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(Name, Matr.-No., Signature)

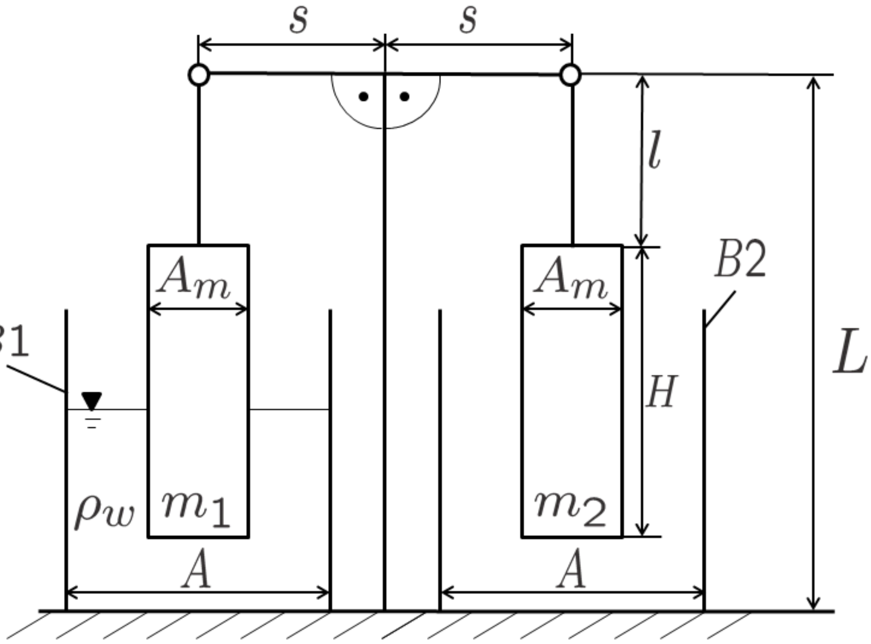
Exam: „Fluid Dynamics“

6 August 2019

Empty

Problem 1 (8 Points)

Two different m_1 and m_2 ($m_1 > m_2$) are attached to a beam balance. The base area A_m and the height H of them are identical. Below each mass, there is an empty container with a cross-sectional area of A.



a) Container $B1$ is filled with water of volume V_W, such that the bars of the balance are in the horizontal position. Determine the difference of the masses $\Delta m = m_1 - m_2$.

b) Then, container $B2$ is filled with the same water volume V_W. Determine the inclination angle of the bars against the horizontal line.

c) How does the angle change qualitatively when container $B2$ is filled with a salt solution instead of water ($\rho_{SL} > \rho_W$)?

Given:

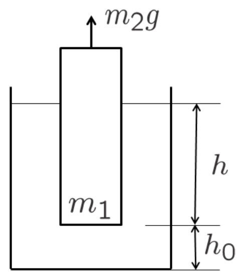
A_m, H, A, L, l, s, V_W, $V_{SL} = V_W$, ρ_W

Hints:

- Neglect the influence of the surrounding air!
- Check the units and signs of your results!

Problem 1

a) Force equilibrium for m_1:



$F_A = m_1 g - m_2 g$ und $F_A = \rho_W g h A_m$

Volume balance for the water:

$$V_W = A h_0 + (A - A_m) h$$
$$\Rightarrow h = \frac{V_W - A h_0}{A - A_m}$$

The same applies: $L = h_0 + H + l \quad \Leftrightarrow \quad h_0 = L - l - H$
Insert into volume balance:

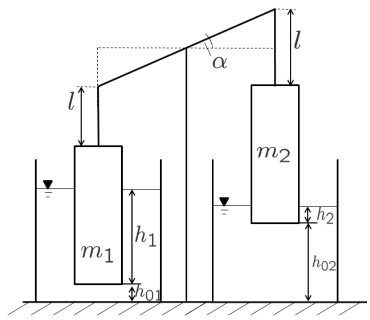
$$h = \frac{V_W - A(L - l - H)}{A - A_m}$$

