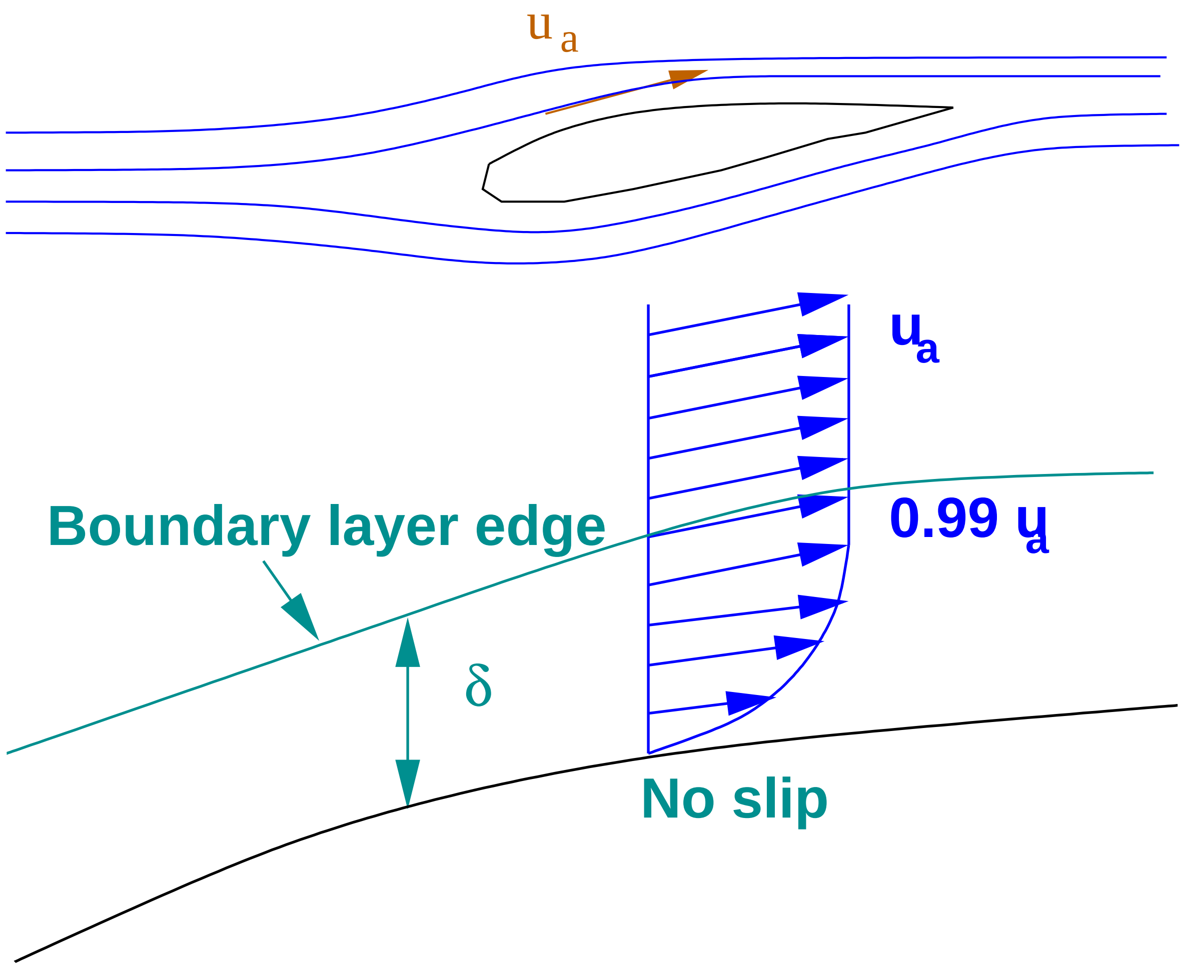
laminar boundary layers

Why Boundary layer theory?

- Only pressure distributions can be computed with the potential theory
  $\rightarrow$ Lift force
- The drag force cannot be described with the potentila theory. Viscous forces have to be considered

The pressure ditribution at slender bodies fits well with the theoretical distribution from the potential theory, if Re $\gg 1$. The influence of the viscous forces is limited to a thin layer at the wall $\rightarrow$ boundary layer

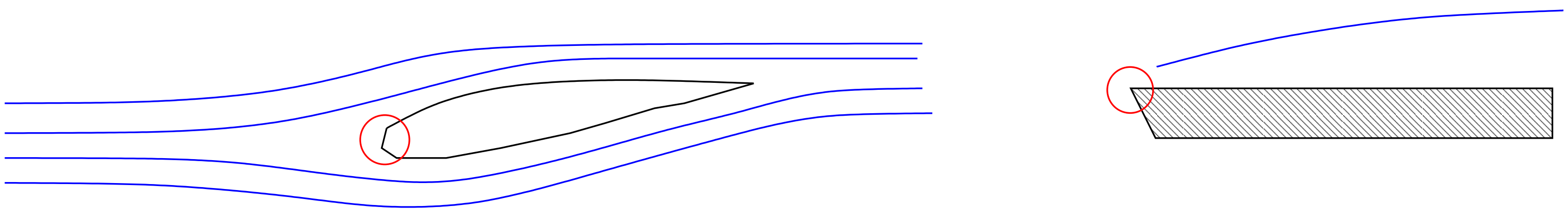
Beispiel



laminar boundary layers

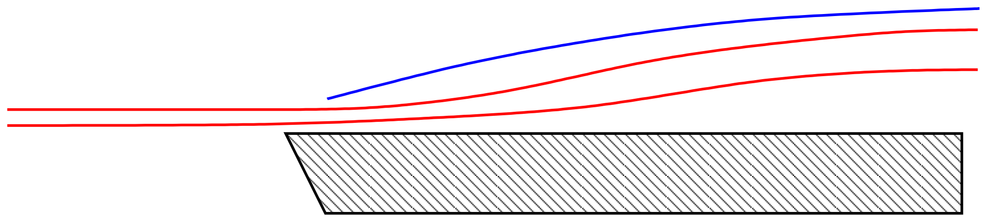
- The segmentation of the flow into two parts (the frictionless outer part + the viscous boundary layer) allows a complete description of the flow field.

