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\right]=\eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$\Longrightarrow$ Navier- Stokes Gleichungen für ein ink. Fluid

$$\boxed{\begin{aligned}
\rho\frac{du}{dt} &= \rho g_x - \frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\\
\rho\frac{dv}{dt} &= \rho g_y - \frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)\\
\rho\frac{dw}{dt} &= \rho g_z - \frac{\partial p}{\partial z} + \eta\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}}$$

Energiegleichungen

1. Hauptsatz der Thermodynamik :

$$\frac{dQ}{dt} = \frac{dE}{dt} + \frac{dW}{dt}$$

Q : Wärme; E : Energie; W : Arbeit

Volumenelement :

$$\Delta V = \Delta x \Delta y \Delta z$$

$$\Delta m = \rho \Delta V$$

Wärmeleitung nach Fourier :

$$\frac{1}{A}\frac{dQ}{dt} = q = -\lambda \frac{\partial T}{\partial n}$$

λ : Wärmeleitfähigkeit

Betrachtung für die Fläche $\Delta y\ \Delta z$ (x-Richtung)

aufgenommene Wärme:

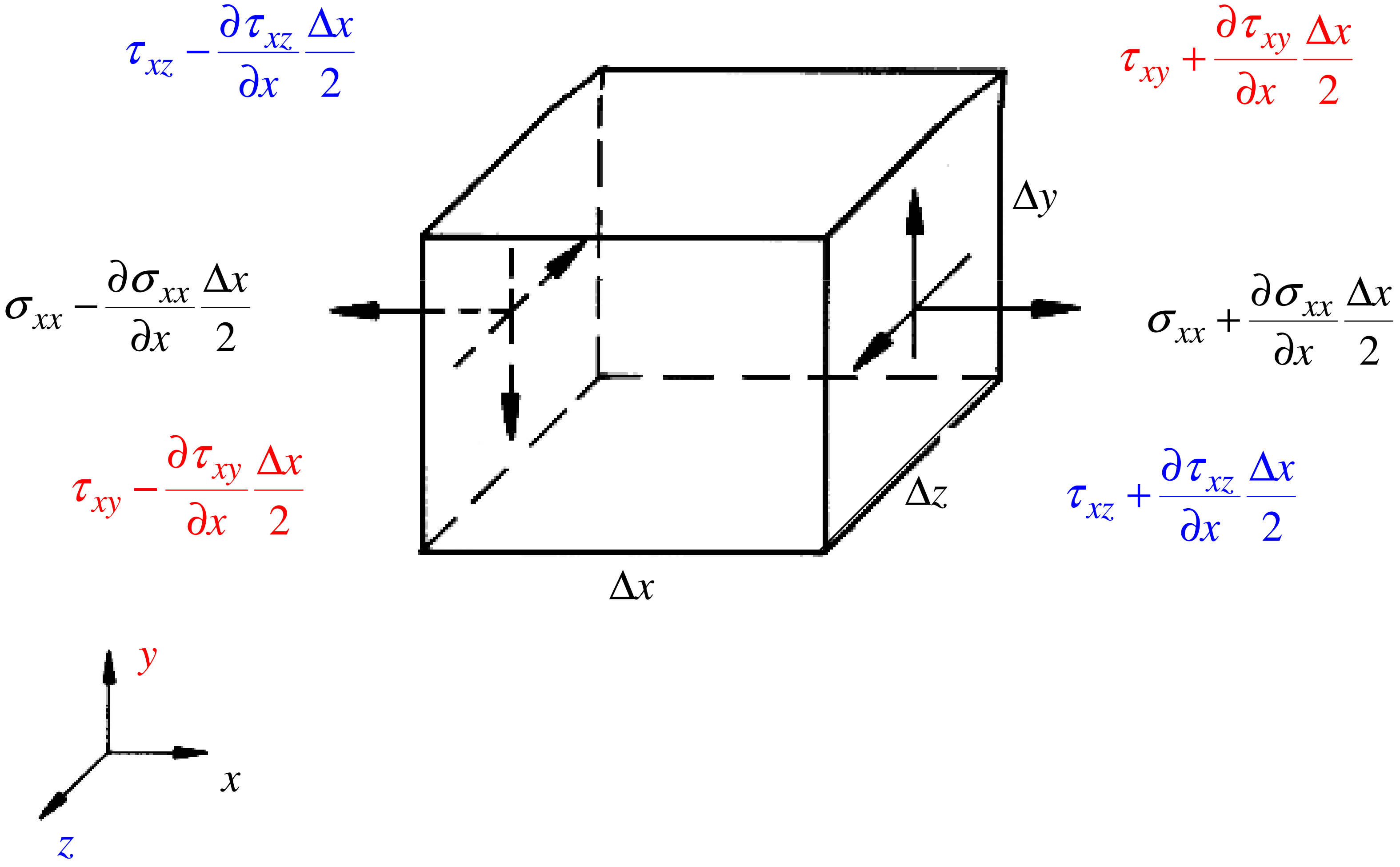
$$-\left(\lambda \frac{\partial T}{\partial x} - \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x} \right) \frac{\Delta x}{2} \right) \Delta y \Delta z$$

