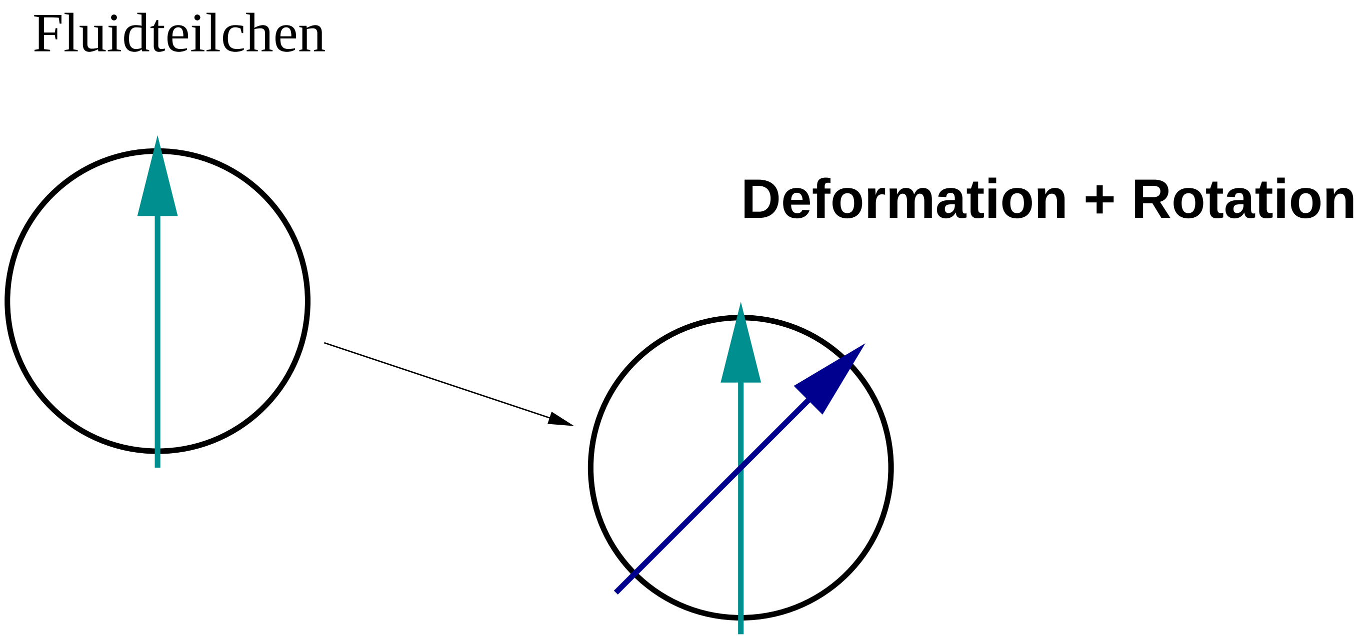
Vortical Flows

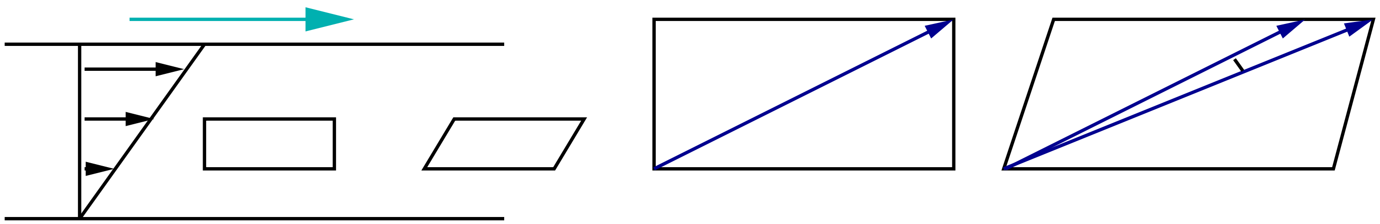
- Usually flowfields are with rotation
- Rotation free flows → Simplification of the Navier-Stokes equati-ons → analytical solution is an approximation of real flow



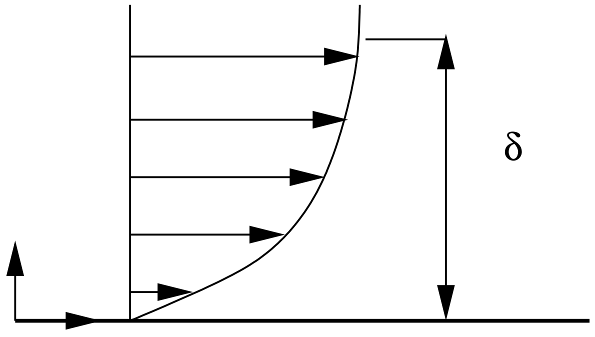
Rotation: Particles rotate around their <u>own</u> axis

Vortical flows

Example: Couette flow



- Rotation happens in frictional flows
- films, pipes, boundary layers.