- The boundary layer theory is <u>not</u> valid in the nose region!



laminar boundary layers

- Due to the deceleration in the boundary layer the streamlines are pushed away from the wall. The streamlines are no longer parallel to the wall.



- The line $\delta(x)$, describing the edge of the boundary layer (boundary layer thickness) is not a streamline. It denotes the line where the velocity reaches the value of the outer flow up to a certain amount. Usually 99 %

$$\frac{u(y)}{u_a} = 0.99 \qquad \text{arbitrary}$$

Boundary layer thickness

In the boundary layer:

$$\mathcal{O}(\text{Inertia}) \approx \mathcal{O}(\text{viscous forces})$$

$$\rho u \frac{\partial u}{\partial x} \approx \eta \frac{\partial^2 u}{\partial y^2}$$

dimensionless values $\rightarrow \mathcal{O}(1)$

$$\bar{u} = \frac{u}{u_\infty} \qquad \bar{x} = \frac{x}{L} \qquad \bar{y} = \frac{y}{\delta}$$

$$\rightarrow \rho u_\infty \frac{u_\infty}{L} \approx \eta \frac{u_\infty}{\delta^2}$$

$$\rightarrow \delta \sim \frac{L}{\sqrt{\frac{u_\infty \rho L}{\eta}}} = \frac{L}{\sqrt{\text{Re}_L}} \qquad \boxed{\delta \sim \sqrt{L}}$$

Boundary layer thickness

Introduction of dimensionless variables in the Navier-Stokes equations for 2-d steady, incompressibles flows. Dimension varaiables are $\mathcal{O}(1)$.

Neglect all terms with the factor $\dfrac{1}{\underline{\text{Re}}}$ or smaller.

$\rightarrow$ Boundary layer equations are valid for Re $\gg 1$ without strong curvatures.

continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

x-mom: $$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\underbrace{\frac{\partial p}{\partial x}}_{= \frac{dp}{dx}} + \eta \frac{\partial^2 u}{\partial y^2}$$

y-mom: $$\frac{\partial p}{\partial y} = 0$$

y-mom.: The pressure is constant normal to the main stream direction. It is impressed from the frictionless outer flow.

Boundary layer edge for a flat plate ($y = \delta$)

$$u = U \rightarrow \frac{\partial u}{\partial y} = 0 \rightarrow \frac{\partial^2 u}{\partial y^2} = 0$$

frictionless outer flow field

x-mom:

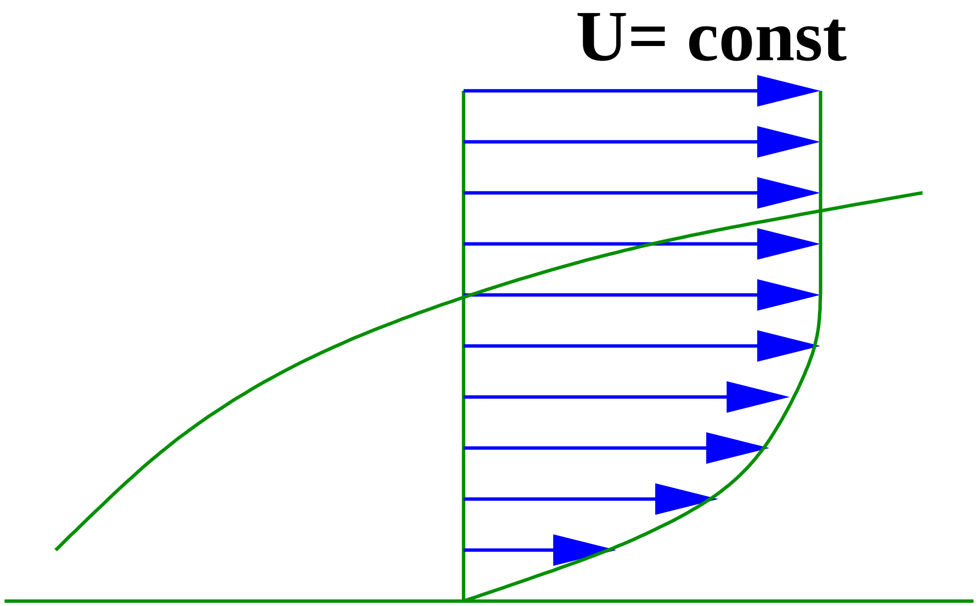
$$\boxed{\rho U \frac{\partial U}{\partial x} = -\frac{\partial p}{\partial x}}$$

Euler equation for $y = \delta$

3 cases depending on the pressure gradient

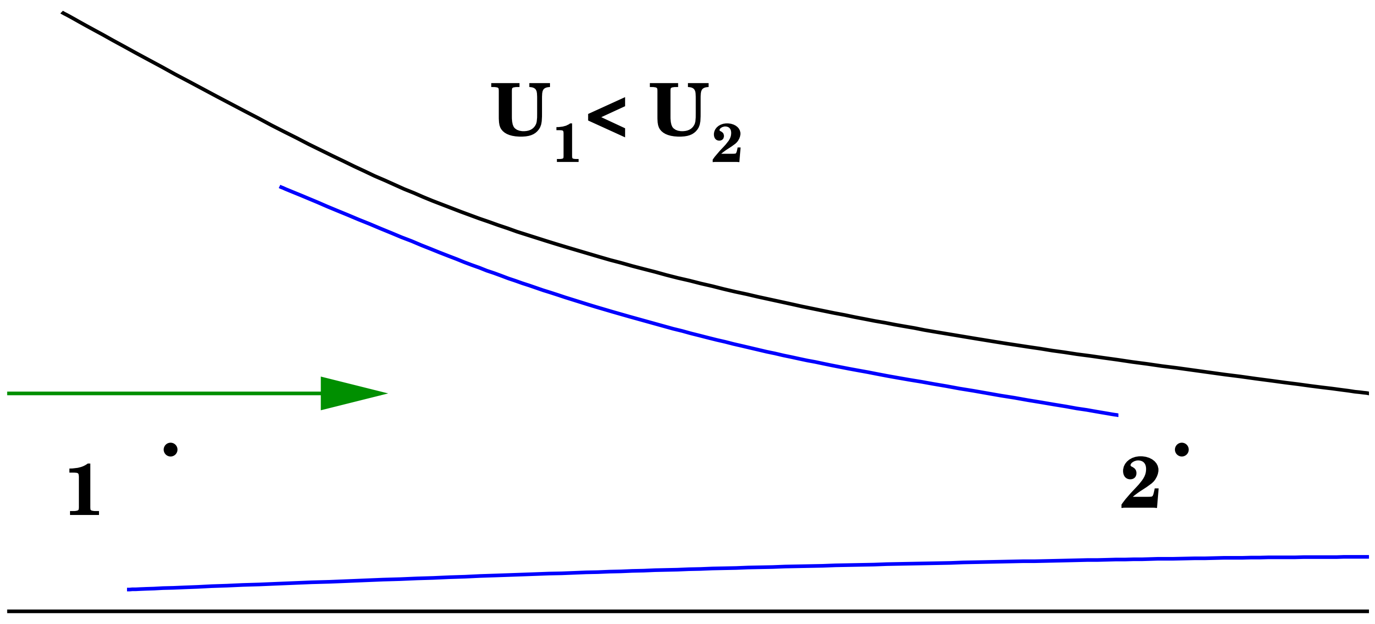
1 $\frac{\partial p}{\partial x} = 0 \rightarrow \frac{\partial U}{\partial x} = 0 \rightarrow U = \text{const}$