Insert into force equilibrium:

$$\begin{aligned} m_1 - m_2 &= \rho_W A_m h \\ \Leftrightarrow \quad \Delta m &= m_1 - m_2 = \frac{\rho_W A_m}{A - A_m}[V_W - A(L - l - H)] \end{aligned}$$

b) Equilibrium of moments:

$$\begin{aligned} (m_1 g - F_{A1}) s\, cos(\alpha) &= (m_2 g - F_{A2}) s\, cos(\alpha) \\ \Leftrightarrow m_1 - \rho_W h_1 A_m &= m_2 - \rho_W h_2 A_m \\ \Leftrightarrow m_1 - m_2 &= \rho_W A_m (h_1 - h_2) \end{aligned}$$



Inclination angle α:

$$L - s\, sin(\alpha) = h_{01} + H + l \quad \text{and} \quad L + s\, sin(\alpha) = h_{02} + H + l$$
$$\Rightarrow h_{02} - h_{01} = 2s\, sin(\alpha)$$

Insert into equilibrium of moments:

$$m_1 - m_2 = \rho_W \frac{A \cdot A_m}{A - A_m} 2s\, sin(\alpha)$$

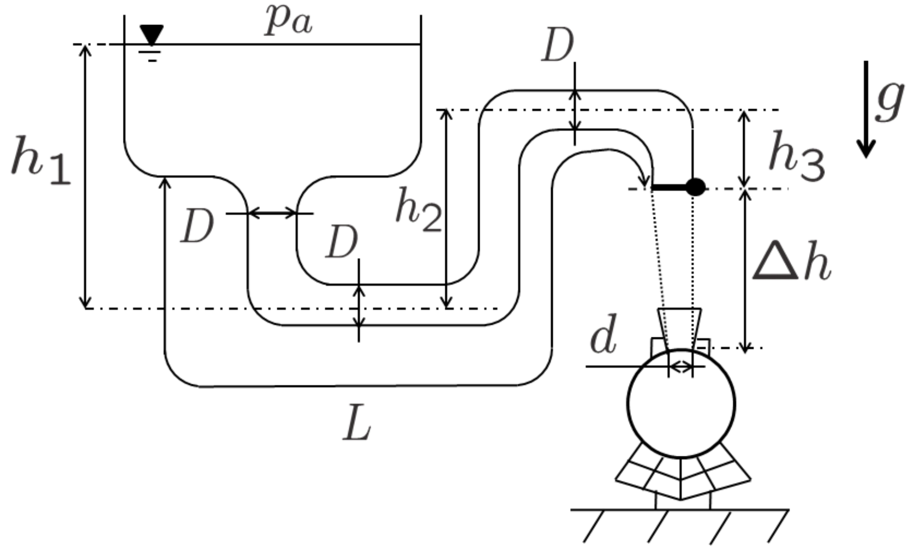
$$\Leftrightarrow \alpha = arcsin\left(\frac{m_1 - m_2}{2s\rho_w} \cdot \frac{A - A_m}{A \cdot A_m}\right) = arcsin\left(\frac{\frac{\rho_W A_m}{A - A_m}[V_W - A(L - l - H)]}{2s\rho_w} \cdot \frac{A - A_m}{A \cdot A_m}\right)$$

c) Equilibrium of moments in b) shows:

$$F_{A2} \sim \rho h_2$$
$$\rho_{SL} > \rho_W \Rightarrow F_{A2} \uparrow \Rightarrow h_{2c)} \downarrow \Rightarrow \alpha \uparrow$$

Problem 2 (12 Points)

To fill a steam engine with water, the valve at the end of the pipe is suddenly opened. The diameter D of the pipe is constant and the length of the pipe is L. The orifice of the large water tank of the steam engine has the diameter d and is located at a distance Δh vertically below the exit of the pipe.



a) Carefully sketch the distribution of the static pressure as a function of the length of the pipe **before** and **after** opening the valve, when a steady state flow has developed.

b) Determine the time ΔT until 99% of the steady state velocity is reached after opening the valve.

c) Determine the distance $\Delta h > 0$ under the condition that no water passes by the orifice of the water tank of the steam engine.

d) What limits must be satisfied for h_2 and h_3 such that ste static pressure in the pipe is greater than the steam pressure p_D?