Increase of the velocity

$$u(x, y = 0) = 0$$
$$u(x, y \neq 0) \neq 0$$

$$\frac{\partial u}{\partial y} \neq 0$$

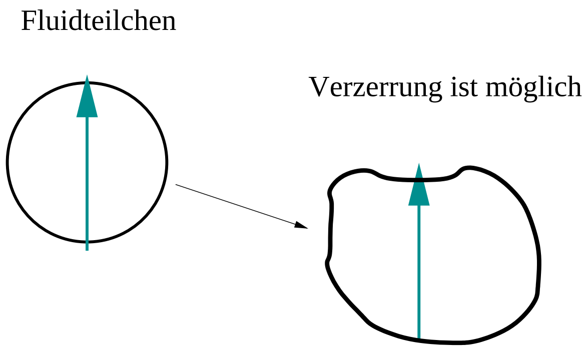
Vortical flows

at small distances to the wall the flow is $\approx$ parallel to the wall $\rightarrow v = 0$

$$\rightarrow \omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \neq 0$$

in frictional flows usually $\vec{\omega} \neq 0$

- Rotation free (Potential flow)



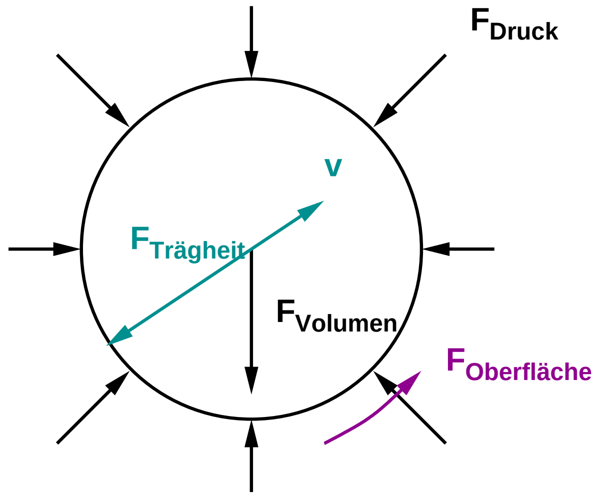
Rotation is usually only the case in frictionless flow fields

Vortical flows

Conservation laws for frictionless flows

$$\frac{d\vec{I}}{dt} = \sum \vec{F}_a = \text{pressure and volume forces}$$

Fluid element is a sphere



- Friction forces lead to rotation
- The direction of forces in a frictionless flow is through the center

Vortical flows

- in a frictionless flow (without discontinuities) no rotattion can be produced
- if a flow field is rotation free at the beginning it remains rotation free

<u>vortex vektor</u>

$$\vec{\omega} = \frac{1}{2}\text{rot}\vec{v} = \frac{1}{2}(\nabla \times \vec{v}) \text{ with } \vec{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\vec{\omega} = \frac{1}{2}\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