flat plate, boundary layer (Blasius)



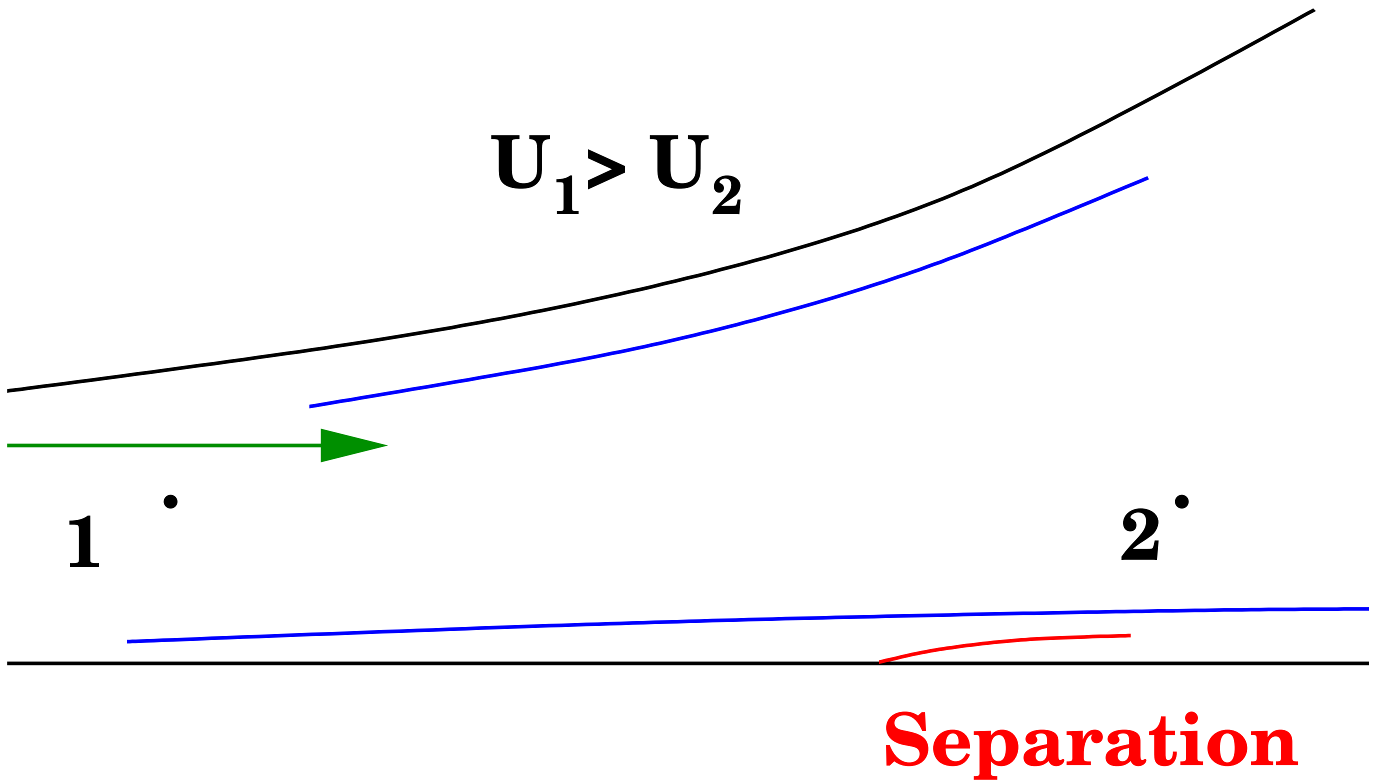
2 $\frac{\partial p}{\partial x} < 0 \rightarrow \frac{\partial U}{\partial x} > 0 \triangleq$ accelerated flow

convergent channel (nozzle)



3 $\frac{\partial p}{\partial x} > 0 \rightarrow \frac{\partial U}{\partial x} < 0 \triangleq$ decelerated flow

divergent channel (diffusor)



At the wall ($y = 0$)

No-slip condition: $u = v = 0$

$$\rightarrow \quad \text{x-mom.:} \quad \frac{\partial p}{\partial x} = \eta \frac{\partial^2 u}{\partial y^2}|_{y=0} \qquad \tau_w = \eta \frac{\partial u}{\partial y}$$

$$\rightarrow \quad \boxed{\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y}|_{y=0} =}$$

