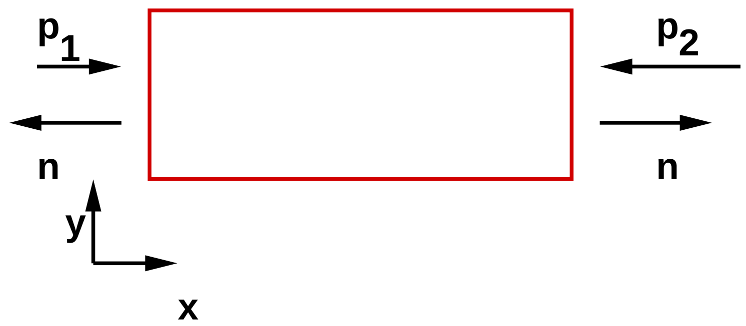
momentum equation

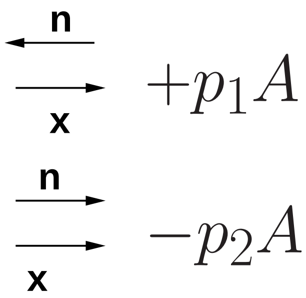
vector equation of motion for a continuum

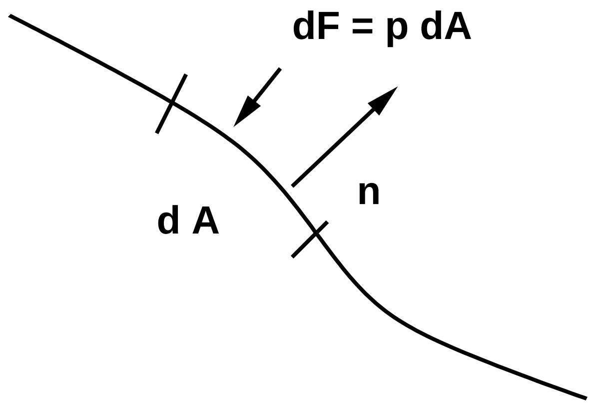
steady flow:

$$\frac{\partial}{\partial t} = 0 : \frac{d\vec{I}}{dt} = \int_A \rho \vec{v}(\vec{v} \cdot \vec{n}) dA = \sum F_a = \vec{F}_p + \vec{F}_g (+\vec{F}_R) + \vec{F}_S$$

pressure: $\vec{F}_p = \int_A -\vec{n} p dA$







volume force: $\vec{F}_g = \int_\tau \vec{g} dm = \int_\tau \vec{g} \rho d\tau$ (incompressible)

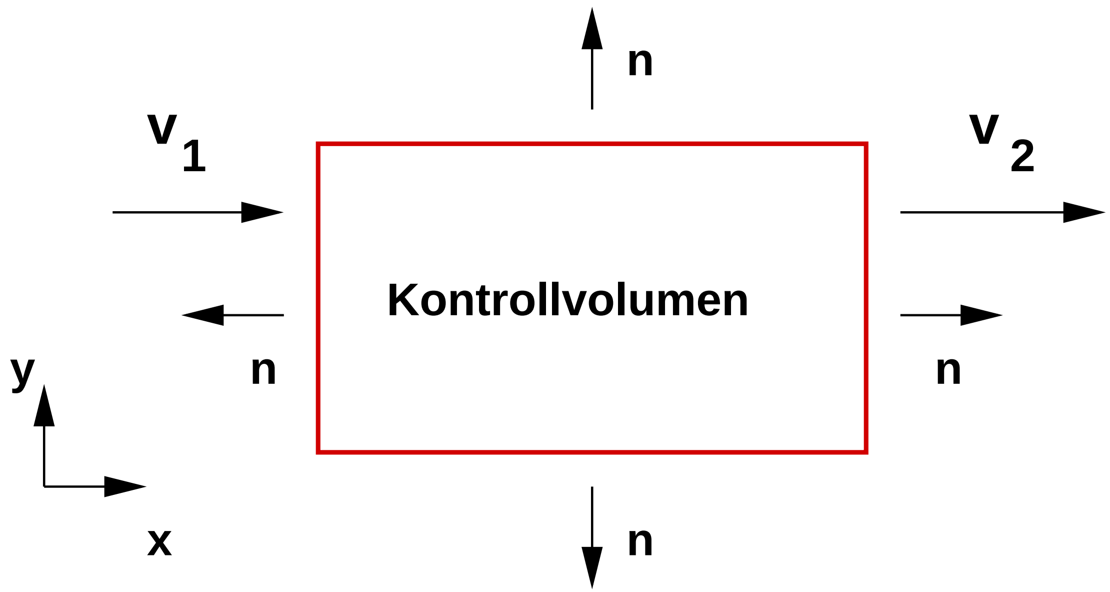
acceleration is parallel to the coordinate direction

friction force: $\vec{F}_R = -\int_A (\vec{\sigma}' \cdot \vec{n}) dA$

$\underline{\vec{F}_S}$: Force from the fitting onto the flow

Scalar product: $\vec{v} \cdot \vec{n}$

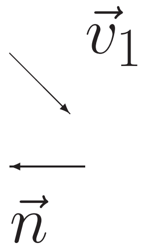
- the part of the mass that moves normal to the surface and flows through the boundary of the control volume
- influences the sign



incoming mass
has a negative sign

outflowing mass
has a positive sign

Scalar product: $\vec{v} \cdot \vec{n}$

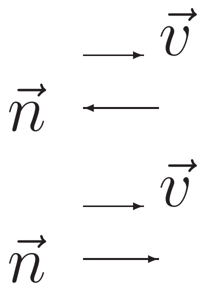


$$\vec{v}_1 = \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix}$$

$$\vec{n} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\longrightarrow \quad -v_{1x} = |\vec{v}_1| \, |\vec{n}| \, \cos\left(\angle(\vec{v}_1, \vec{n})\right)$$

The sign of the scalar product is not depending on the orientation of $\vec{n}$ in the coordinate system