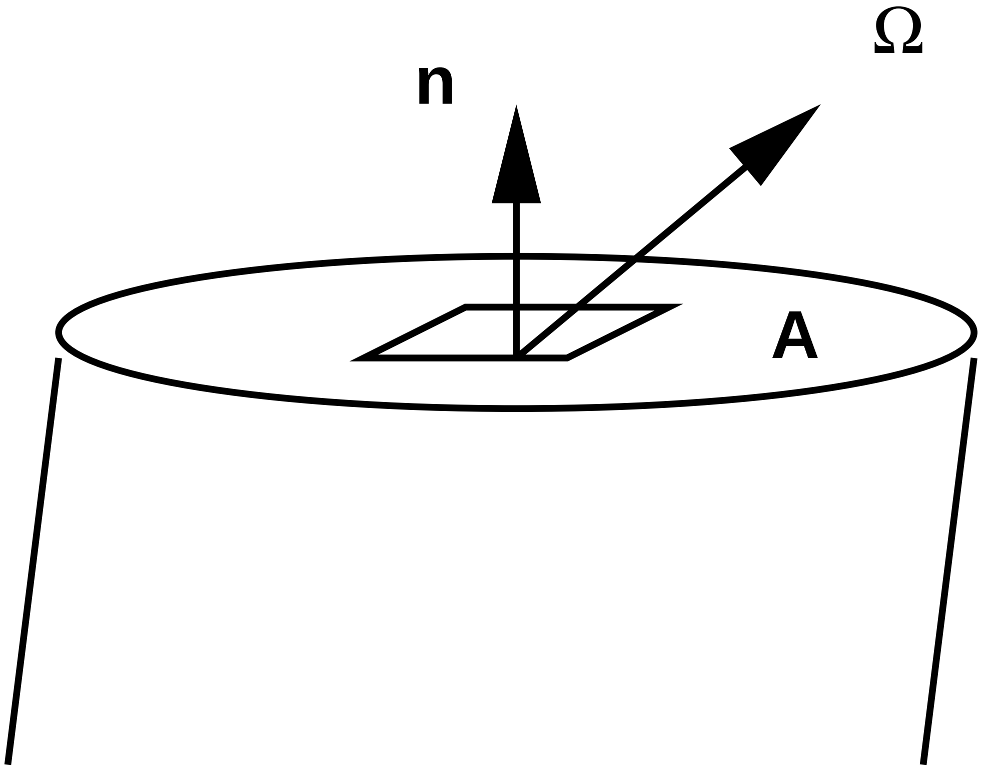
Vortical flows

plane 2d flow ($w = 0, \frac{\partial}{\partial z} = 0$)

$$\vec{\omega} = \frac{1}{2} \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}$$

vortical flux Ω

Integral of the vortex vector moving through a surface A tretenden Wirbelvektor (analogue to the integral of the velocity vector = volume flux)



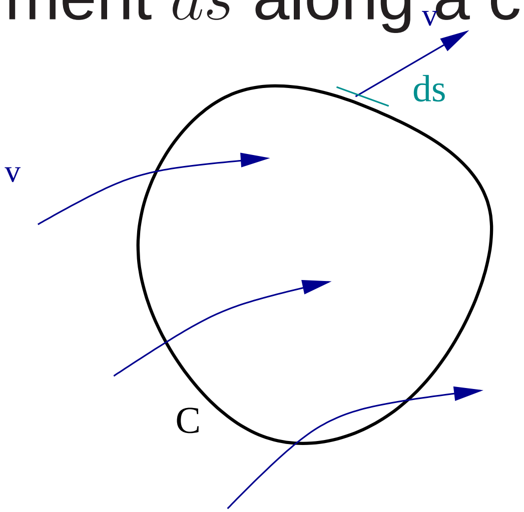
$$\Omega = \int_A \vec{\omega} \cdot \vec{n} \, dA$$

$$\dot{Q} = \int_A \vec{v} \cdot \vec{n} \, dA$$

Vortical flows

Zirkulation Γ

Line integral of the scalar product from velocity $\vec{v}$ and the line element $\vec{ds}$ along a closed curve C.