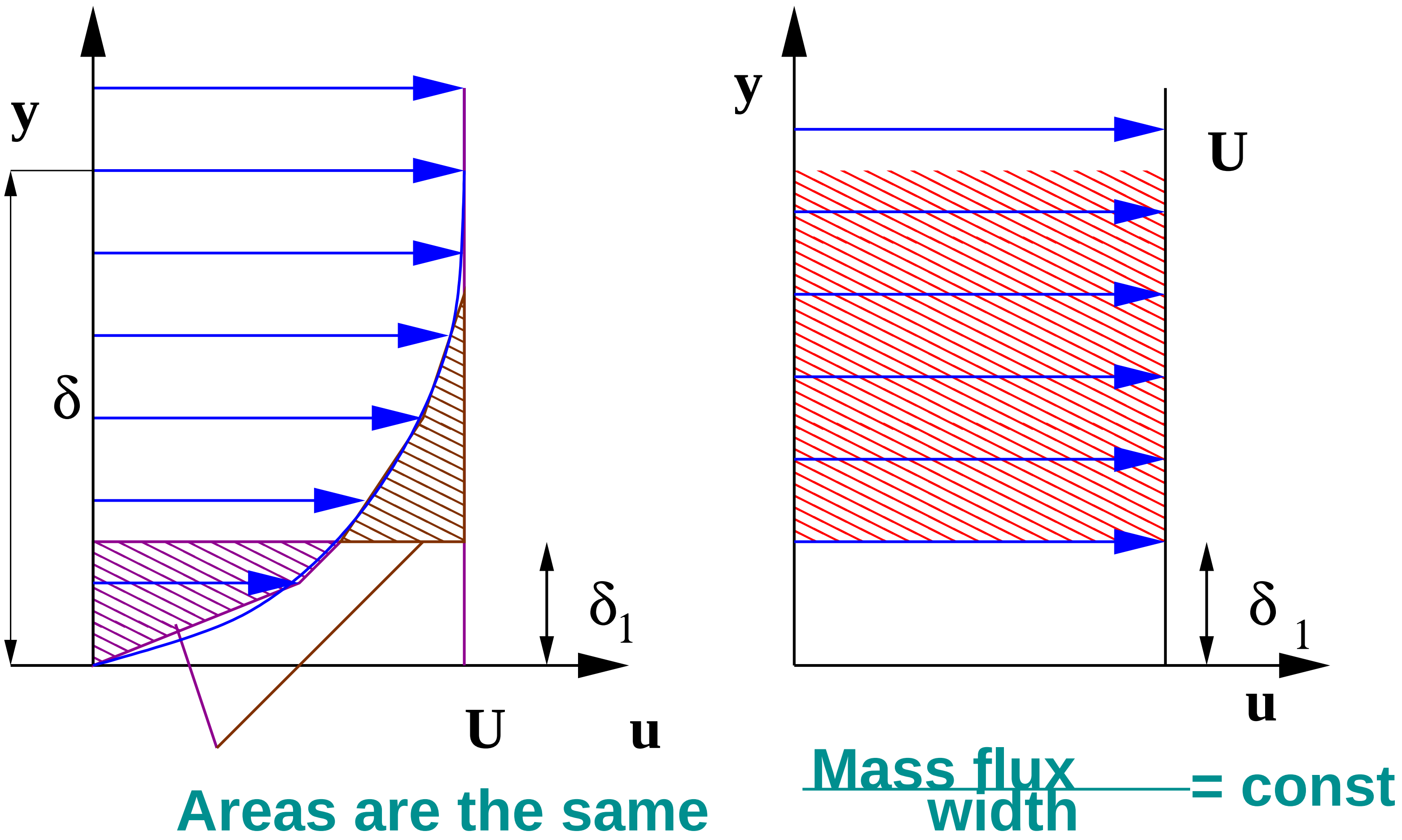
flat plate: $\frac{\partial p}{\partial x} = 0$ (no pressure gradient, $U =$ const.)

$$\frac{\partial^2 u}{\partial y^2}|_{y=0} = 0 \qquad (\frac{\partial u}{\partial y}|_{y=0} = const)$$

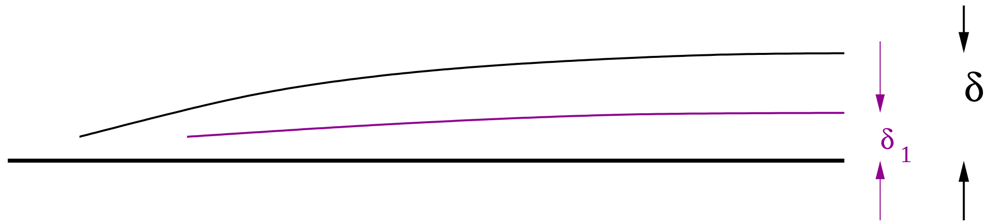
$\rightarrow$ no curvature of the velocity profile at the wall

Displacement-/Momentum thickness

δ_1 (displacemant thickness): characteristic measure for the displacement of an undisturbed streamline



Displacement-/Momentum thickness



δ_1 from $\dot{m}$ = const.

$$U(\delta - \delta_1) = \int_0^\delta u\, dy \rightarrow \int_0^\delta U - u\, dy = U\delta_1$$

$$\rightarrow \qquad \boxed{\delta_1 = \int_0^\delta (1 - \frac{u(y)}{U})\, dy}$$

in dimensionless form $\frac{\delta_1}{\delta} = \int_0^1 (1 - \frac{u(y)}{U})\, d\frac{y}{\delta}$ $\qquad (\delta,\ \delta_1 \neq f(y))$

Displacement-/Momentum thickness

Due to friction some momentum losses occur.

From the momentum balance

$$\delta_2 = \int_0^\infty \frac{u}{U}(1 - \frac{u}{U})\ dy$$

$$\frac{\delta_2}{\delta} = \int_0^1 \frac{u}{U}(1 - \frac{u}{U})\ d\frac{y}{\delta}$$

To compute the drag, these two measures +
the von Kàrmàn integral equation are used.

Displacement-/Momentum thickness

Integration of the x-momentum equation

$$\frac{d}{dx}(U^2\delta_2) + \delta_1 U \frac{dU}{dx} = \frac{\tau_w}{\rho}$$

or $$\frac{d\delta_2}{dx} + \frac{1}{U}\frac{dU}{dx}(2\delta_2 + \delta_1) = \frac{\tau_w}{\rho U^2}$$

- Assume a function (polynomial, . . .) for the velocity

- Use the boundary conditions to compute the coefficients

- Compute δ_1 and δ_2
- Use the von Kàrmàn integral equation to compute $\tau_w(x)$ or $\delta(x)$.