Given: h_1, h_2, h_3, L, D, $L >> D$, d, $d < D$, ρ, g, p_a, p_D

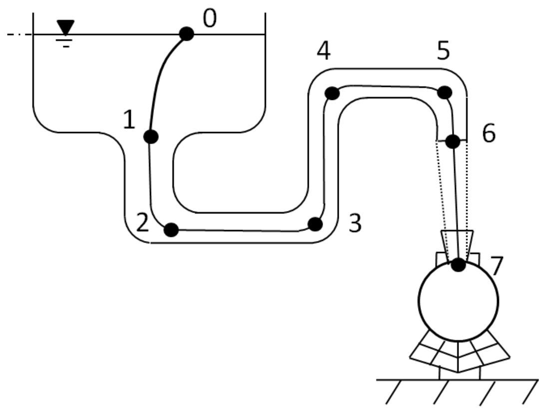
Hints:

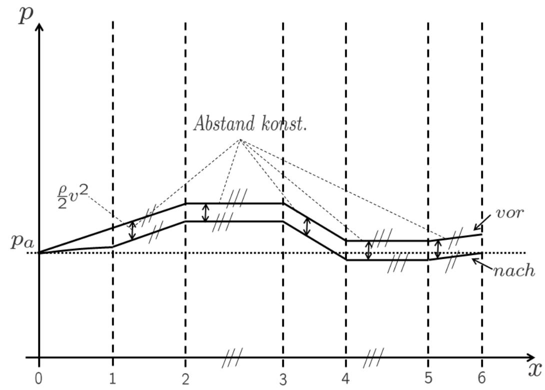
- The flow is incompressible and inviscid.
- The following integrals are given:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} ln \frac{a+x}{a-x} \quad \text{for} \quad |x| < a$$
$$\int \frac{dx}{ax+b} = \frac{1}{a} ln(ax+b) \quad \text{for} \quad a \neq 0$$
$$\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2} ln(ax+b) \quad \text{for} \quad a \neq 0$$

- Check the units and signs of your results!

Problem 2





a)

b) Steady state Bernoulli from $0 - 6$:

$$\begin{aligned} p_a + \rho g h_1 &= p_a + \frac{\rho}{2}v_{steady}^2 + \rho g(h_2 - h_3) \\ p_a &= p_a + \frac{\rho}{2}v_{steady}^2 + \rho g(-h_1 + h_2 - h_3) \\ \Rightarrow v_{steady} &= \sqrt{2g(h_1 - h_2 + h_3)} \end{aligned}$$

Unsteady Bernoulli from $0 - 6$:

$$\begin{aligned} p_a &= p_a + \frac{\rho}{2}v^2 + \rho g(-h_1 + h_2 - h_3) + \rho \int \frac{\partial v}{\partial t} ds \\ \text{with} \quad \int \frac{\partial v}{\partial t} ds &= \frac{dv}{dt}L \qquad \text{since} \qquad D << L \\ \Rightarrow \frac{dv}{dt} &= \frac{2g(-h_1 + h_2 - h_3) - v^2}{2L} = \frac{v_{steady}^2 - v^2}{2L} \\ \int_0^{0.99 v_{steady}} \frac{dv}{v_{steady}^2 - v^2} &= \int_0^{\Delta T} \frac{dt}{2L} \\ \Rightarrow \Delta T &= \frac{L}{v_{steady}} ln\left(\frac{1 + \frac{v}{v_{steady}}}{1 - \frac{v}{v_{steady}}}\right) \\ \Leftrightarrow \Delta T &= \frac{L}{v_{steady}} ln\left(\frac{1.99}{0.01}\right) = \frac{L}{\sqrt{2g(h_1 - h_2 + h_3)}} ln(199) \end{aligned}$$