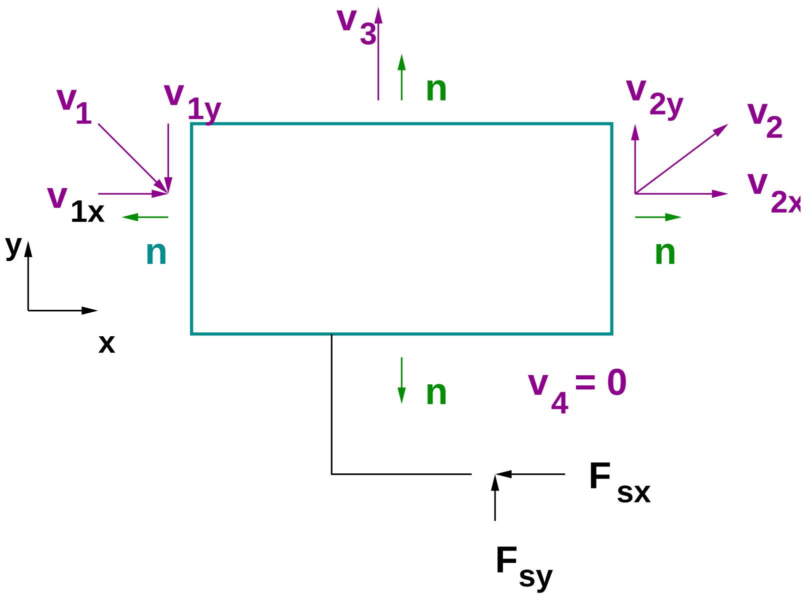


negative

positive

momentum equation

The respective component of the velocity is used for the computation of the momentum. The sign of the velocity depends on the coodinate system.



x-direction: $\frac{dI_x}{dt} = -F_{sx} = \rho\, v_{1x} \underbrace{(-v_{1x})}_{\vec{v}_1 \cdot \vec{n}} A_1 + \rho\, v_{2x} \underbrace{(v_{2x})}_{\vec{v}_2 \cdot \vec{n}} A_2$

y-direction: $\frac{dI_y}{dt} = F_{sy} = \rho\, v_{1y}\,(-v_{1x})\,A_1 + \rho\, v_{2y}\,(v_{2x})\,A_2 + \rho v_{3y} v_{3y} A_3$

momentum equation

The choice of the control surface or the control volume is quite important

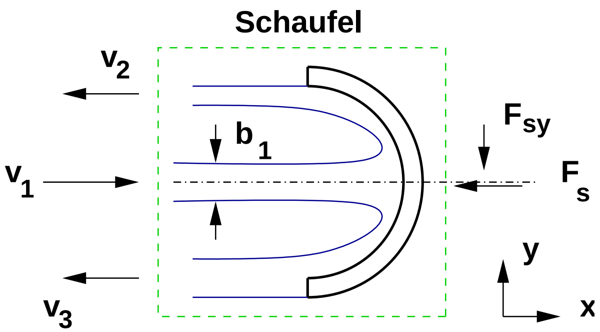
procedure/criterions

1. sketch the flow
2. define the coordinate system
3. choose the control surface such that
   - the integrands in the different directions are known
   - the integrands are zero (symmetry plane)
   - the geometry of the control surface is simple
   - fitting forces are included (or not)
   - if necessary use a moving control surface
4. determine the integrals for the special problem

1st example

2-dimensional, frictionless symmetrical

Bernoulli



$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$
$$= p_3 + \frac{1}{2}\rho v_3^2$$
$$p_1 = p_2 = p_3$$
$$\longrightarrow v_1 = v_2 = v_3$$
$$B_1 v_1 = B_2 v_2 + B_3 v_3$$
$$B_2 = B_3 = \frac{1}{2} B_1$$

given: ρ, v_1, B_1

momentum equation in x-direction: $\frac{dI_x}{dt} = \int_A \rho \vec{v}(\vec{v} \cdot \vec{n}) dA = \sum F_x$

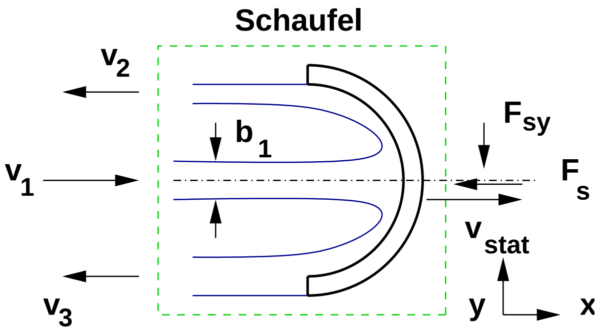
$$\underbrace{\rho(+v_1)(-v_1)B_1}_{\text{inflow}} + \underbrace{\rho(-v_2)(+v_2)B_2}_{\text{outflow}} + \underbrace{\rho(-v_3)(+v_3)B_3}_{\text{outflow}} = -F_{sx}$$

$$\rho v_1^2 \left(-B_1 - \frac{1}{2}B_1 - \frac{1}{2}B_1\right) = -F_{sx} \rightarrow F_{sx} = 2\rho v_1^2 B_1$$

2nd example

2-dimensional, frictionless, symmetrical

the same as in example no. 1, but with a moving blade



F_{sx} = ?

$$v_{abs} = v_{rel} + v_{stat}$$
$$v_{rel,1} = v_{abs,1} - v_{stat}$$
$$v_{rel,2} = v_{abs,2} - v_{stat}$$
$$v_{rel,3} = v_{abs,3} - v_{stat}$$

given: $\rho, v_{1,abs}, B_1, v_{stat}$

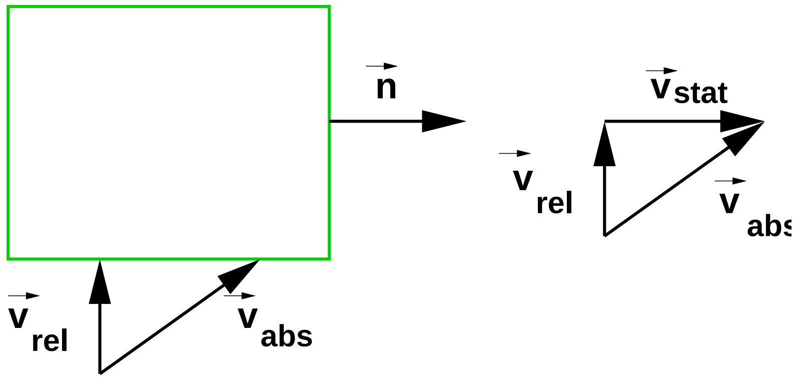
<u>Bernoulli, conti, symmetry</u> $v_{rel,1} = v_{rel,2} = v_{rel,3} \rightarrow B_2 = B_3 = \frac{1}{2} B_1$