Ansatz: polynom for the velocity profile

$$\frac{u(x,y)}{U(x)} = \sum_{i=0}^{n} a_i \left(\frac{y}{\delta}\right)^i = f(x, \frac{y}{\delta})$$

selfsimilar profile $a_i(x), \delta(x)$

boundary conditions

1. no-slip condition (Stokes) for $\frac{y}{\delta} = 0 \rightarrow u = v = 0 \qquad (u_B = u_w)$

2. boundary layer edge $\frac{y}{\delta} = 1 \rightarrow u = U$

3. from x-momentum

$$\eta \frac{\partial^2 u}{\partial y^2}|_{y=0} = \frac{\partial p}{\partial x} \quad (= 0 \text{ for a flat plate})$$

$\frac{\partial p}{\partial x}$ from Euler equation (Bernoulli)

Only: if the degree of the polynomial is > 2, other boundary conditions are necessary from the continuity at the boundary layer edge

4. $\frac{y}{\delta} > 1 \rightarrow \frac{\partial u}{\partial y} = 0$
   continous transition from the boundary layer to the outer flow

5. $\frac{y}{\delta} = 1 \rightarrow \frac{\partial^2 u}{\partial y^2} = 0$
   frictionless flow

Example 15.8

In the stagnation point of a flat plate that is flown against normally to the outer flow $u_a(x)$ is accelerated in such a way that a constant boundary layer thickness δ_0 is generated. The velocity profile is assumed to be linear as a first approximation.

$$\frac{u(x,y)}{u_a(x)} = a_0 + a_1 \frac{y}{\delta_0}$$

Determine:

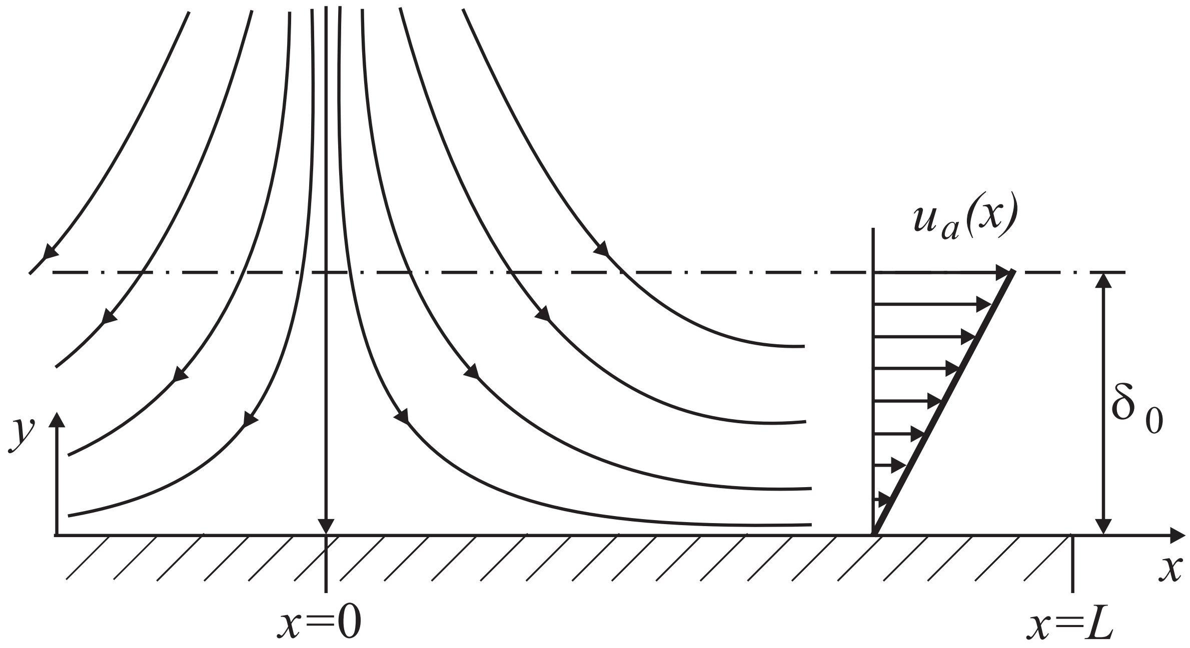
a) the constants $a_0, \ a_1$
b) the distribution of the outer velocity $u_a(x)$ using the von Kármán integral equation.
c) the tangential force that is applied between $x = 0$ and $x = L$ on the plate with the width B.

Given: $\delta_0, \ L, \ \eta, \ \rho, \ B$

Example 15.8

von Kármán integral equation

$$\frac{d\delta_2}{dx} + \frac{1}{u_a}\frac{du_a}{dx}(2\delta_2 + \delta_1) = \frac{\tau_w}{\rho u_a^2}$$



Example 15.8

Remark: $u_a = U_a = U = u_e = \ldots$ different notations

1) $a_0,\ a_1 = ?$

no-slip condition $u(y=0) = 0 \rightarrow a_0 = 0$

at the boundary layer edge: $\frac{u}{U}|_{\frac{y}{\delta_0}=1} = 1 \rightarrow a_1 = 1$

$$\rightarrow \frac{u(x,y)}{U(x)} = \frac{y}{\delta_0}$$

Example 15.8

2) displacement thickness: $$\frac{\delta_1}{\delta_0} = \int_0^1 (1 - \frac{u(y)}{U})\, d\frac{y}{\delta}$$

$$= \left[\frac{y}{\delta_0} - \frac{1}{2}\left(\frac{y}{\delta_0}\right)^2\right]_0^1 = \frac{1}{2} \quad \rightarrow \quad \delta_1 = \frac{1}{2}\delta_0$$

momentum thickness: $$\frac{\delta_2}{\delta_0} = \int_0^1 \frac{u}{U}(1 - \frac{u}{U})\, d\frac{y}{\delta}$$

$$= \frac{1}{2}\left(\frac{y}{\delta_0}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta_0}\right)^3\Bigg|_0^1 = \frac{1}{6} \rightarrow \delta_2 = \frac{1}{6}\delta_0$$

$$\rightarrow \frac{\partial \delta_2}{\partial \delta_0} = \frac{1}{6} = \text{const} \rightarrow \frac{\partial \delta_2}{\partial x} = 0$$