abgegebene Wärme:

$$\left(\lambda \frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x} \right) \frac{\Delta x}{2} \right) \Delta y \Delta z$$

Nettowärmestrom in x- , y- und z-Richtung

$$dQ = dt\Delta V \left\{ \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z} \right) \right\}$$



zeitliche Änderung der Gesamtenergie:

$$\frac{dE}{dt} = \rho \Delta V \left[\frac{de}{dt} + \frac{1}{2} \frac{d}{dt} (u^2 + v^2 + w^2) \right]$$

e : massenbezogene innere Energie

Arbeit pro Zeit anhand von σ_{xx} **:**

$$dW_{\sigma_{xx}} = -\Delta y \Delta z dt \left\{ \underline{-\left(u - \frac{\partial u}{\partial x} \frac{\Delta x}{2} \right)\left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x} \frac{\Delta x}{2} \right)} \right.$$

$$\left. \underline{+\left(u + \frac{\partial u}{\partial x} \frac{\Delta x}{2} \right)\left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} \frac{\Delta x}{2} \right)} \right\}$$

$$= -\Delta y \Delta z dt \left(u \frac{\partial \sigma_{xx}}{\partial x} + \sigma_{xx} \frac{\partial u}{\partial x} \right) \Delta x$$

$$= -\Delta V dt \underline{\underline{\frac{\partial}{\partial x} (u \sigma_{xx})}}$$

analoge Vorgehensweise für den gesamten Spannungstensor :

$$\frac{dW}{dt} = -\Delta V\left[\frac{\partial}{\partial x}\left(\underline{\underline{u\sigma}}_{xx} + v\tau_{xy} + w\tau_{xz}\right) + \frac{\partial}{\partial y}\left(u\tau_{yx} + v\sigma_{yy} + w\tau_{yz}\right) + \frac{\partial}{\partial z}\left(u\tau_{zx} + v\tau_{zy} + w\sigma_{zz}\right)\right]$$

Mittels Impulserhaltung (z.B. x-Impuls) :

$$\underline{\rho\frac{du}{dt}} = \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}$$

Umformung der $\frac{dE}{dt}$ und $\frac{dW}{dt}$ -Therme :

$$\frac{dE}{dt} = \left[\rho\frac{de}{dt} + \underline{u\underline{\rho\frac{du}{dt}}} + v\rho\frac{dv}{dt} + w\rho\frac{dw}{dt}\right]\Delta V$$

$$\frac{dW}{dt}\frac{1}{\Delta V} = -\underline{u\frac{\partial \sigma_{xx}}{\partial x}} - \sigma_{xx}\frac{\partial u}{\partial x}\ldots$$

$$-\underline{u\frac{\partial \tau_{xy}}{\partial y}} - \tau_{xy}\frac{\partial u}{\partial y}\ldots$$

$$-\underline{u\frac{\partial \tau_{zx}}{\partial z}} - \tau_{zx}\frac{\partial u}{\partial z}\ldots$$

