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(Matr.-o, Signature)

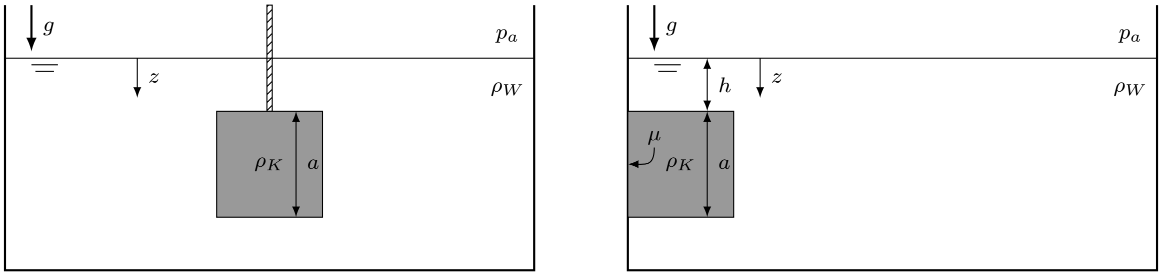
Exam „Fluid Dynamics“

14 September 2023

Leer

Problem 1 (9 Points)

A cube of the density ρ_K and the edge length a is submerged using a negligibly thin rope into a basin filled with a liquid of the density ρ_W. It is $\rho_K > \rho_W$. The cube does not touch the wall of the pool (see left figure below).



a)+b) c)+d)

a) Draw the pressure distribution in the projected plane along the horizontal and vertical surfaces of the body. To do so, transfer the left figure below to your solution sheet.

b) Determine the buoyancy force as a function of the cube volume by integrating the pressure forces on the 6 surfaces of the cube.

Next, the cube is submerged into the fluid on the left wall of the pool. The contact surface between the cube and the wall is ideal and free of fluid. The friction coefficient between the wall and the cube is μ.

c) Again, draw the pressure distribution in the projected plane along the surface of the cube carefully. Transfer the above right figure from the task to your solution sheet.

d) Determine the depth h at which the cube reaches its equilibrium state. Assume a steady state problem and neglect the inertia of the cube.

Given:

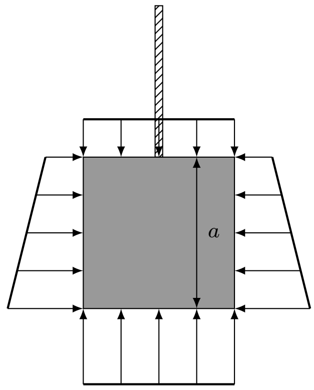
$\rho_W, \quad \rho_K, \quad \rho_K > \rho_W, \quad a, \quad g, \quad p_a, \quad \mu$

Hints:

- The friction force can be determined with $F_R = \mu F_N$, where F_N is the contact force. Note that, no distinction is made between sliding and static friction.
- Check your results for unit and sign plausibility!

Problem 1

a) Pressure distribution:



b) Surface integral

$\vec{F} = -\int_A p\, d\vec{A}$

$$\vec{F} = -\left[\underbrace{\int_{left} p\, d\vec{A} + \int_{right} p\, d\vec{A}}_{=0} + \underbrace{\int_{front} p\, d\vec{A} + \int_{back} p\, d\vec{A}}_{=0} + \int_{top} p\, d\vec{A} + \int_{bottom} p\, d\vec{A}\right]$$

$F_A = -\left[\int_{a^2} p_{top}\, d\vec{A} + \int_{a^2} p_{bottom}\, d\vec{A}\right]$

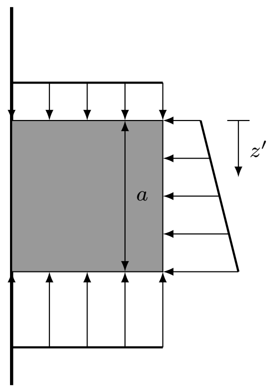
The hydrostatic equation provides:

$p_{top} = p_a + \rho_W g h_{top}$ and $p_{bottom} = p_a + \rho_W g(h_{top} + a)$

Substituting:

$F_A = -a^2[(p_a + \rho_W g h_{top}) - (p_a + \rho_W g(h_{top} + a))] = \rho_W g a^3 \ (= \rho_W g V_K)$

c) Pressure distribution:



d) Surface Integral:

$$\vec{F} = \underbrace{\int_{left} p\, d\vec{A} + \int_{right} p\, d\vec{A}}_{\neq 0} + \underbrace{\int_{front} p\, d\vec{A} + \int_{back} p\, d\vec{A}}_{=0} + \int_{top} p\, d\vec{A} + \int_{bottom} p\, d\vec{A}$$

The upward buoyant force is determined similarly to b) as

$F_A = \rho_W g a^3$.