Example 15.8

Wall shear stress: $\tau_w = \eta \left.\dfrac{\partial u}{\partial y}\right|_{y/\delta=0}$

$$\rightarrow \tau_w = \eta \frac{U}{\delta_0} \frac{\partial(\frac{u}{U})}{\partial(\frac{y}{\delta_0})} = \eta \frac{U}{\delta_0}$$

Example 15.8

von Kàrmàn integral equation

$$\frac{dU}{dx} = U \, \frac{1}{\rho U^2} \, \eta \, \frac{U}{\delta_0^2} \left(\frac{1}{2\frac{1}{6} + \frac{1}{2}} \right) = \frac{\eta}{\rho \delta_0^2} \frac{6}{5}$$

$$U(x) = \frac{\eta}{\rho \delta_0^2} \frac{6}{5} x$$ (usually $\delta = \delta(x)$)

3) $F = \int\limits_0^L \tau(x) \, B \, dx = \frac{6}{5} \frac{\eta}{\rho \delta_0^2} \frac{\eta}{\delta_0} B \int\limits_0^L x \, dx$

$$\boxed{F = \frac{3}{5} \frac{\eta^2}{\rho} \frac{B L^2}{\delta_0^3}}$$

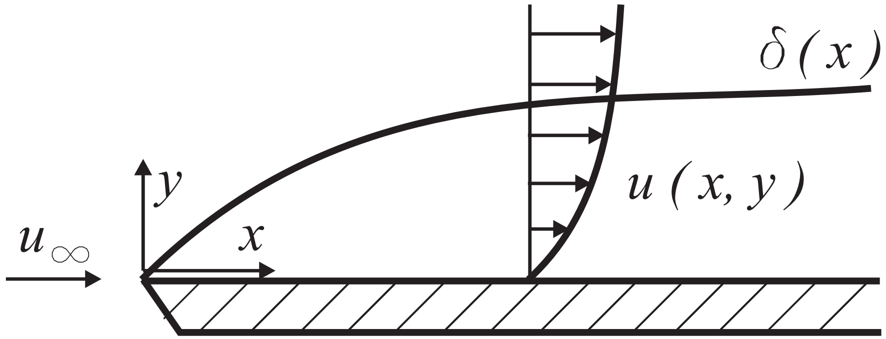
From exercise 15.2 the drag of a flate plate, wetted on both sides, of length x and width B can be determined with

$$W = \int_0^x \tau_w(x) B dx = \rho \int_0^{\delta(x)} u(u_\infty - u) B dy.$$

Using this equation and with the approximation of the velocity profile

$$\frac{u(x,y)}{u_\infty} = a_0 + a_1 \frac{y}{\delta} + a_2 \left(\frac{y}{\delta}\right)^2$$

the boundary layer thickness $\delta(x)$ is to be determined and compared with the Blasius solution $\delta(x) = 5.2\sqrt{\frac{\nu x}{u_\infty}}$.



Given: $\nu,\ u_\infty$

Determination of the coefficients $a_0,\ a_1,\ a_2$ by using the boundary conditions:

B.C.1: no slip $u(x, y = 0) = 0$

B.C.2: boundary layer edge $u(x, y = \delta) = u_\infty$

R.B 3: at the wall (from x-momentum) $\eta \dfrac{\partial^2 u(x, y = 0)}{\partial y^2} = \dfrac{\partial p}{\partial x} = 0$

If additional boundary conditions are necessary, a steady transition at $\frac{y}{\delta} = 1$ can be assumed, i.e.

$$\left.\frac{\partial^n u}{\partial y^n}\right|_{y=\delta} = 0 \text{ with } n \geq 1$$

$\Longrightarrow$ from B.C.1 follows $a_0 = 0$

from B.C.2 follows $a_1 + a_2 = 1$

from B.C.3 follows

$$\frac{\partial^2 u}{\partial y^2} = u_\infty \frac{1}{\delta^2} \frac{\partial^2 (u/u_\infty)}{\partial (y/\delta)^2} = 2u_\infty \frac{1}{\delta^2} a_2 = 0$$

$$\Longrightarrow a_2 = 0, \quad a_1 = 1$$

$$\Longrightarrow \frac{u(x,y)}{u_\infty} = \frac{y}{\delta} \qquad \text{linear distribution.}$$

Permutation of the order of boundary conditions, e,g,:

B.C.1 $u(x, y = 0) = 0$

B.C.2 $u(x, y = \delta) = u_\infty$

B.C.3 $\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0$

$\Longrightarrow a_0 = 0; \quad a_1 = 2; \quad a_2 = -1$ parabolic distribution

Since the approximation of the velocity profiles is not an exact solution of the boundary layer equations, the boundary conditions are not satisfiable in all cases. Hence, it has to be paid attention, that the physical boundary conditionsa re satisfied first.

Computation of the boundary layer thickness $\delta(x)$:

$$\tau_w = \eta \left.\frac{\partial u}{\partial y}\right|_{y=0} = \eta \frac{u_\infty}{\delta} \left.\frac{\partial u/u_\infty}{\partial y/\delta}\right|_{y=0} = \eta \frac{u_\infty}{\delta}$$

$$u(u_\infty - u) = u_\infty \frac{y}{\delta}\left(u_\infty - u_\infty \frac{y}{\delta}\right) = u_\infty^2 \left(\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)$$

with the equation

$$B\int_0^x \tau_w(x)dx = B\rho\int_0^{\delta(x)} u(u_\infty - u)dy$$

$$\Longrightarrow \int_0^x \eta\frac{u_\infty}{\delta(x)}dx = \rho u_\infty^2 \int_0^1 \delta\left(\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right) d\left(\frac{y}{\delta}\right)$$

$$\eta u_\infty \int_0^x \frac{dx}{\delta(x)} = \rho u_\infty^2 \delta(x)\left(\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right)\Bigg|_0^1$$

$$= \rho u_\infty^2 \delta(x)\frac{1}{6}$$

Differentiating gives

$$\frac{1}{\delta(x)} = \frac{1}{6}\frac{\rho}{\eta}u_\infty\frac{d\delta(x)}{dx}$$

$$\Longrightarrow dx = \frac{1}{6}\frac{\rho}{\eta}u_\infty\delta(x)d\delta(x)$$

Integration:

$$x = \frac{1}{12}\frac{\rho}{\eta}u_\infty\delta^2(x)$$

$$\Longrightarrow \delta(x) = \sqrt{\frac{12\nu x}{u_\infty}}$$

$$\Longrightarrow \delta(x) = \sqrt{12}\sqrt{\frac{\nu x}{u_\infty}} \approx 3,5\sqrt{\frac{\nu x}{u_\infty}}$$