c) Steady state Bernoulli: 6 - 7

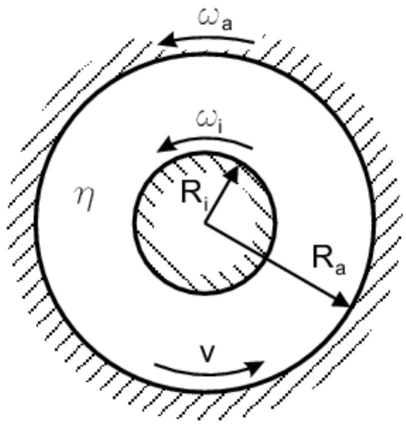
$$\begin{aligned} p_a + \rho g \Delta h + \frac{\rho}{2}v_{steady}^2 &= p_a + \frac{\rho}{2}v_7^2 \\ \text{Continuity:} \quad \frac{\pi D^2}{4} v_{steady} &= \frac{\pi d^2}{4} v_7 \quad \Rightarrow \quad v_7 = \frac{D^2}{d^2} v_{steady} \\ \Rightarrow \Delta h &= \frac{v_7^2 - v_{steady}^2}{2g} = \frac{v_{steady}^2}{2g}\left(\frac{D^4}{d^4} - 1\right) \\ \Leftrightarrow \Delta h &= (h_1 - h_2 + h_3)\left(\frac{D^4}{d^4} - 1\right) \end{aligned}$$

d) p_{min} in steady state flow between 4 and 5

$$\begin{aligned} p_a + \rho g h_1 &= p_4 + \frac{\rho}{2} v_{steady}^2 + \rho g h_2 \\ \text{with} \quad p_{4min} &= p_D \quad \text{and} \quad v_{steady}^2 = 2g(h_1 - h_2 + h_3) \\ h_3 &= \frac{p_a - p_D}{\rho g} \\ \Rightarrow h_2 \quad \text{is arbitrarily chosen} : h_{3max} &= \frac{p_a - p_D}{\rho g} \end{aligned}$$

Problem 3 (13 Points)

A pump consists of a cylindrical shaft with radius R_i, which rotates in an inner cylinder with radius R_a. Both cylinders have the length L. In the gap between the cylinders is a fluid of viscosity η. The shaft rotates with the angular velocity ω_i, at the outer cylinder the torque M_a is measured at the angular velocity ω_a.



a) Formulate the momentum equilibrium for a cylindrical element to show the validity of:

$$\frac{\partial(r^2\tau)}{\partial r} = 0$$

.

b) Consider the hints and determine the velocity distribution $v(r, \omega_a)$ as a function of the angular velocity ω_a.

c) Determine the maximum torque that can be converted and the associated angular velocity ω_a.

d) Determine the angular velocity ω_a for which the maximum power is converted.

Given:

$R_i, \quad R_a, \quad \omega_i, \quad L, \quad \eta, \quad 0 \leq \omega_a \leq \omega_i$

Hints:

- Higher-order terms are negligible.
- The flow in the gap is steady and fully developed.
- The following equations are given:

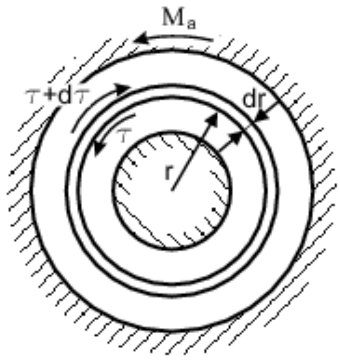
$$\frac{1}{r^2}\frac{d(r^2\tau)}{dr} = -\eta\frac{d}{dr}\left(\frac{1}{r}\frac{d(rv)}{dr}\right)$$

$$\tau = -\eta r\frac{d(\frac{v}{r})}{dr}$$

- Check the units and signs of your results!

Problem 3

a) Equilibrium of moments:



$$\tau \cdot 2\pi r L \cdot r - (\tau + \frac{\partial \tau}{\partial r} dr) \cdot 2\pi L(r + dr) \cdot (r + dr) = 0$$

$$\tau r^2 - (\tau + \frac{\partial \tau}{\partial r} dr)(r^2 + 2rdr + dr^2) = 0$$

$$-2\tau r dr - \tau dr^2 - r^2 \frac{\partial \tau}{\partial r} dr - 2rdr \frac{\partial \tau}{\partial r} dr - \frac{\partial \tau}{\partial r} dr dr^2 = 0$$

$$2\tau r dr + r^2 \frac{\partial \tau}{\partial r} dr = dr(2\tau r + \frac{\partial \tau}{\partial r} r^2) = 0 \quad \text{da Terme } O(2) \approx 0$$

$$\frac{\partial (r^2 \tau)}{\partial r} = 0$$

b) Use given velocity distribution: $\Rightarrow \eta \frac{d}{dr}\left(\frac{1}{r}\frac{d(rv)}{dr}\right) = 0$