Contact with the wall results in a pressure force directed towards the wall:

$F_N = \int_A p_{right}(z')\, dA$

The hydrostatic equation yields:

$p_{right}(z') = p_{top} + \rho_W g z' = p_a + \rho_W g h + \rho_W g z'$

Substituting:

$F_N = a \int_a (p_a + \rho_W g h + \rho_W g z') dz' = a^2(p_a + \rho_W g h + \frac{1}{2}\rho_W g a)$.

The force balance around the cube leads to:

$0 = F_A + F_R - F_G$

Where the frictional force acts against the direction of motion.

Substituting:

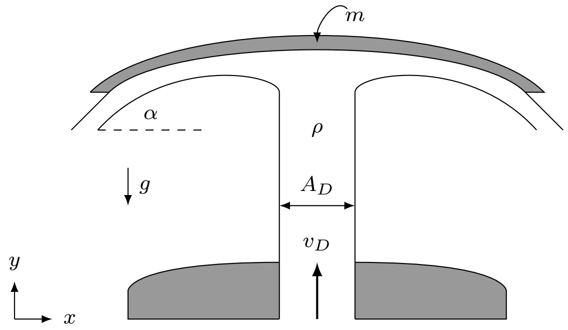
$0 = \rho_W g a^3 + \mu F_N - \rho_K g a^3 = \rho_W g a^3 + \mu a^2(p_a + \rho_W g h + \frac{1}{2}\rho_W g a) - \rho_K g a^3$

Rearranging:

$h = \frac{\frac{\rho_K g a^3 - \rho_W g a^3}{\mu a^2} - p_a - \frac{1}{2}\rho_W g a}{\rho_W g} = \frac{a}{\mu}(\frac{\rho_K}{\rho_W} - 1) - \frac{p_a}{\rho_W g} - \frac{1}{2}a$

Problem 2 (11 Points)

A Frisbee with mass m enters the stream of a fountain. The water jet impinges upon the disc and is divided into two equal-sized partial streams. The problem can be assumed two-dimensional. Initially, the Frisbee does not move.



a) Determine the mass m of the Frisbee.

To remove the Frisbee from the water jet, the velocity of the jet is reduced to v_D^*. Therefore, the Frisbee descends at a speed of $\frac{1}{4}v_D^*$.

b) Determine the new velocity of the water jet v_D^*.

We consider the momentum equation in the moving coordinate system.

c) Show through a short calculation that

$$\frac{d\vec{I}}{dt} = \int_{KF} \rho \vec{v}_{abs}(\vec{v}_{rel} \cdot \vec{n})dA = \int_{KF} \rho \vec{v}_{rel}(\vec{v}_{rel} \cdot \vec{n})dA$$

holds.

Given:

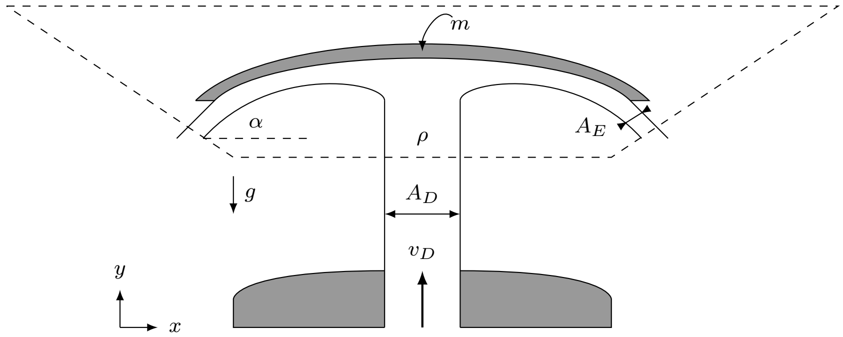
$\rho, \quad g, \quad \alpha, \quad v_D, \quad A_D$

Hints:

- The flow is frictionless.
- The gravitational force of the fluid can be neglected.
- Check your results for unit and sign plausibility!