Compare with Blasius solution: $\delta(x) = 5.2\sqrt{\frac{\nu x}{u_\infty}}$

The velocity profile of a laminar incompressible boundar layer with constant viscosity η can be described with a polynomial:

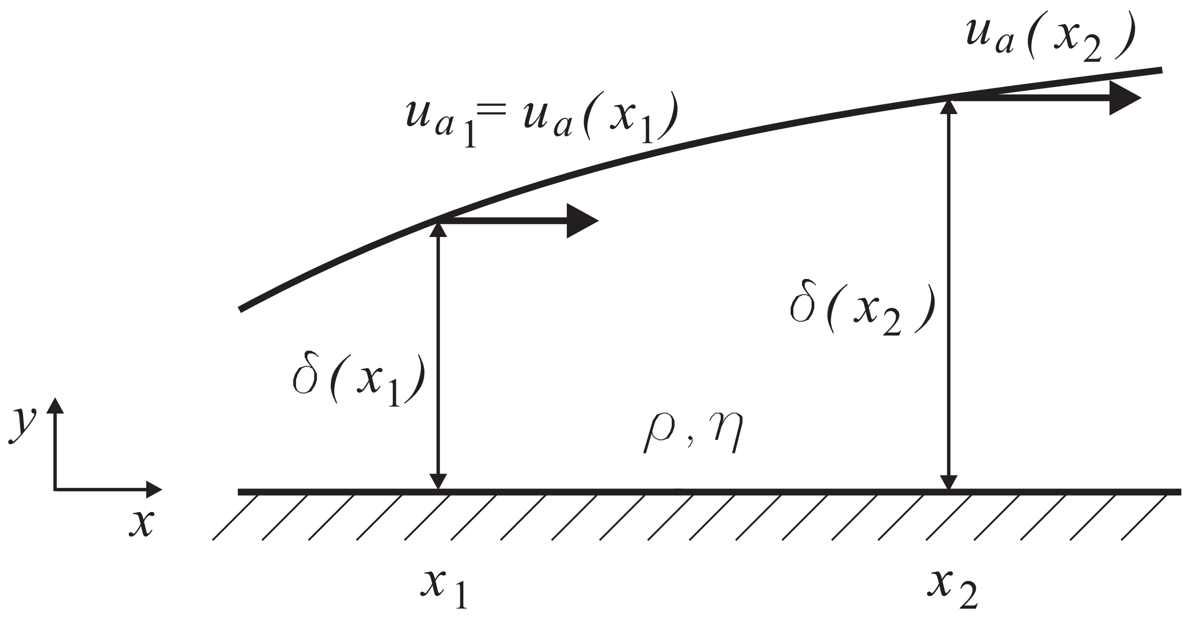
$$\frac{u(x,y)}{u_a(x)} = a_0 + a_1(x)\left(\frac{y}{\delta}\right) + a_2(x)\left(\frac{y}{\delta}\right)^2 + a_3(x)\left(\frac{y}{\delta}\right)^3$$

The outer velocity $u_a(x)$ is given by the following approach:

$$u_a(x) = u_{a1} - C \cdot (x - x_1)^2.$$

u_{a1} is the outer velocity at x_1 and C is a positive constant. The boundary layer thickness at x_2 is $\delta(x_2)$.

15.4



Given: $\rho,\ \eta,\ x_1,\ u_{a1},\ \delta(x_2),\ C,$ mit: $C > 0$

Determine:

a) the pressure gradient $\partial p/\partial x$ in the flow as a function of x.
b) the coefficient a_0 and the coefficients $a_1(x),\ a_2(x),\ a_3(x)$.

15.4

a) $\dfrac{\partial p}{\partial x}$ in the flow:

frictionless outer flow: $\rho u_a \dfrac{\partial u_a}{\partial x} = -\dfrac{\partial p}{\partial x}$

$$\Longrightarrow \frac{\partial p}{\partial x} = -\rho u_a \frac{\partial u_a}{\partial x} = -\rho(u_{a_1} - C(x - x_1)^2) \cdot (-2C(x - x_1))$$

$$\frac{\partial p}{\partial x} = +2\rho C(x - x_1)(u_{a_1} - C(x - x_1)^2)$$

$$\left.\frac{\partial p}{\partial x}\right|_{x_1} = 0$$

$$\left.\frac{\partial p}{\partial x}\right|_{x_2} = 2\rho C(x_2 - x_1)(u_{a_1} - C(x_2 - x_1)^2)$$

15.4

b) 4 B.C:

I $\frac{y}{\delta} = 0 \quad u = 0 \quad H.B. \Longrightarrow a_0 = 0$

II $\frac{y}{\delta} = 1 \quad u = u_a$

III $\frac{y}{\delta} = 0 \quad \frac{\partial p}{\partial x} = \eta \frac{\partial^2 u}{\partial y^2}$ at the wall

IV $\frac{y}{\delta} = 1 \quad \frac{\partial u}{\partial y} = 0$

$$u = \left(a_1 \left(\frac{y}{\delta}\right) + a_2 \left(\frac{y}{\delta}\right)^2 + a_3 \left(\frac{y}{\delta}\right)^3 \right) u_a$$

$$\frac{\partial u}{\partial y} = \left(a_1 \frac{1}{\delta} + 2a_2 \frac{y}{\delta^2} + 3a_3 \frac{y^2}{\delta^3} \right) u_a$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\qquad + 2a_2 \frac{1}{\delta^2} + 6a_3 \frac{y}{\delta^3} \right) u_a$$