<u>momentum equation in the absolute or in the relative system</u>

$$\frac{dI_x}{dt} = \int_A \rho \vec{v}_{abs} (\vec{v}_{rel} \cdot \vec{n}) dA = \sum F_x$$

velocity mass flux

moving control surface



$$\vec{v}_{abs} = \vec{v}_{rel} + \vec{v}_{stat}$$

$$\frac{dI_x}{dt} = \int_A \rho \vec{v}_{abs}(\vec{v}_{rel} \cdot \vec{n})dA = \int_A \rho(\vec{v}_{rel} + \vec{v}_{stat})(\vec{v}_{rel} \cdot \vec{n})dA$$

$$= \underbrace{\int_A \rho \vec{v}_{stat}(\vec{v}_{rel} \cdot \vec{n})dA}_{=0} + \int_A \rho \vec{v}_{rel}(\vec{v}_{rel} \cdot \vec{n})dA$$

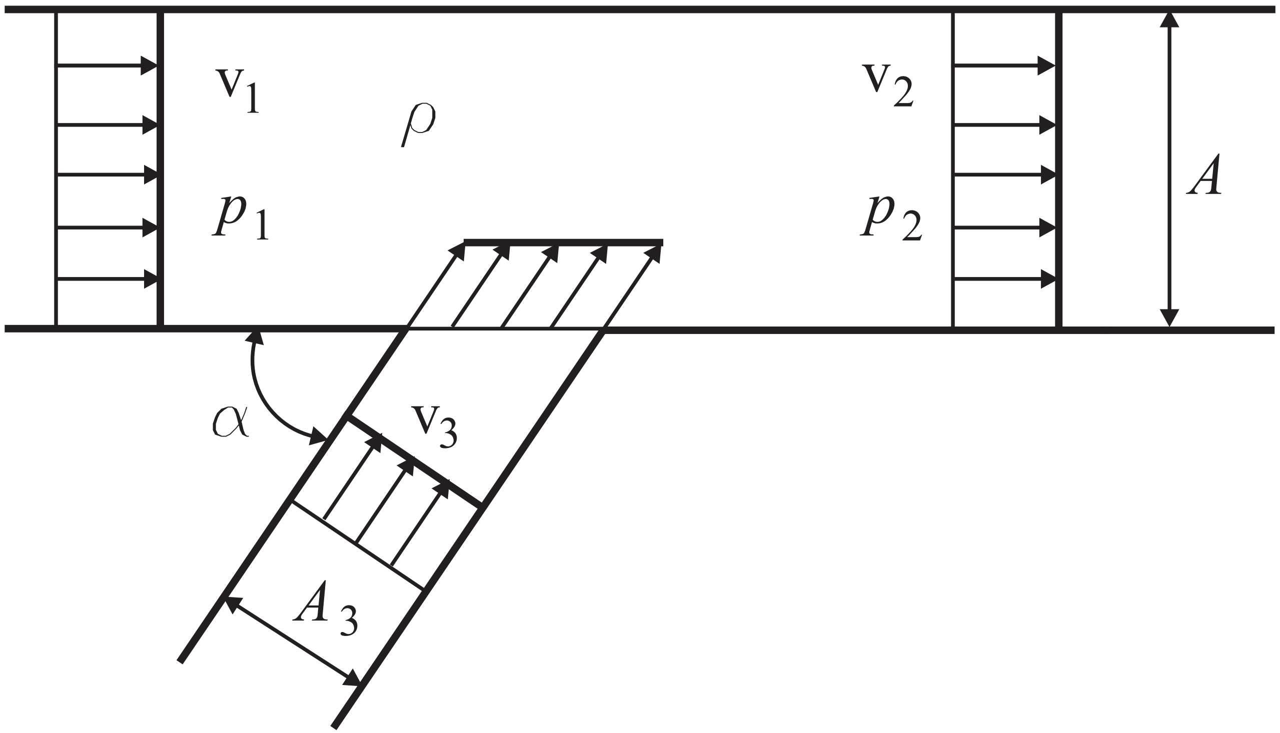
$$\boxed{\frac{dI_x}{dt} = \int_A \rho \vec{v}_{rel}(\vec{v}_{rel} \cdot \vec{n})dA =} \qquad \int_A \rho \vec{v}_{abs}(\vec{v}_{rel} \cdot \vec{n})dA$$

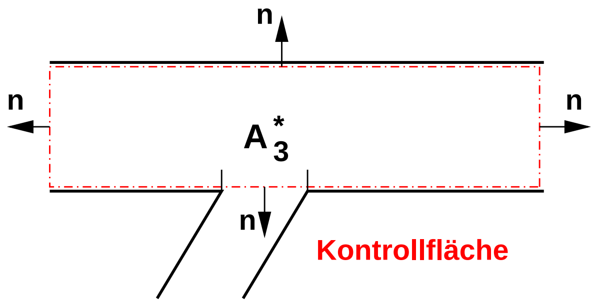
$$F_{sx} = 2\rho v_{rel,1}^2 B_1$$

7.2

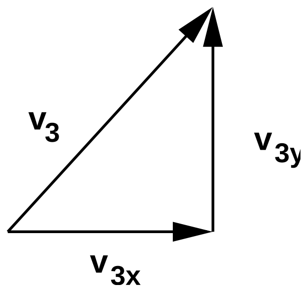
Determine the pressure difference $\Delta p = p_2 - p_1$ in the plotted bifurcation by neglecting the friction.