Problem 2

a) Momentum equation in the fixed coordinate system



In the y-direction:

$\frac{dI}{dt} = -\rho A_D v_D^2 - 2\rho v_E^2 A_E \sin\alpha = -mg$

Using Bernoulli and $p_D = p_E = p_a$, we get:

$v_E = v_D$

Also, from symmetry and continuity:

$A_E = \frac{1}{2}A_D$

Rearranging:

$m = \frac{\rho}{g} v_D^2 A_D (1 + \sin\alpha)$

b) Momentum equation in the moving coordinate system

In the y-direction:

$\frac{dI}{dt} = -\rho A_D v_{D,rel}^{*2} - 2\rho v_{E,rel}^{*2} A_E \sin\alpha = -mg$

Still, $A_E = \frac{1}{2}A_D$.

Using Bernoulli in the moving system and $p_D = p_E = p_a$:

$v_{E,rel}^* = v_{D,rel}^*$

Rearranging and substituting from part (a):

$\rho v_{D,rel}^{*2} A_D (1 + \sin\alpha) = \rho v_D^2 A_D (1 + \sin\alpha)$

With $v_{D,rel}^* = v_D^* + \frac{1}{4}v_D^* = \frac{5}{4}v_D^*$:

$\Rightarrow v_D^* = \frac{4}{5}v_D$

c) It holds:

$\frac{d\vec{I}}{dt} = \int_{KF} \rho \vec{v}_{abs} (\vec{v}_{rel} \cdot \vec{n}) dA$

where $\vec{v}_{abs} = \vec{v}_F + \vec{v}_{rel}$, yielding:

$\int_{KF} \rho \vec{v}_F (\vec{v}_{rel} \cdot \vec{n}) dA + \int_{KF} \rho \vec{v}_{rel} (\vec{v}_{rel} \cdot \vec{n}) dA$

Since $\vec{v}_F$ is constant, the first term becomes:

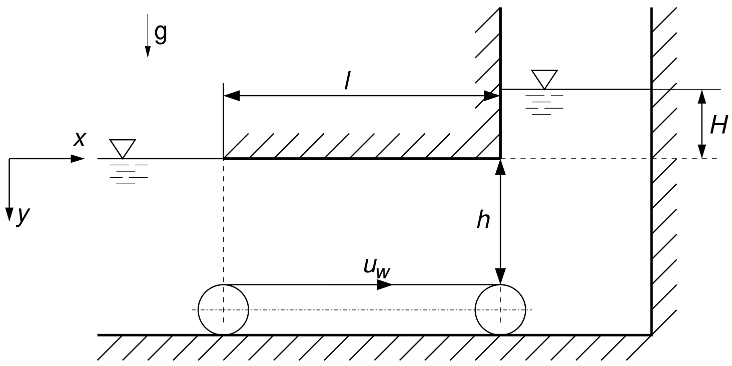
$\vec{v}_F \int_{KF} \rho (\vec{v}_{rel} \cdot \vec{n}) dA = 0$

due to mass conservation of the system. What remains is:

$\int_{KF} \rho \vec{v}_{abs} (\vec{v}_{rel} \cdot \vec{n}) dA = \int_{KF} \rho \vec{v}_{rel} (\vec{v}_{rel} \cdot \vec{n}) dA$

Problem 3 (11 Points)

A conveyor belt with a circumferential velocity of u_w conveys oil through a laminar flow in a gap. The oil has the viscosity η and the density ρ. Outside the oil, the ambient pressure is uniform.



a) Establish the force equilibrium in the direction of flow for a fluid element and simplify it to formulate the differential equation for the shear stress.

b) Determine the velocity profile $u(y)$ and the shear stress profile $\tau(y)$ as a function of the pressure gradient $\frac{\partial p}{\partial x}$.

c) What maximum height difference H can be achieved?