$$\Gamma = \oint_C \vec{v} \cdot d\vec{s}$$

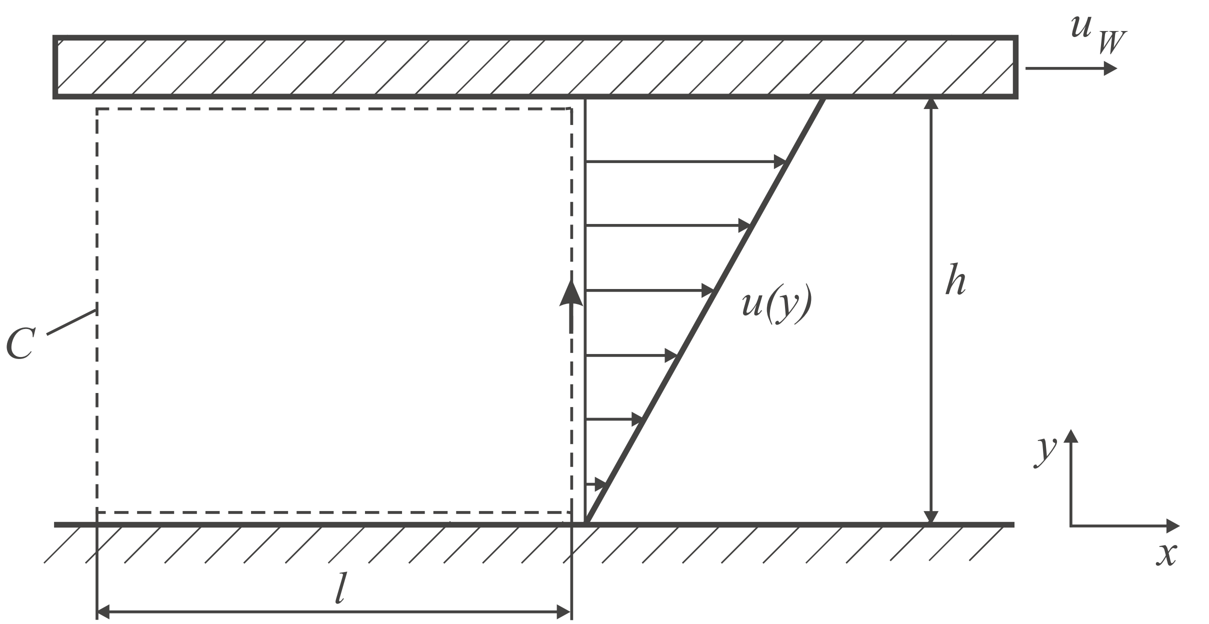
Propositiom of Stokes

$$\Gamma = \oint_C \vec{v} \cdot d\vec{s} = \int_A (\nabla \times \vec{v})\vec{n}\, dA = \int_A 2(\vec{\omega} \cdot \vec{n})\, dA = 2\Omega$$

13.3

The 2d Couette flow without pressure gradient is analysed. Compute $\vec{\omega},\ \Gamma_C,\ \Omega$.

Given: $u_W,\ h,\ l$



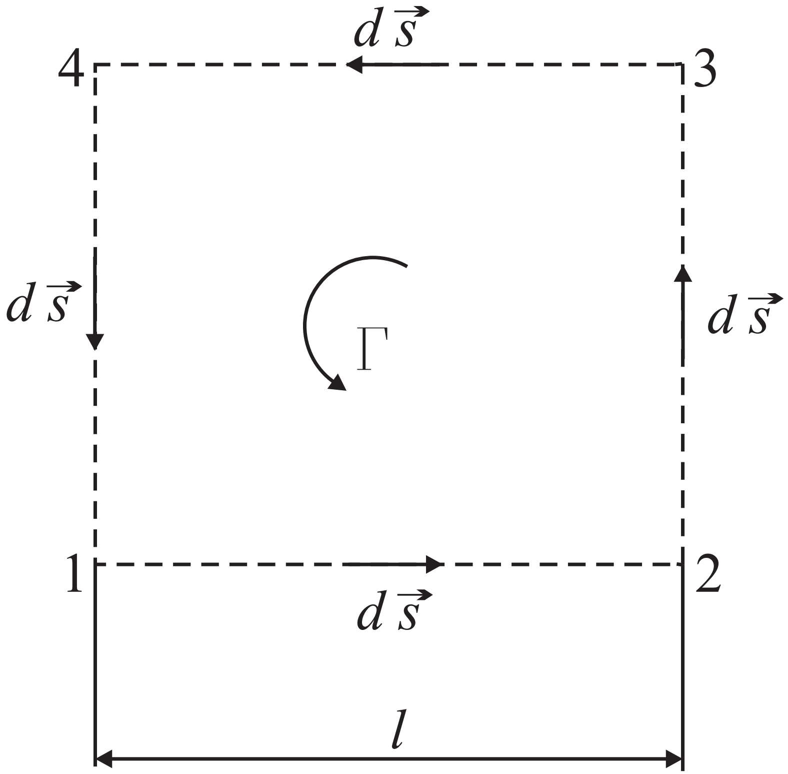
Couette flow $\Longrightarrow$ no pressure gradient

$$\Longrightarrow u(y) \;=\; u_W \,\frac{y}{h}$$

2d flow

$$\Longrightarrow \vec{\omega} \;=\; \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix} \qquad \omega_z \;=\; \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \;=\; -\,\frac{1}{2}\,\frac{u_W}{h}$$

$$\Longrightarrow \vec{\omega} \;=\; \begin{pmatrix} 0 \\ 0 \\ \dfrac{-u_W}{2h} \end{pmatrix}$$



$$\Gamma_C = \oint_C \vec{v}\, d\vec{s} = \int_1^2 \vec{v}\, d\vec{s} + \int_2^3 \vec{v}\, d\vec{s} + \int_3^4 \vec{v}\, d\vec{s} + \int_4^1 \vec{v}\, d\vec{s}$$

$$\int_1^2 \vec{v}\, d\vec{s} = 0, \qquad \text{since } u = v = 0$$

$$\int_2^3 \vec{v}\, d\vec{s} = 0, \qquad \text{since } \vec{v} \perp d\vec{s}$$

$$\int_3^4 \vec{v}\, d\vec{s} = \int_3^4 \begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} -dx \\ dy \end{pmatrix} = \int_3^4 -u_W dx = -u_W l$$

$$\int_4^1 \vec{v}\, d\vec{s} = 0, \qquad \text{since } \vec{v} \perp d\vec{s}$$

$$\Longrightarrow \Gamma = -u_W \cdot l$$

$$\Omega = \int_A \vec{\omega}\, \vec{n}\, dA = \omega_z \cdot A = \frac{-u_W}{2h} \cdot l \cdot h = -\frac{1}{2} u_W \cdot l = \frac{1}{2}\, \Gamma$$