Given: $v_1,\ v_2,\ A_3 = \frac{1}{4}A,\ \alpha,\ \rho = const.$





momentum in x-direction: $\frac{dI_x}{dt} = \int_A \rho \vec{v}_x (\vec{v} \cdot \vec{n}) dA = \Sigma F_x \quad (\Delta p_x)$



$$A_3^* = A_3 / \sin\alpha$$
$$\vec{v} \cdot \vec{n} = v_3 \sin\alpha = v_3*$$

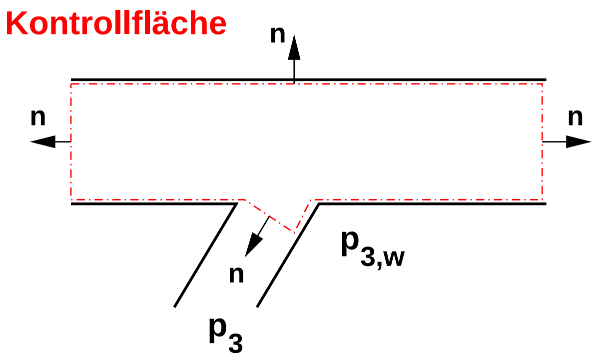
$$\frac{dI_x}{dt} = \rho v_1(-v_1)A_1 + \rho v_2 v_2 A + \rho v_3 \cos\alpha(-v_3 A_3) = \Sigma F_x$$

only pressure force: $\Sigma F_x = -\int p\vec{n} dA = (p_1 - p_2)A$

conti: $v_1 A_1 + v_3 A_3 = v_2 A_2 \rightarrow v_3 = 4(v_2 - v_1)$

$$\longrightarrow \Delta p = p_2 - p_1 = \rho(v_1^2 - v_2^2 + 4(v_2 - v_1)^2 \cos\alpha)$$

alternative



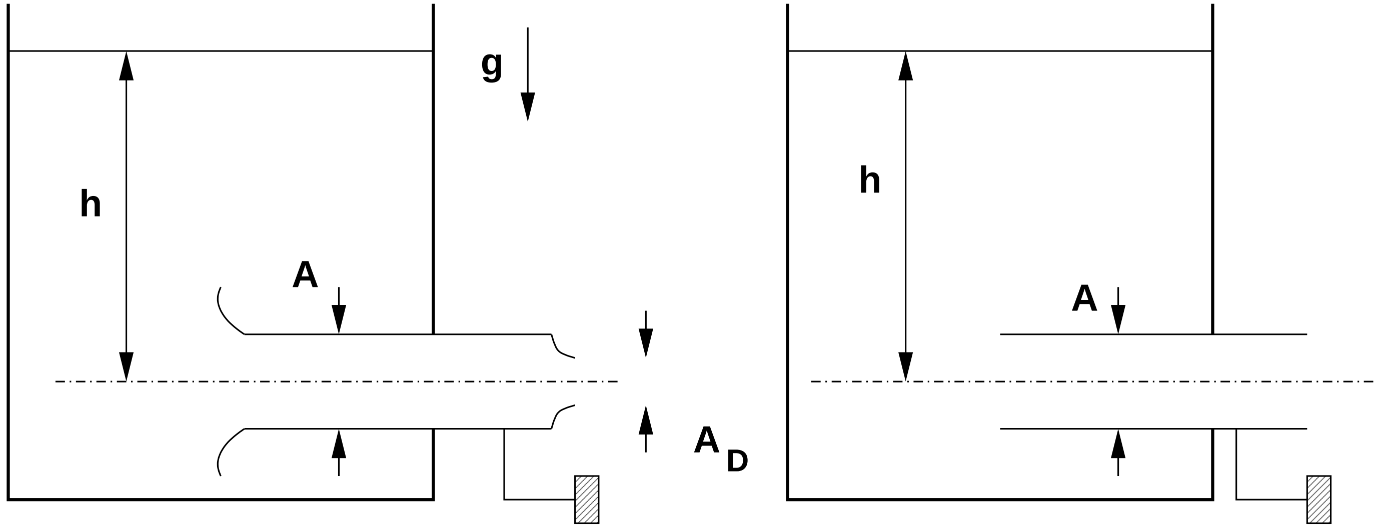
$$\int \rho \underbrace{\vec{v}}_{v_3 \cos\alpha} \underbrace{(\vec{v}\cdot\vec{n})dA}_{-v_3 A_3}$$

$p_3, p_{3,w}$ are unknown

$\longrightarrow \int p\vec{n}\, dA$ cannot be computed

example

water is flowing steadily from a large container into the open air. The inlet is well rounded. At the exit is a nozzle



Given: A, A_D, h, ρ, g

Determine the fitting force

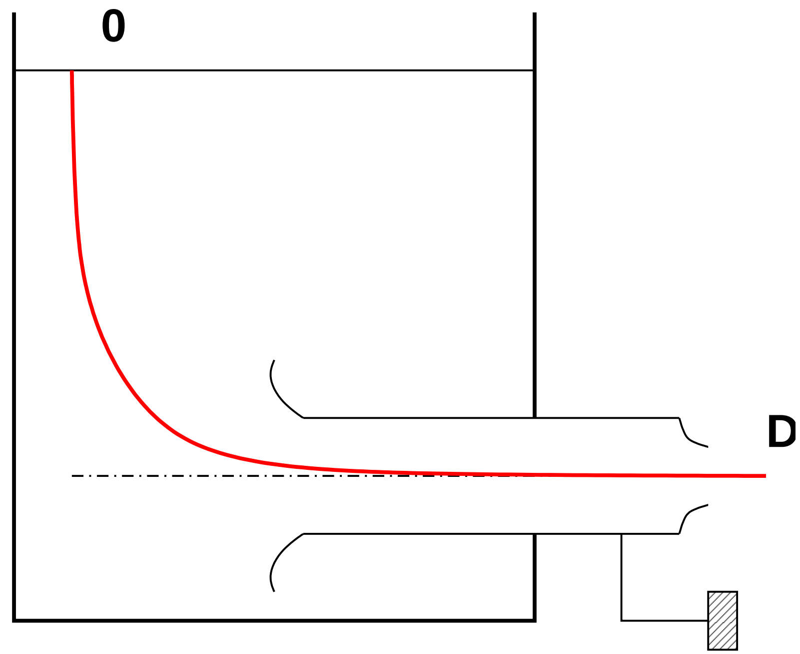
a) for the standard configuration
b) without inlet and nozzle

example

1.) mass flux

$$\dot{Q} = vA = v_D A_D$$

a) well rounded inlet and nozzle
$\longrightarrow$ no losses
$\longrightarrow$ Bernoulli