Given:

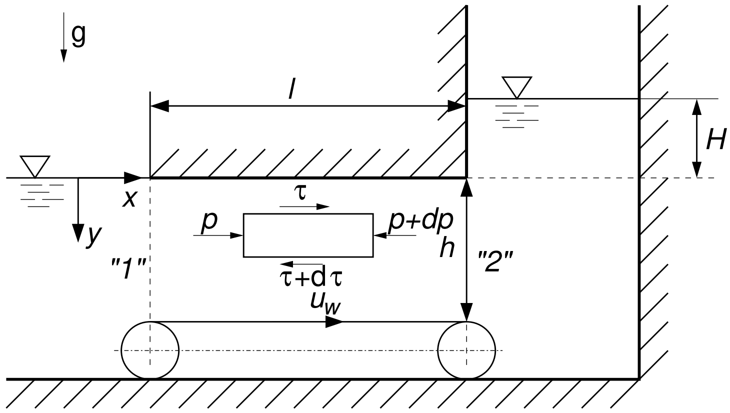
u_w, ρ, η, h, l, g

Hints:

- Assume that the flow in the gap (height h) is fully developed along the length of the belt l.
- Check your results for unit and sign plausibility!

Problem 3

a) Force balance on the volume element in the x-direction:



$$\left(p - \left(p + \frac{\partial p}{\partial x}dx\right)\right)dy + \left(\tau - \left(\tau + \frac{d\tau}{dy}dy\right)\right)dx = 0$$

$$\Rightarrow \frac{d\tau}{dy} = -\frac{\partial p}{\partial x}$$

b) double integration:

$$\rightarrow \tau(y) = -\frac{\partial p}{\partial x}y + c_1$$

With $\tau = -\eta\frac{\partial u}{\partial y}$

$$\rightarrow u(y) = \frac{\frac{\partial p}{\partial x}}{2\eta}y^2 - \frac{c_1}{\eta}y + c_2$$

Boundary conditions:

$$\left.\begin{array}{ll} y = 0: & u = 0 \\ y = h: & u = u_w \end{array}\right\} \rightarrow c_1 = \frac{\frac{\partial p}{\partial x}}{2}h - \frac{u_w\eta}{h} \quad \text{and} \quad c_2 = 0$$

$$\Rightarrow \tau(y) = \frac{\frac{\partial p}{\partial x}}{2}\left(\frac{h}{2} - y\right) - \frac{u_w\eta}{h}$$

$$\Rightarrow u(y) = \frac{\frac{\partial p}{\partial x}}{2\eta}\left(y^2 - yh\right) + y\frac{u_w}{h}$$

c) Maximum height $\rightarrow \quad \dot{V} = 0$

Bernoulli/Hydrostatics :

$$p_2 = p_1 + \rho g H\ , \quad \frac{\partial p}{\partial x} = \frac{\rho g H}{l}$$

$$\rightarrow u(y) = \frac{\rho g H}{2\eta l}\left(y^2 - yh\right) + y\frac{u_w}{h}$$

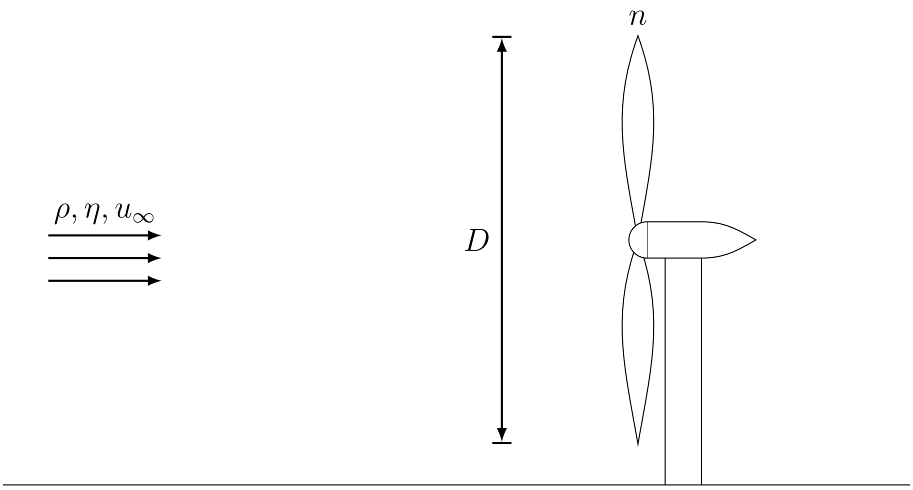
$$\rightarrow \dot{V} = 0 = \int_0^h \left[\frac{\rho g H}{2\eta l}\left(y^2 - yh\right) + y\frac{u_w}{h}\right]dy$$

$$\rightarrow 0 = \frac{\rho g H}{2\eta l}\left(-\frac{h^3}{6}\right) + \frac{h u_w}{2}$$

$$\rightarrow H = \frac{6\eta l u_w}{\rho g h^2}$$

Problem 4 (8 Points)

The upstream airflow (density ρ, dynamic viscosity η) of a wind turbine with rotor diameter D is defined at constant freestream velocity. The rotational speed of the turbine is n. The incoming flow is considered incompressible.



a) How many dimensionless numbers describe the flow field?

b) Determine the dimensionless numbers of this flow field using dimensional analysis (the Π-theorem).

c) Express the dimensionless numbers determined in b) using standard similarity parameters in fluid mechanics.