1st Integration: $\frac{1}{r}\frac{d(rv)}{dr} = C_1$; 2nd Integration: $rv = \frac{1}{2}C_1 r^2 + C_2$

Boundary conditions: $v(r = R_i) = \omega_i R_i$; $v(r = R_a) = \omega_a R_a$;

$$R_i^2 \omega_i = \frac{1}{2} C_1 R_i^2 + C_2; \quad R_a^2 \omega_a = \frac{1}{2} C_1 R_a^2 + C_2$$

$$\Rightarrow C_1 = \frac{2(R_i^2 \omega_i - R_a^2 \omega_a)}{R_i^2 - R_a^2}; \quad C_2 = \frac{R_i^2 R_a^2 (\omega_a - \omega_i)}{R_i^2 - R_a^2}$$

$$v(r, \omega_a) = \frac{R_i^2 \omega_i - R_a^2 \omega_a}{R_i^2 - R_a^2} r + \frac{R_i^2 R_a^2 (\omega_a - \omega_i)}{R_i^2 - R_a^2} \frac{1}{r}$$

c) Maximum moment at $\omega_a = 0 \Rightarrow M_a = 2\pi R_a^2 L \tau(r = R_a)$

$$\tau = -\eta r \frac{d}{dr}\left[\frac{1}{2}C_1 + C_2 \frac{1}{r^2}\right] = -\eta r \left(-\frac{2C_2}{r^3}\right) = 2\eta \frac{C_2}{r^2} \Rightarrow \tau(r = R_a) = 2\eta \frac{C_2}{R_a^2}$$

$$M_{a,max} = -4\pi L \eta \frac{R_i^2 R_a^2 \omega_i}{R_i^2 - R_a^2}$$

d) Maximum power: $P = M_a \omega_a = 2\pi R_a^2 L \tau(r = R_a)\omega_a \propto \omega_a C_2$

$$\frac{\partial P}{\partial \omega_a} \doteq 0 = \frac{\partial(\omega_a^2 - \omega_i \omega_a)}{\partial \omega_a} = 2\omega_a - \omega_i \Rightarrow \omega_a = \omega_i / 2$$

Problem 4 (9 Points)

The symmetrical flow around two identical half bodies is considered, whose singularities have the distance b from each other and whose connecting line is perpendicular to the flow direction.

a) Specify the complex potential function $F(z)$ and the sign(s) of the constants of the elementary function(s) used for the described flow.

b) Calculate, preferably from the conjugated complex velocity field $\bar{w}(x, y)$, the velocity components $u(x, y)$ and $v(x, y)$.

c) Qualitatively sketch the calculated flowfield. Draw the coordinate system used and mark the stagnation point streamline(s), the stagnation point(s) and the body contours.

Given: a, b, all necessary constants of the elementary functions

Known complex potential functions:

Potential vortex:	$F(z) = -\frac{i\Gamma}{2\pi} \ln z$
Source/Sink:	$F(z) = \frac{E}{2\pi} \ln z$
Dipole:	$F(z) = \frac{M}{2\pi z}$
stagnation point flow:	$F(z) = \alpha z^2$
Parallel flow:	$F(z) = (u_\infty - iv_\infty) z$

Hints:

- $z = x + iy = r \cdot e^{i\varphi} = r(\cos\varphi + i \sin\varphi)$
- $\dfrac{1}{z} = \dfrac{\bar{z}}{z\bar{z}}$

Problem 4

a) 2 half bodies: 2 sources at distance b + parallel flow
Symmetrical flow field: Sources of equal strength

$$\begin{aligned} F(z) &= F_{Parallel} + F_{Source_1} + F_{Source_2} \\ \Rightarrow F(z) &= u_\infty z + \frac{E}{2\pi}\ln z + \frac{E}{2\pi}\ln(z - ib) \text{ with } u_\infty, E, b > 0 \end{aligned}$$