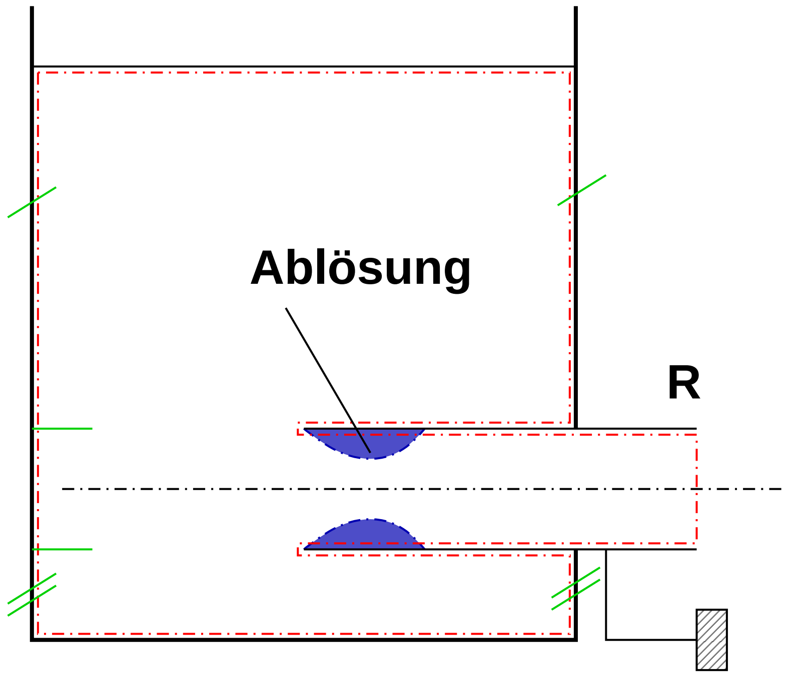
$$p_a + \rho g h = p_a + \frac{1}{2}\rho v_D^2$$

sharp edged exit

$$\longrightarrow v_D = \sqrt{2gh}$$

Beispiel

b) Borda estuary (separation)



$\longrightarrow$ losses
$\longrightarrow$ no Bernoulli
$\longrightarrow$ momentum equation

example

choice of control surface

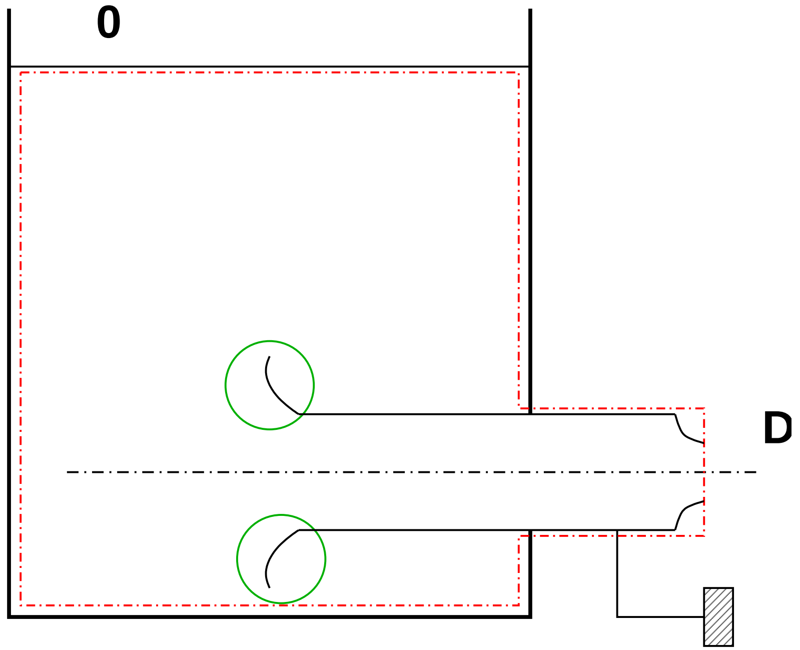
- exit
- no forces
- no pressure difference
- outside of the pipe
- within the pipe along the wall $v = 0$
- $p(z) = p_a + \rho g(h - z)$

$$\frac{dI_x}{dt} = \int_A \rho \vec{v}_x(\vec{v} \cdot \vec{n})dA = \int_{A_R} \rho \vec{v}_R(\vec{v}_R \cdot \vec{n})dA = \rho v_R A_R v_R = \dot{m} v_R$$
$$= \sum F_a = F_{p,x} = (p_a + \rho g h)A_R - p_a A_R$$

$$\longrightarrow v_{D,b} = \sqrt{gh} < v_{D,a}$$

2.) forces

a)



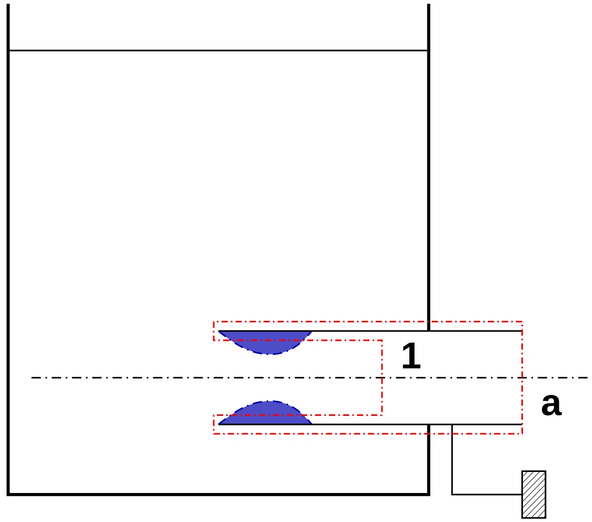
forces
exit
not cutting the tank

$$\rho v_D^2 A_D = (p_a + \rho g h) A_R - p_a A_R + F_x$$

$$v_D = \sqrt{2gh} \longrightarrow F_x = \rho g h (2A_D - A_R)$$

example

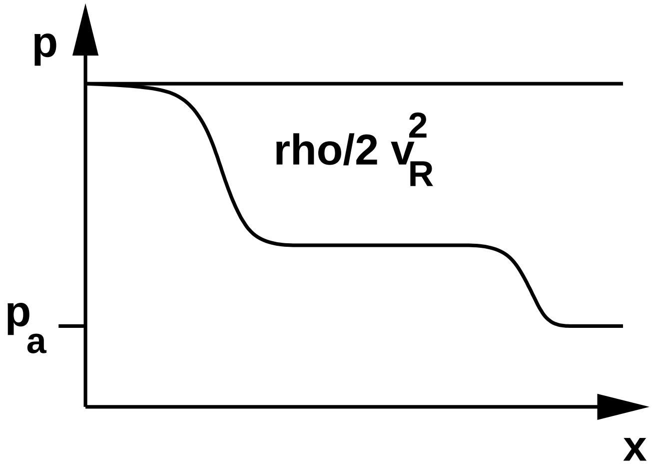
b)