A model of the system with a rotor diameter of D' is investigated in a wind tunnel. Findings for the real system are to be determined by measuring the rotational speed n' and torque M' on the model. The fluid in the wind tunnel is also air (ρ, η).

d) Determine the rotational speed n and torque M of the real system under the assumption that the inflow conditions in the wind tunnel have been chosen such that the data can be extrapolated to the real case.

Given:

$\rho, \eta, u_\infty, D, D', n', M'$

Problem 4

a)

$$\begin{aligned} &\text{Parameters: } k: \quad u_\infty, \rho, \eta, D, n \\ &\text{Dimensions: } r: \quad M, L, T \quad \text{(3 repeating variables)} \\ &\Rightarrow k = 5, r = 3 \rightarrow k - r = 2 \text{ dimensionless numbers} \qquad 1_{1} \end{aligned}$$

b) Choice of repeating variables, e.g., n, D, ρ

$$\begin{aligned} \Pi_1 =& u_\infty n^{\alpha_1} D^{\beta_1} \rho^{\gamma_1} \\ & M: 0 + 0\alpha_1 + 0\beta_1 + 1\gamma_1 = 0 \rightarrow \gamma_1 = 0 \\ & L: 1 + 0\alpha_1 + 1\beta_1 - 3\gamma_1 = 0 \rightarrow \beta_1 = 3\gamma_1 - 1 = -1 \\ & T: -1 - 1\alpha_1 + 0\beta_1 + 0\gamma_1 = 0 \rightarrow \alpha = -1 \\ & \Rightarrow \Pi_1 = \frac{u_\infty}{nD} \qquad 1_{2} \end{aligned}$$

$$\begin{aligned} \Pi_2 =& \eta\, n^{\alpha_2} D^{\beta_2} \rho^{\gamma_2} \\ & M: 1 + 0\alpha_2 + 0\beta_2 + 1\gamma_2 = 0 \rightarrow \gamma_2 = -1 \\ & L: -1 + 0\alpha_2 + 1\beta_2 - 3\gamma_2 = 0 \rightarrow \beta_2 = 3\gamma_2 + 1 = -2 \\ & T: -1 - 1\alpha_2 + 0\beta_2 + 0\gamma_2 = 0 \rightarrow \alpha = -1 \\ & \Rightarrow \Pi_2 = \frac{\eta}{nD^2\rho} \qquad 1_{3} \end{aligned}$$

c) With the circumferential velocity of the blade tips $u_\Theta = \pi D n$, the transformation of the second dimensionless number is evident.

$$\Pi_1 = \frac{u_\infty}{nD} = \frac{1}{Sr} \qquad 1_{4}$$

$$\Pi_2 = \frac{\eta}{nD^2\rho} = \frac{\eta}{\rho D(nD)} = \frac{1}{Re} \qquad 1_{5}$$

d)

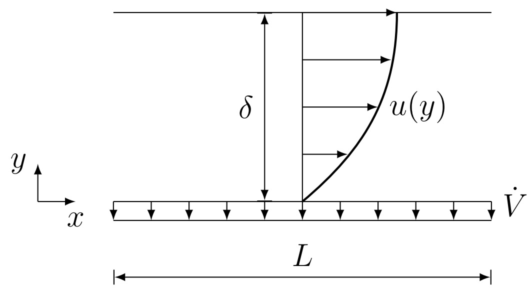
$$Re = Re' : \frac{\rho D^2 n}{\eta} = \frac{\rho D'^2 n'}{\eta} \rightarrow D^2 n = D'^2 n' \rightarrow n = n' \left(\frac{D'}{D}\right)^2 \qquad 1_{6}$$

$$Sr = Sr' : \frac{nD}{u_\infty} = \frac{n'D'}{u'_\infty} \rightarrow u'_\infty = u_\infty \frac{n'}{n} \frac{D'}{D} = u_\infty \frac{D}{D'} \qquad 1_{7}$$

$$C_M = C'_M : \frac{M}{\rho u_\infty^2 D^3 \pi} = \frac{M'}{\rho u'^2_\infty D'^3 \pi} \rightarrow M = M' \left(\frac{u_\infty}{u'_\infty}\right)^2 \left(\frac{D}{D'}\right)^3 = M' \frac{D}{D'} \qquad 1_{8}$$