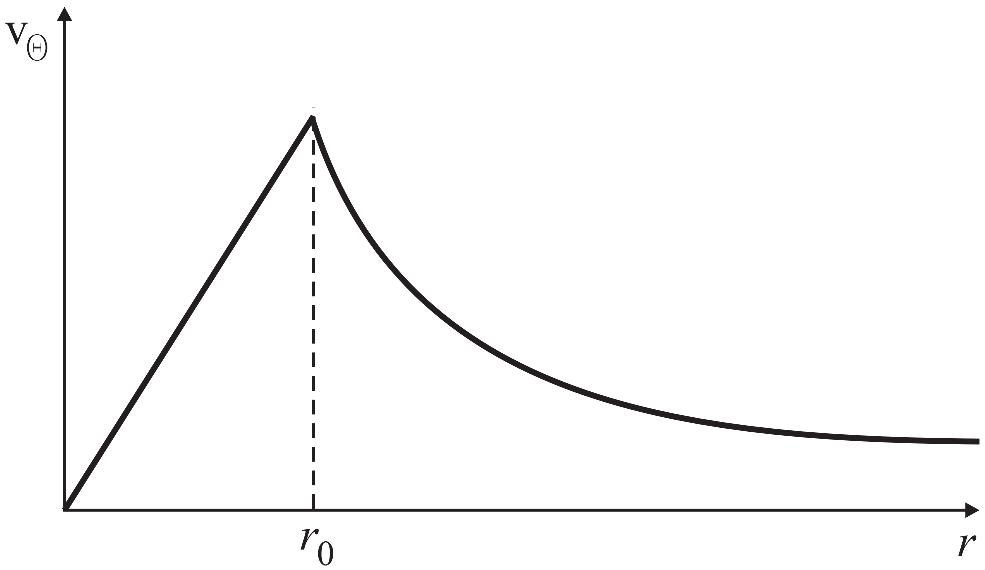
A tornado has the following velocity distribution:

$$v_\Theta(r) = \begin{cases} \omega\, r & r \leq r_0 \\ \dfrac{\omega\, r_0^2}{r} & r > r_0 \end{cases} \qquad v_r = 0$$

$$r_0 = 10\ m \qquad \omega = 10\ \frac{1}{s} \qquad H = 100\ m \qquad \rho = 1,25\ kg/m^3$$

a) Sketch $v_\Theta(r)$!
b) Determine the circulation on a circle around the center with Ursprung für $r < r_0, r = r_0$ and $r > r_0$!
c) Proof that the flow is rotation free for $r > r_0$!
d) What is the kinetic energy in a cylinder with the radius $R = 2\ r_0$ and height H?

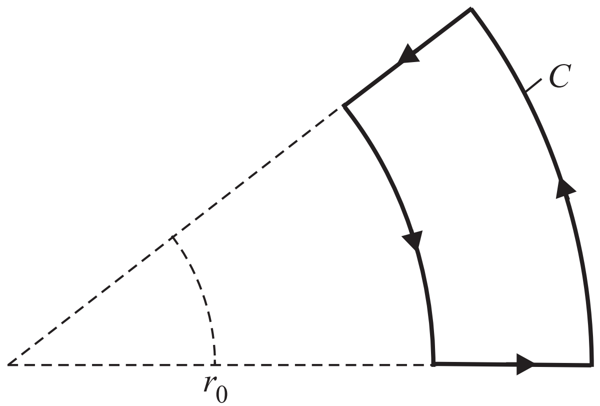
a)



b)

$$\Gamma = \oint_C \vec{v}\cdot d\vec{s} = \int_0^{2\pi} v_\Theta(r)\, r\, d\,\Theta = \begin{cases} 2\pi\,\omega\, r^2 & r \le r_0 \\ 2\pi\,\omega\, r_0^2 & r > r_0 \end{cases}$$

c)



$$\Gamma = \oint_C \vec{v} \cdot d\vec{s} = 0$$

$$\vec{\omega} = 0$$

d) $$E = \int_0^{2r_0} \frac{\rho}{2} v_\Theta^2 \, H \, 2\pi \, r \, d \, r =$$

$$\pi \, \rho \, H \, \omega^2 \, r_0^4 \, (0,25 + \ln 2) = 3,7 \cdot 10^8 \, Nm$$