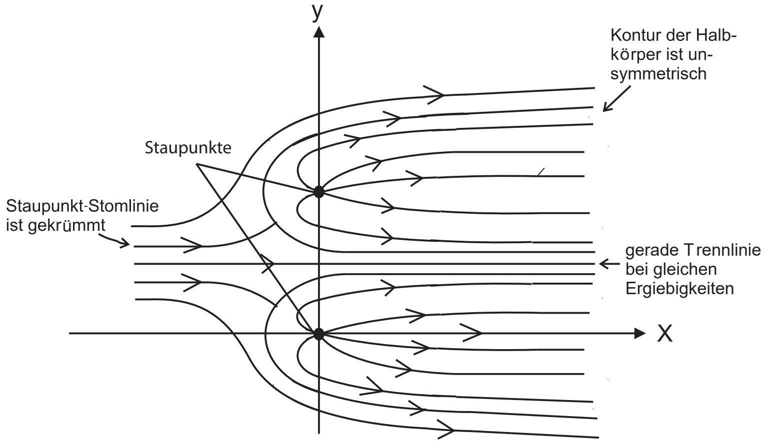
Note: arithmetic operation at $\ln(z \pm ib)$ depending on the coordinate system from the sketch in part c)

b) complex conjugated speed $\bar{w} = \dfrac{dF}{dz} = u - iv$

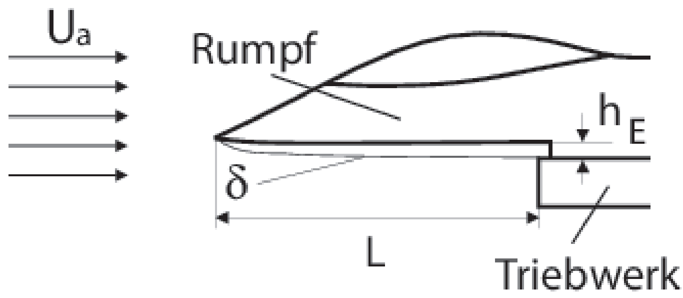
$$\begin{aligned} \bar{w} = \frac{dF}{dz} &= u_\infty + \frac{E}{2\pi z} + \frac{E}{2\pi(z - ib)} \\ \text{with } \frac{1}{z} &= \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{x^2 + y^2} \\ \text{and } \frac{1}{(z - ib)} &= \frac{1}{x + i(y - b)}\frac{(x - i(y - b))}{(x - i(y - b))} = \frac{(x - i(y - b))}{x^2 + (y - b)^2} \end{aligned}$$

$$\begin{aligned} \rightarrow \bar{w} &= u_\infty + \frac{E}{2\pi}\left(\frac{x - iy}{x^2 + y^2} + \frac{(x - i(y - b))}{x^2 + (y - b)^2}\right) \\ u(x, y) &= u_\infty + \frac{E}{2\pi}\left(\frac{x}{x^2 + y^2} + \frac{x}{x^2 + (y - b)^2}\right) \\ v(x, y) &= \frac{E}{2\pi}\left(\frac{y}{x^2 + y^2} + \frac{y - b}{x^2 + (y - b)^2}\right) \end{aligned}$$

c) Sketch:



Problem 5 (11 Points)



An aircraft is powered by a jet engine, the inlet of which is located on the lower side. For an angle of attack $\alpha = 0$, consider the boundary layer that forms on the lower side of the fuselage during a subsonic flight at speed U_a. The upstream effect of the engine inlet can be neglected. The velocity profile in the boundary layer can be represented by the following polynomial approach:

$$\frac{u(x,y)}{U_a} = \sum_{i=0}^{4} a_i \left(\frac{y}{\delta}\right)^i .$$

a) Determine the coefficients a_i assuming an even flow.

b) Explain and sketch without specifying the formula the physical meaning of the displacement thickness δ_1.

c) For the ratio of the momentum thickness to the boundary layer thickness in the case under consideration $\delta_2/\delta = 37/315$ applies. Prove with the help of von Kármán's integral relation the connection

$$\frac{\delta}{x} = \frac{5,84}{\sqrt{Re_x}} .$$

d) Determine the distance h_E between the engine and the fuselage (see sketch) such that no fuselage boundary layer enters the engine inlet.