Problem 5 (12 Points)

On the upper surface of a flat plate with a width B a constant external velocity u_a generates a laminar, incompressible boundary layer. Using a suction device, which sucks a constant volume flow rate $\dot{V}$ distributed evenly over a length L, the drag force is reduced, resulting in a constant boundary layer thickness δ.



The following approach is used to approximate the velocity profile in the laminar boundary layer:

$$\frac{u(x,y)}{u_a} = a_0 + a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 - a_1\left(\frac{y}{\delta}\right)^3 .$$

a) Determine the coefficients a_0, a_1, a_2, and thus the velocity profile $u(y/\delta)$ in the boundary layer.

b) Determine the tangential force F_T acting on the upper surface of the plate.

c) Calculate the tangential force that occurs when the suction system is turned off. Assume a linear velocity profile in the boundary layer.

d) From a technical point of view, what is the maximum reasonable volume flow rate $\dot{V}$? State the relevant condition. An explicit solution for $\dot{V}$ is not necessary.

Given:

$\eta,\ \rho,\ u_a = const.,\ \dot{V},\ B,\ L$

Hint:

- Boundary layer equations (2D, incompressible)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \eta\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0$$

- von Kármán integral relationship (extended with suction velocity v_a)

$$\frac{d\delta_2}{dx} + \frac{1}{u_a}\frac{du_a}{dx}(2\delta_2 + \delta_1) - \frac{\tau_W}{\rho u_a^2} = \frac{v_a}{u_a}$$

Problem 5

a) Three boundary conditions to determine the unknowns a_i.

$$\text{No-slip condition:} \quad \frac{y}{\delta} = 0 : u = 0 \to a_0 = 0$$

$$\text{Boundary layer edge:} \quad \frac{y}{\delta} = 1 : u = u_a \to a_2 = 1 \qquad 1_1$$

$$\text{Wall boundary from x-momentum:} \quad \frac{y}{\delta} = 0 : v \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\eta}{\rho} \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} \qquad 1_2$$

Substituting the known coefficients into the velocity distribution and using the definition of the volume flow rate $v|_{y=0} = -\frac{\dot{V}}{BL}$ $\quad 1_3$

$$\frac{u}{u_a} = a_1 \frac{y}{\delta} + \left(\frac{y}{\delta}\right)^2 - a_1 \left(\frac{y}{\delta}\right)^3 \to \frac{\partial}{\partial y}\left(\frac{u}{u_a}\right) = a_1 \left(\frac{1}{\delta} - \frac{3y^2}{\delta^3}\right) + \frac{2y}{\delta^2}$$

$$\to \frac{\partial^2}{\partial y^2}\left(\frac{u}{u_a}\right) = a_1 \left(-\frac{6y}{\delta^3}\right) + \frac{2}{\delta^2}$$

Plugging into wall boundary condition: $-\frac{\dot{V}}{BL}\frac{a_1 u_a}{\delta} = \frac{\eta}{\rho}\frac{2u_a}{\delta^2} \to a_1 = -\frac{2BL\eta}{\rho\delta\dot{V}}$ $\quad 1_4$

b) For the given problem, the von Kármán integral relationship simplifies to

$$\frac{du_a}{dx} = 0\,, \quad \frac{d\delta_2}{dx} = 0\,, \quad v_a = -\frac{\dot{V}}{BL} \quad \Rightarrow \frac{\tau_W}{\rho u_a} = \frac{\dot{V}}{BL} \qquad 1_5$$

The tangential force is given by

$$F_T = \int_L \tau_W B dx = \int_L \frac{\rho \dot{V} u_a}{BL} B dx = \rho \dot{V} u_a \qquad 1_6$$

c) For the switched-off system (here subscript off), <u>no</u> constant boundary layer thickness is established. $\quad 1_7$