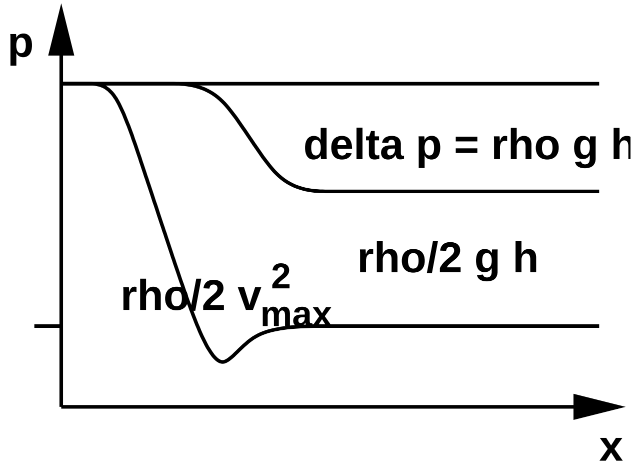


$$-\rho v_R^2 A_R + \rho v_R^2 A_R = F_x + (p_1 - p_a) A_R$$

$$p_1 + \frac{1}{2}\rho v_1^2 = p_a + \frac{1}{2}\rho v_a^2 \rightarrow v_1 = v_a = v_R$$

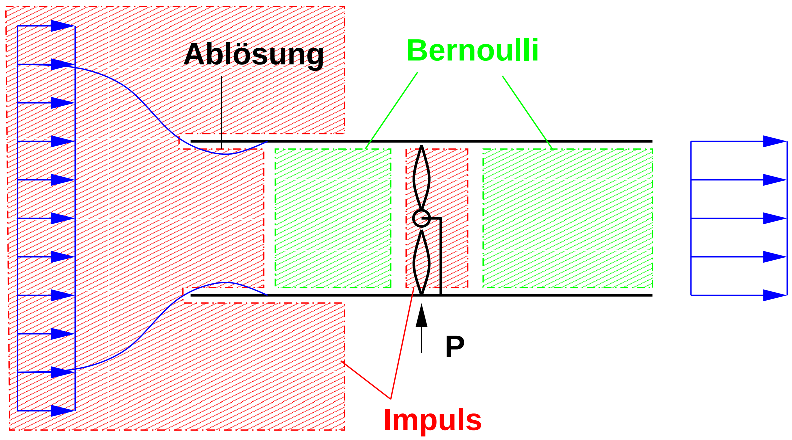
$$\longrightarrow p_1 = p_a \longrightarrow F_x = 0$$



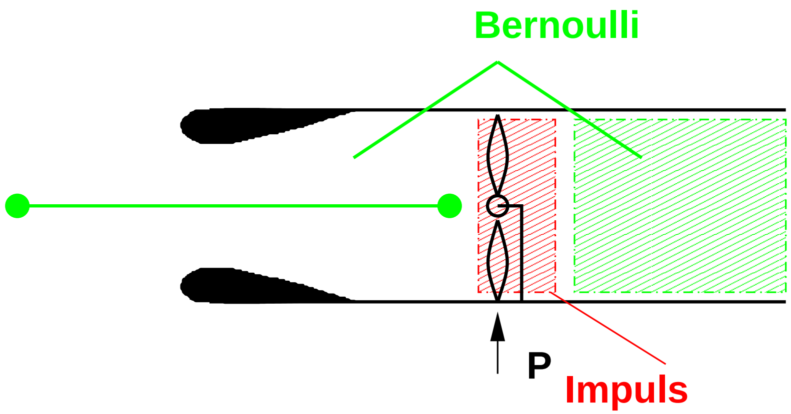


different forms of propellers

1.) sharp edged inlet

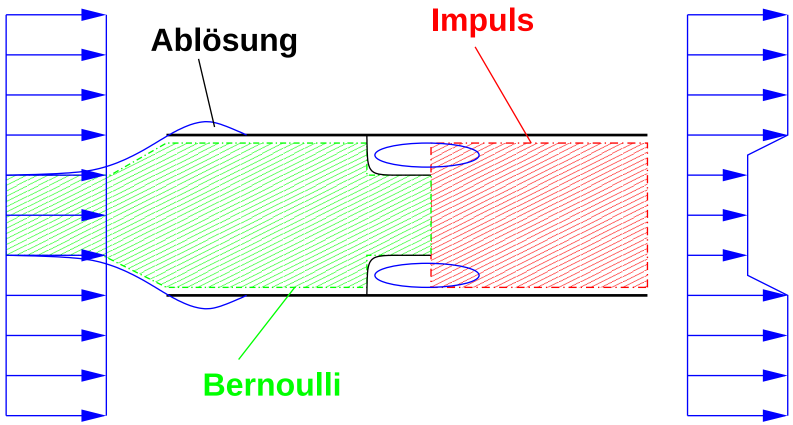


2.) well rounded inlet



different form of propellers

3.) pipe with nozzle



summary

- well rounded inlet $\longrightarrow$ Bernoulli
- sharp edged inlet $\longrightarrow$ momentum
- sharp edged exit $\longrightarrow$ Bernoulli
- losses (separation, mixing, ...) $\longrightarrow$ momentum
- power $\longrightarrow$ momentum
- outer forces $\longrightarrow$ momentum

but:

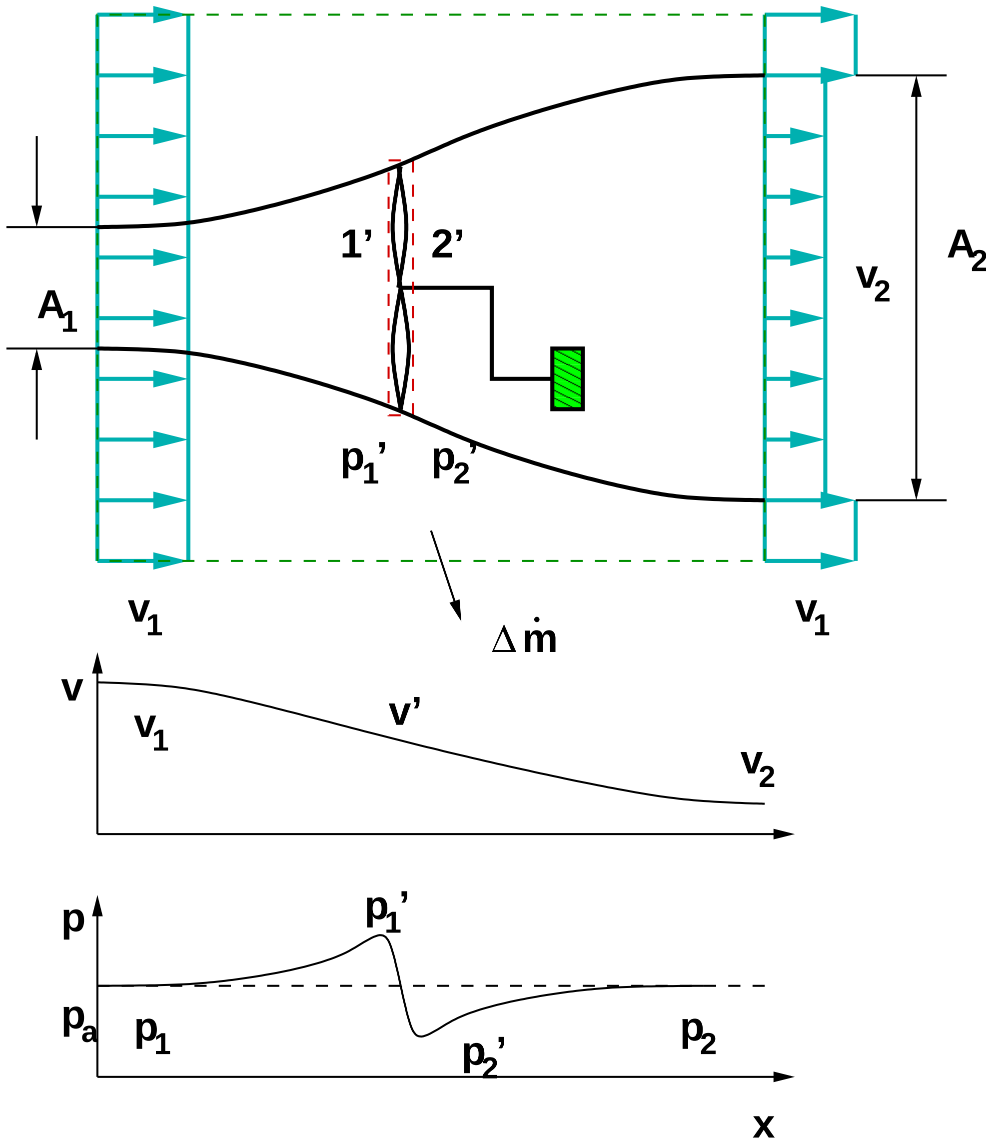
- for special problems both equations are necessary
- if Bernoulli is valid the momentum equations is also valid
- don't forget the continuity equation

Rankine‘s theory of jets

Propeller, windmills, ship’s screws

- 1-dimensional flow
- no influence of the rotation
- distribution of force is constant across the cross section
- acceleration or deceleration

Rankine's theory of jets



Rankine‘s theory of jets

continuity:

$$\rho v_1 A_1 = \rho v_1^{'} A^{'} = \rho v_2^{'} A^{'} = \rho v_2 A_2$$

Bernoulli:

$$1 \longrightarrow 1' \; : \; p_a + \frac{\rho}{2} v_1^2 = p_1^{'} + \frac{\rho}{2} v_1^{'2}$$

$$2' \longrightarrow 2 \; : \; p_2^{'} + \frac{\rho}{2} v_2^{\prime 2} = p_a + \frac{\rho}{2} v_2^2$$

momentum: control volume

$$-\rho v^{'2} A' + \rho v^{'2} A' = (p_1^{'} - p_2^{'}) A' + F$$

$$\longrightarrow F = (p_2^{'} - p_1^{'}) A' < 0$$

continuity:

$$\Delta \dot{m} = \rho A_2 (v_1 - v_2)$$

Rankine's theory of jets

momentum: control volume

$$-\rho v_1^2 A_\infty + \rho v_2^2 A_2 + \rho v_1^2 (A_\infty - A_2) + \Delta \dot{m} v_1 = F$$

$$\longrightarrow F = \rho v_2 A_2 (v_2 - v_1) = \rho v' A' (v_2 - v_1)$$

⋮

theorem of Froude: $v' = \frac{1}{2}(v_1 + v_2)$

power: $P = \dot{Q}\Delta p_0 = \frac{\rho}{4} A' v_1^3 (1 + \frac{v_2}{v_1})(1 - \frac{v_2^2}{v_1^2}) \sim v_1^3$

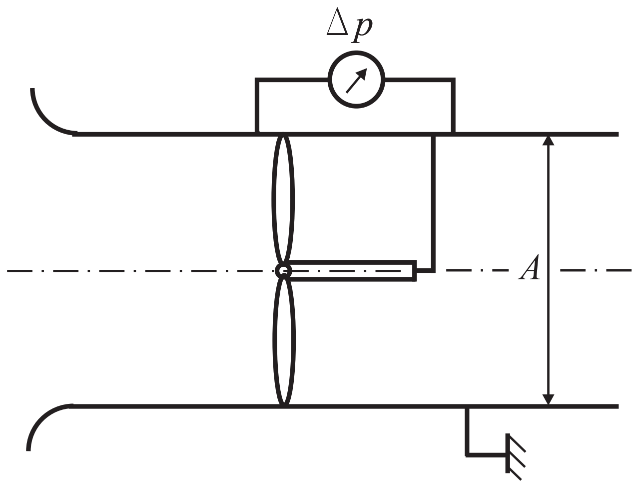
maximum power: $\frac{\partial P}{\partial(\frac{v_2}{v_1})} = 0 \longrightarrow \frac{P_{max}}{A'} = \frac{8}{27}\rho v_1^3$