$$\left(\frac{dE}{dt} + \frac{dW}{dt}\right)\frac{1}{\Delta V} = \rho\frac{de}{dt} - \sigma_{xx}\frac{\partial u}{\partial x} - \tau_{xy}\frac{\partial v}{\partial x} - \tau_{xz}\frac{\partial w}{\partial x}$$

$$-\tau_{yx}\frac{\partial u}{\partial y} - \sigma_{yy}\frac{\partial v}{\partial y} - \tau_{yz}\frac{\partial w}{\partial y}$$

$$-\tau_{zx}\frac{\partial u}{\partial z} - \tau_{zy}\frac{\partial v}{\partial z} - \sigma_{zz}\frac{\partial w}{\partial z}$$

Einführung der Spannungen ergibt die Energiegleichung

$$\rho \frac{de}{dt} + p\, div\, \vec{v} = \frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right) + \eta \Phi$$

mit Φ für $\hat{\eta} = 0$

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2$$

$$+ \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 > 0$$

$\Phi = \text{Dissipatio nsfunktion}$

mechanische Energie $\longrightarrow$ thermische Energie

Energiegleichungen für ideale Gase

$$e = f(T),\ \ h = e + \frac{p}{\rho} = f(T)$$

kalorische Zustandsgleichungen:

$$de = \left(\frac{\partial e}{\partial T}\right)_{\frac{1}{\rho}} dT = c_v dT \qquad\qquad dh = \left(\frac{\partial h}{\partial T}\right)_p dT = c_p dT$$

$$\Rightarrow \quad \frac{de}{dt} = \frac{dh}{dt} - \frac{d\left(\frac{p}{\rho}\right)}{dt} = c_p \frac{dT}{dt} - \frac{1}{\rho}\left[\frac{dp}{dt} - \frac{p d\rho}{\rho dt}\right]$$

Kontinuitätsgleichung:

$$-\frac{1}{\rho}\frac{d\rho}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = div\,\vec{v}$$

$$\Rightarrow \quad \boxed{\rho c_p \frac{dT}{dt} = \frac{dp}{dt} + \left[\frac{\partial}{\partial x}\left(\lambda \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda \frac{\partial T}{\partial z}\right)\right] + \eta \Phi}$$

Form der Energiegleichung in CFD-Untersuchungen

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x}\left[u(E+p)\right]+\frac{\partial}{\partial y}\left[v(E+p)\right]+\frac{\partial}{\partial z}\left[w(E+p)\right]=$$

$$\frac{\partial}{\partial x}\left[u\tau_{xx}+v\tau_{xy}+w\tau_{xz}-q_x\right]+$$

$$\frac{\partial}{\partial y}\left[u\tau_{xy}+v\tau_{yy}+w\tau_{yz}-q_y\right]+$$

$$\frac{\partial}{\partial z}\left[u\tau_{xz}+v\tau_{yz}+w\tau_{zz}-q_z\right]$$

bzw. mit $H=h+\frac{\|\vec{v}\|^2}{2}$

$$\rho H=p+\rho\left(e+\frac{\|\vec{v}\|^2}{2}\right)=p+E$$

lauten die räumlichen 1. Ableitungen der linken Seite

$$\frac{\partial}{\partial x}(\rho uH)+\frac{\partial}{\partial y}(\rho vH)+\frac{\partial}{\partial z}(\rho wH)$$

Formen der Erhaltungsgleichungen

Vektorschreibweise unter Berücksichtigung des Nabla-Operators

Masse:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

oder

$$\frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$$

Impuls:

Spannungstensor τ

$$\tau = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

$$\sigma_x = \sigma_{xx} + p = \eta \left(2\dot{\varepsilon}_x - \frac{2}{3} div\ \vec{v} \right) + \hat{\eta} div\ \vec{v}$$

$$\sigma_y = \sigma_{yy} + p$$

$$\sigma_z = \sigma_{zz} + p$$

$\Rightarrow$ Formen der Navier-Stokes Gleichungen:

$$\rho \frac{d\vec{v}}{dt} = \rho\vec{g} - \vec{\nabla}p + \vec{\nabla} \cdot \tau$$

oder:
$$\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}\right) = \rho\vec{g} - \vec{\nabla}p + \vec{\nabla} \cdot \tau$$

oder:
$$\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla} \cdot (\rho\,\vec{v}\vec{v}) = \rho\vec{g} - \vec{\nabla}p + \vec{\nabla} \cdot \tau$$

mit $\vec{v}\vec{v}$
$$\vec{v}\vec{v} = \begin{pmatrix} u^2 & uv & uw \\ vu & v^2 & vw \\ wu & wv & w^2 \end{pmatrix}$$

inkompressibles Fluid mit η = konstant:

$$\rho \frac{d\vec{v}}{dt} = \rho\vec{g} - \vec{\nabla}p + \eta\nabla^2\vec{v}$$

\+ stationär
$$\rho\left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} = \rho\vec{g} - \vec{\nabla}p + \eta\nabla^2\vec{v}$$

Energie

$$E = \rho\left(e + \frac{\|\vec{v}\|^2}{2}\right) \quad : \quad \text{Gesamtenergie}$$

$$\frac{\partial E}{\partial t} + \vec{\nabla}\cdot(E\vec{v}) = \rho\vec{g}\cdot\vec{v} - \vec{\nabla}\cdot\vec{q} - \vec{\nabla}\cdot(p\vec{v}) + \vec{\nabla}\cdot(\tau\cdot\vec{v})$$

plus Kontinuitätsgleichung

$$\rho\frac{d}{dt}\left(e + \frac{\|\vec{v}\|^2}{2}\right) = \rho\vec{g}\cdot\vec{v} - \vec{\nabla}\cdot\vec{q} - \vec{\nabla}\cdot(p\vec{v}) + \vec{\nabla}\cdot(\tau\cdot\vec{v})$$

$$H = \frac{p+E}{\rho} \quad : \quad \text{Gesamtenthalpie}$$

$$\rho\frac{dH}{dt} = \rho\vec{g}\cdot\vec{v} + \frac{\partial p}{\partial t} - \vec{\nabla}\cdot\vec{q} + \vec{\nabla}(\tau\cdot\vec{v})$$

$$\frac{d}{dt}\left(e + \frac{\|\vec{v}\|^2}{2}\right) = \frac{dH}{dt} - \frac{1}{\rho}\left(\frac{\partial p}{\partial t} + \vec{\nabla}\cdot(p\vec{v})\right)$$

e : innere Energie

$$\rho \frac{de}{dt} = -\vec{\nabla} \cdot \vec{q} - p\vec{\nabla} \cdot \vec{v} + \tau \cdot \vec{\nabla}\vec{v}$$

mit

$$\begin{aligned}\tau \cdot \vec{\nabla}\vec{v} &= \sigma_x \frac{\partial u}{\partial x} + \tau_{xy} \frac{\partial u}{\partial y} + \tau_{xz} \frac{\partial u}{\partial z} + \tau_{yx} \frac{\partial v}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \tau_{yz} \frac{\partial v}{\partial z} + \tau_{zx} \frac{\partial w}{\partial x} + \tau_{zy} \frac{\partial w}{\partial y} + \sigma_z \\ &= \sigma_x \frac{\partial u}{\partial x} + \sigma_y \frac{\partial v}{\partial y} + \sigma_z \frac{\partial u}{\partial z} + \tau_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \tau_{yz}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \tau_{zx}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \\ &= \eta \Phi\end{aligned}$$

h : innere Enthalpie

$$\rho \frac{dh}{dt} = -\vec{\nabla} \cdot \vec{q} + \frac{dp}{dt} + \tau \cdot \vec{\nabla}\vec{v}$$

bzw. bei idealem Gas: $$\rho c_p \frac{dT}{dt} = -\vec{\nabla} \cdot \vec{q} + \frac{dp}{dt} + \tau \cdot \vec{\nabla}\vec{v}$$