The velocity profile is given by $\frac{u_{off}}{u_a} = \frac{y}{\delta_{off}(x)}$. It follows:

$$\frac{du_a}{dx} = 0\,, \quad v_a = 0\,, \quad \frac{\delta_2}{\delta} = \int_0^1 \frac{u}{u_a}\left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 d\left(\frac{y}{\delta}\right) = \frac{1}{6} \qquad 1_8$$

$$\Rightarrow \frac{1}{6}\frac{d\delta}{dx} = \frac{\tau_W}{\rho u_a^2} = \frac{1}{\rho u_a}\eta \left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\eta}{\rho\delta} \quad \to \quad \delta d\delta = \frac{6\eta}{\rho}dx \quad \to \quad \frac{1}{2}\delta^2 = \frac{6\eta}{\rho}x$$

$$\to \delta_{off} = \sqrt{\frac{12\eta}{\rho}}\sqrt{x} \quad \Rightarrow \quad \tau_{W,off} = \frac{\eta u_a}{\delta_{off}} = \frac{\sqrt{\eta\rho}u_a}{\sqrt{12}}\frac{1}{\sqrt{x}} \qquad 1_9$$

$$\Rightarrow F_{T,off} = \int_L \tau_{W,off} B dx = \frac{\sqrt{\eta\rho}u_a}{\sqrt{3}}\sqrt{x}\Big|_0^L = \frac{\sqrt{\eta\rho}u_a}{\sqrt{3}}\sqrt{L} \qquad 1_{10}$$

d) The application is meaningful if the power savings ΔP_W are at least equal to the applied power P_a, i.e., $\Delta P_W \geq P_a$. 1 11

$$\Delta P_W = (F_{T,off} - F_T)u_a$$

$$P_a = \dot{V}\Delta p_{tot} = \dot{V}\frac{\rho}{2}(u_a^2 - v_a^2) = \dot{V}\frac{\rho}{2}\left(u_a^2 - \left(\frac{\dot{V}}{BL}\right)^2\right)$$ 1 12

Problem 6 (9 Points)

a) Describe the terms streamlines, pathlines, and streaklines.

b) Name three non-Newtonian fluid types and sketch the shear stress as a function of shear rate for each.

Considerations are now made regarding boundary layer flows over flat plates with different relative roughnesses k/L.

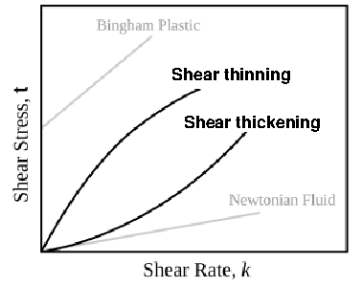
c) Sketch in a diagram the skin friction coefficient c_f against the Reynolds number Re_L for i) laminar flow, ii) turbulent flow over a smooth plate, and iii) fully turbulent flow over a rough plate.

d) Name one advantage and one disadvantage of turbulent flow compared to laminar flow in technical applications.

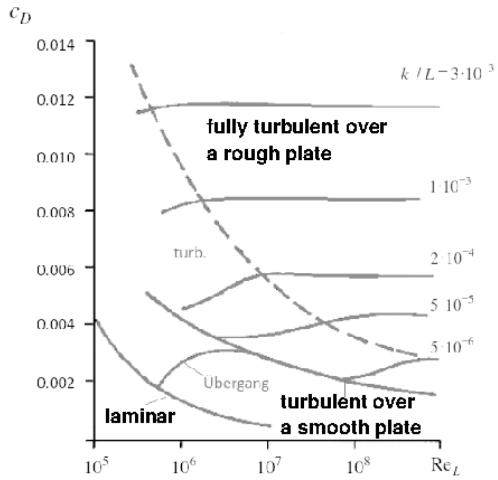
Problem 6

a)
- Streamline: The lines that are tangential to the velocity vector field are called streamlines.
- Pathline: The pathline is the trajectory of a specific fluid particle over a time interval.
- Streakline: The streakline defines the instantaneous location of fluid particles that have passed by the same fixed spatial point at a previous time.

b) Shearthickening, shear thinning, Bingham plastic



c)



d) Advantage: Higher mixing is important for heat exchangers or mixing different fluids, increased detachment tolerance due to stronger momentum exchange in the boundary layer.
Disadvantage: Increased friction force 1 1