Given: $\rho,\ \eta,\ L,\ U_a = \text{const.}$

Hint:

von Kármán integral relation: $\dfrac{d\delta_2}{dx} + \dfrac{1}{U_a}\dfrac{dU_a}{dx}\left(2\delta_2 + \delta_1\right) = \dfrac{\tau_w}{\rho U_a^2}$

Problem 5

a)

$$\begin{aligned}
\frac{u}{U_a} &= a_0 + a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3 + a_4\left(\frac{y}{\delta}\right)^4 \\
\frac{\partial\left(\frac{u}{U_a}\right)}{\partial\left(\frac{y}{\delta}\right)} &= a_1 + 2a_2\left(\frac{y}{\delta}\right) + 3a_3\left(\frac{y}{\delta}\right)^2 + 4a_4\left(\frac{y}{\delta}\right)^3 \\
\frac{\partial^2\left(\frac{u}{U_a}\right)}{\partial\left(\frac{y}{\delta}\right)^2} &= 2a_2 + 6a_3\left(\frac{y}{\delta}\right) + 12a_4\left(\frac{y}{\delta}\right)^2
\end{aligned}$$

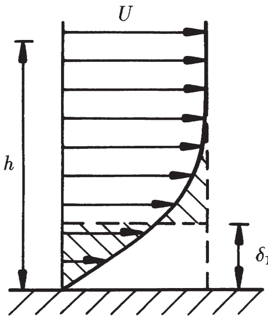
Boundary conditions:

$$\begin{array}{lll}
1st\,BC & \frac{y}{\delta}=0 & : \quad \frac{u}{U_a} = 0 \Rightarrow a_0 = 0 \\
2nd\,BC & \frac{y}{\delta}=1 & : \quad \frac{u}{U_a} = 1 \Rightarrow a_1 + a_2 + a_3 + a_4 = 1 \\
3rd\,BC & \frac{y}{\delta}=0 & : \quad \frac{\partial^2\left(\frac{u}{U_a}\right)}{\partial\left(\frac{y}{\delta}\right)^2} = 0 \Rightarrow a_2 = 0 \\
4th\,BC & \frac{y}{\delta}=1 & : \quad \frac{\partial\left(\frac{u}{U_a}\right)}{\partial\left(\frac{y}{\delta}\right)} = 0 \Rightarrow a_1 + 3a_3 + 4a_4 = 0 \\
5th\,BC & \frac{y}{\delta}=1 & : \quad \frac{\partial^2\left(\frac{u}{U_a}\right)}{\partial\left(\frac{y}{\delta}\right)^2} = 0 \Rightarrow 6a_3 + 12a_4 = 0
\end{array}$$

$$\begin{aligned}
\text{Thus follows from the } 5th\,BC \quad a_3 &= -2a_4, \\
\text{inserted into the } 4th\,BC \quad a_1 &= 2a_4, \\
\text{inserted into the } 2nd\,BC \quad a_4 &= 1 \rightarrow a_3 = -2 \rightarrow a_1 = 2
\end{aligned}$$

$$\frac{u}{U_a} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$$

b) 1.) Displacement thickness: The distance by which the body must be thickened in a theoretically friction-free flow such that the same mass flow occurs as in the actual frictional flow.



c) von Kármán's integral relationship results for the plane problem: $\dfrac{d\delta_2}{dx} = \dfrac{\tau_w}{\rho U_a^2}$

$$
\begin{aligned}
\text{with} \quad \frac{d\delta_2}{dx} &= \frac{37}{315}\frac{d\delta}{dx} \\
\text{and} \quad \tau_w &= \eta \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\eta U_a}{\delta} \left.\frac{\partial\left(\frac{u}{U_a}\right)}{\partial\left(\frac{y}{\delta}\right)}\right|_{\frac{y}{\delta}=0} = \frac{2\eta U_a}{\delta} \\
\rightarrow \frac{37}{315}\frac{d\delta}{dx} &= \frac{2\eta}{\rho U_a \delta} \\
\delta d\delta &= \frac{630}{37}\frac{\eta}{\rho U_a} dx \\
\frac{\delta^2}{2} &= \frac{630}{37}\frac{\eta x}{\rho U_a} \\
\frac{\delta}{x} &= \sqrt{\frac{1260}{37}}\frac{1}{\sqrt{Re_x}} = \frac{5.84}{\sqrt{Re_x}} \quad q.e.d.
\end{aligned}
$$

d)

$$
h_E = \delta(L) = \frac{5,84}{\sqrt{\frac{\rho U_a}{\eta L}}}
$$

Problem 6 (7 Points)