II: $$1 = a_1 + a_2 + a_3$$

IV: $$0 = a_1 + 2a_2 + 3a_3$$

III: $$\frac{\partial p}{\partial x} = \eta \, u_a \, \frac{2a_2}{\delta^2} \quad \Longrightarrow \quad a_2 = \frac{1}{2} \frac{\delta^2}{\eta \, u_a} \frac{\partial p}{\partial x}$$

$$\Longrightarrow \quad a_2(x) = \frac{\delta^2}{2\eta} \frac{2 \, \rho \, C \, (x - x_1)(u_{a_1} - C \, (x - x_1)^2)}{(u_{a_1} - C \cdot (x - x_1)^2)}$$

$$a_2(x) = \frac{\delta^2 \, \rho \, C}{\eta} \, (x - x_1)$$

II / IV:

$$a_3 = -\frac{1}{2} - \frac{1}{2}\,a_2$$

$$a_1 = \frac{3}{2} - \frac{1}{2}\,a_2$$

therefore:

$$a_1(x) = \frac{3}{2} - \frac{1}{2}\,\frac{\delta^2\,\rho\,C}{\eta}\,(x - x_1)$$

$$a_3(x) = -\frac{1}{2} - \frac{1}{2}\,\frac{\delta^2\,\rho\,C}{\eta}\,(x - x_1)$$

In the boundary layer of a flat plate the velocity profiles and the pressure distribution are measured. Thre pressure distribution on the surface is described with $\frac{p(x)}{p_0} = 1 - k\left(\frac{x}{l}\right)^2$, with $k = const < 1$ and the velocity profiles are presented with

$$\frac{u(x,y)}{u_a(x)} = \left(\frac{y}{\delta_0}\right)^{\frac{1}{2}}$$

with a constant boundary layer thickness δ_0.

Determine the wall shear stress τ_w using the Kármán integral equation

$$\frac{d\delta_2}{dx} + \frac{1}{u_a}\frac{du_a}{dx}(2\delta_2 + \delta_1) + \frac{\tau(y=0)}{\rho u_a^2} = 0$$

Given: $p_0,\ k,\ \delta_0,\ l$

Boundary layer equation (x-momentum):

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\eta}{\rho}\frac{\partial^2 u}{\partial y^2}$$

a)

$$\frac{d\delta_2}{dx} + \frac{1}{u_a}\frac{du_a}{dx}(2\delta_2 + \delta_1) + \frac{\tau(y=0)}{\rho u_a^2} = 0$$

- $\frac{\delta_1}{\delta_0} = \int_0^1 \left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \left(1 - \left(\frac{y}{\delta_0}\right)^{\frac{1}{2}}\right) d\left(\frac{y}{\delta}\right) = \left(\frac{y}{\delta} - \frac{2}{3}\left(\frac{y}{\delta}\right)^{\frac{3}{2}}\right)_0^1$

$$\Longrightarrow \delta_1 = \frac{1}{3}\,\delta_0$$

- $\frac{\delta_2}{\delta_0} = \int_0^1 \frac{u}{u_a}\left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \left(\left(\frac{y}{\delta}\right)^{1/2} - \left(\frac{y}{\delta}\right)\right) d\left(\frac{y}{\delta}\right) = \left(\frac{2}{3}\left(\frac{y}{\delta}\right)^{3/2} - \frac{1}{2}\left(\frac{y}{\delta}\right)^2\right)$

$$\Longrightarrow \delta_2 = \frac{1}{6}\,\delta_0$$

- $\frac{d\delta_2}{dx} = 0$, since $\delta_0 = const.$

from the x-momentum equation for $y = \delta_0$:

$$\rho u_a \frac{du_a}{dx} = -\frac{dp}{dx}$$

$$\Longrightarrow \tau_w = -\frac{dp}{dx}(2\delta_2 + \delta_1)$$

with
$$\frac{dp}{dx} = p_0 \frac{d}{dx}\left(1 - k\left(\frac{x}{l}\right)^2\right) = -p_0\, k\, \frac{2x}{l^2}$$

$$\Longrightarrow \tau_w = p_0 k \frac{2x}{l^2}\frac{2}{3}\delta_0 = \frac{4}{3} p_0 k \frac{\delta_0 x}{l^2}$$

15.7

The velocity profile in the laminar boundary layer of a flat plate (length L) is described by a polynomial of fourth order

$$\frac{u}{u_a} = a_0 + a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3 + a_4\left(\frac{y}{\delta}\right)^4 .$$

a) Determine the coefficient of the polynomial!
b) Proof the following relationships:

$$\frac{\delta_1}{\delta} = 3/10$$

$$\frac{\delta_2}{\delta} = 37/315$$

$$\frac{\delta}{x} = 5.84/\sqrt{Re_x}$$

$$c_w = 1.371/\sqrt{Re_L}$$

a) Boundary conditions:

$$\frac{y}{\delta} = 0: \qquad \frac{u}{u_a} = 0, \qquad \frac{v}{u_a} = 0$$

$$\frac{y}{\delta} = 1: \qquad \frac{u}{u_a} = 1$$

from boundary layer equation

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \eta \frac{\partial^2 u}{\partial y^2}:$$

$$\frac{y}{\delta} = 0: \qquad u = v = 0: \qquad \frac{\partial^2 (u/u_a)}{\partial (y/\delta)^2} = 0$$

$$\frac{y}{\delta} = 1: \qquad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0: \qquad \frac{\partial^2 (u/u_a)}{\partial (y/\delta)^2} = 0$$

frictionless outer flow

$$\frac{y}{\delta} = 1 : \qquad \tau \sim \frac{\partial(u/u_a)}{\partial(y/\delta)} = 0$$

$$\frac{u}{u_a} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$$

b)

$$\frac{\delta_1}{\delta} = \int_0^1 \left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \frac{3}{10}$$

$$\frac{\delta_2}{\delta} = \int_0^1 \frac{u}{u_a}\left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \frac{37}{315}$$

15.7

von Kármánsche integral equation:

$$\frac{d\delta_2}{dx} + \frac{\tau(y=0)}{\rho u_a^2} = 0$$

$$\tau(y=0) = -\frac{\eta u_a}{\delta} \left.\frac{d(u/u_a)}{d(y/\delta)}\right|_{y/\delta=0} = -2\frac{\eta u_a}{\delta}$$

Integration: $\dfrac{\delta}{x} = \dfrac{5.84}{\sqrt{Re_x}}$

$$c_w = \frac{2}{L}\int_0^L \frac{\tau_w}{\rho u_a^2} dx = -\frac{2}{L}\int_0^L \frac{\tau(y=0)}{\rho u_a^2} dx = \frac{1.371}{\sqrt{Re_L}}$$