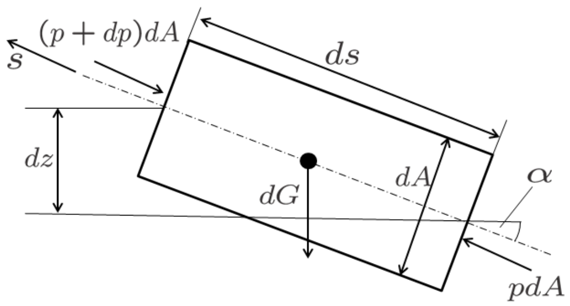
a) Derive the Bernoulli equation without losses for an inclined streamline element.

b) Sketch the mean velocity profile of a turbulent flow in a circular pipe. Mark the individual layers.

Problem 6

a)

$$
\begin{aligned}
\sum d\vec{F} &= dm\frac{d\vec{v}}{dt} \\
\Leftrightarrow -dG sin\alpha + pdA - (p+dp)dA & \\
\text{with} \quad dG &= gdm = \rho g ds dA \quad \text{and} \quad sin\alpha = \frac{dz}{ds} \\
\Rightarrow \rho ds dA\frac{dv}{dt} &= -\rho g ds dA\frac{dz}{ds} - dp dA \\
\Leftrightarrow \rho\frac{dv}{dt} &= -\rho g\frac{dz}{ds} - \frac{dp}{ds}
\end{aligned}
$$



Since $v = F(s,t)$, applies:

$$
dv = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial s}ds
$$

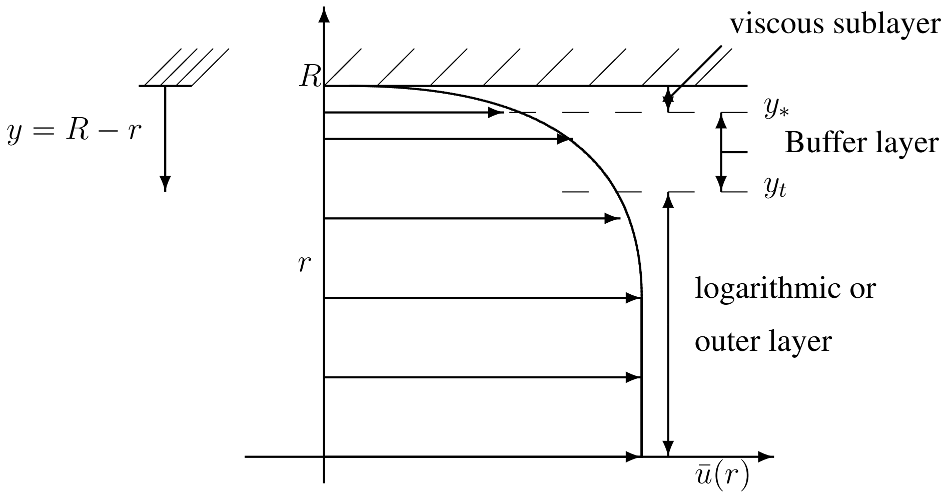
$$
\Rightarrow \frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s} = \frac{\partial v}{\partial t} + \frac{\partial(\frac{v^2}{2})}{\partial s}
$$

Insert results:

$$
\rho\frac{\partial v}{\partial t} + \frac{\rho}{2}\frac{\partial v^2}{\partial s} + \frac{dp}{ds} + \rho g\frac{dz}{ds} = 0
$$

$$
\Rightarrow p + \frac{\rho}{2}v^2 + \rho g z + \rho\int\frac{\partial v}{\partial t}ds = const.
$$

b) Sketch


viscous sublayer
R
$y = R - r$
y_*
Buffer layer
y_t
r
logarithmic or
outer layer
$\bar{u}(r)$