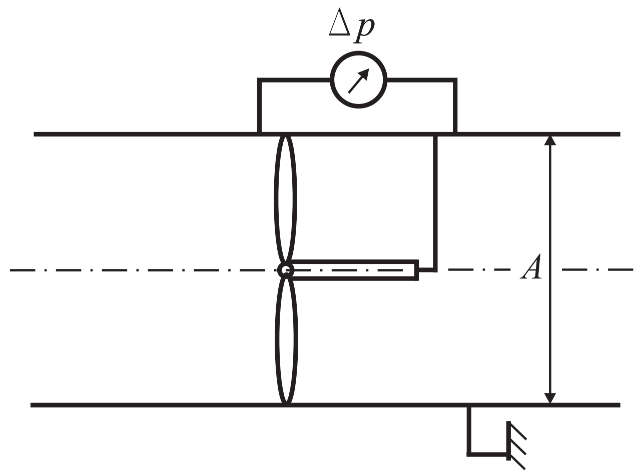
maximum thrust: $\frac{F}{A'} = -\frac{4}{9}\rho v_1^2 \sim v_1^2$

7.8

Two fans sucking air from the surrounding differ in their inlets

Given: $\rho, \; A, \; \Delta p$



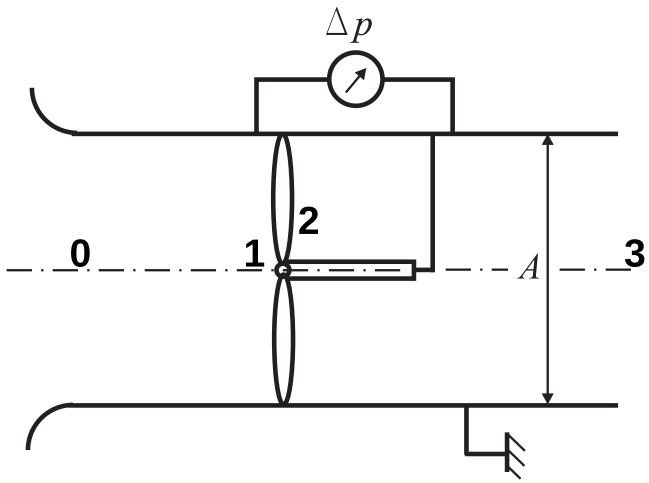


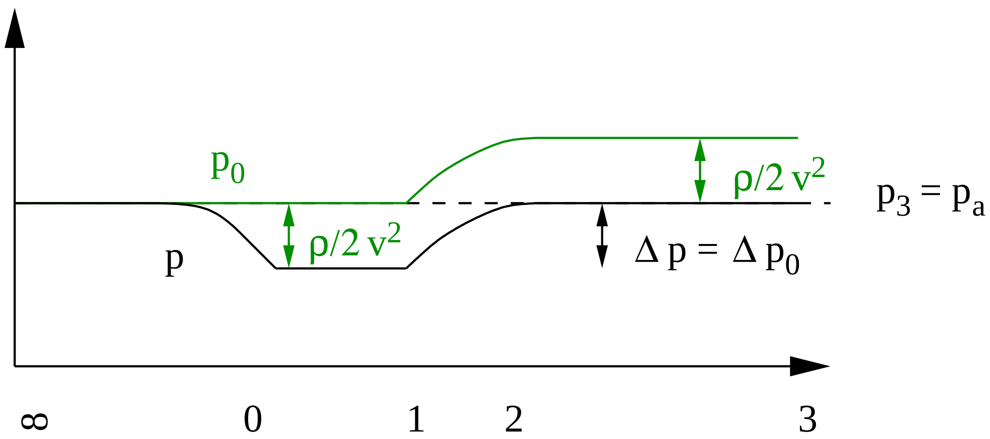
Compute

a) the volume flux,

b) the power of the fans,

c) the force on the fitting.





Bernoulli $2 \longrightarrow 3$:

$$p_2 + \frac{\varrho}{2} v_2^2 = p_3 + \frac{\varrho}{2} v_3^2$$

$$\rightarrow p_2 = p_3 = p_a$$

Bernoulli $-\infty \longrightarrow 1$:

$$p_a + 0 = p_1 + \frac{\varrho}{2} v_1^2$$

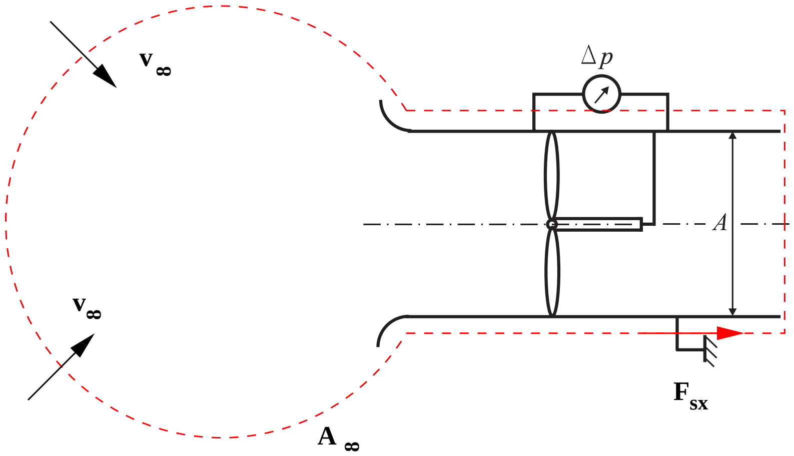
$$(\Delta p = p_2 - p_1)$$

$$\longrightarrow v_1 = \sqrt{\frac{2}{\rho}\Delta p} \longrightarrow \boxed{\dot{Q} = v_1 A = \sqrt{\frac{2}{\rho}\Delta p} A}$$

1st theorem for steady flow processes: $P = \dot{Q}\Delta p_0$

here: $\Delta p_0 = p_{02} - p_{01} = p_2 + \frac{\rho}{2}v_2^2 - p_1 - \frac{\rho}{2}v_1^2 = p_2 - p_1 = \Delta p$

$$\longrightarrow \boxed{P = \Delta p A \sqrt{\frac{2}{\rho}\Delta p}}$$



7.8

assumption: flow field can be described using a point sink

$\rightarrow$ no direction at infinity

$\rightarrow$ the velocity is constant

$\rightarrow A_\infty v_\infty = Av$

$\rightarrow v_\infty = \dfrac{Av}{A_\infty}$

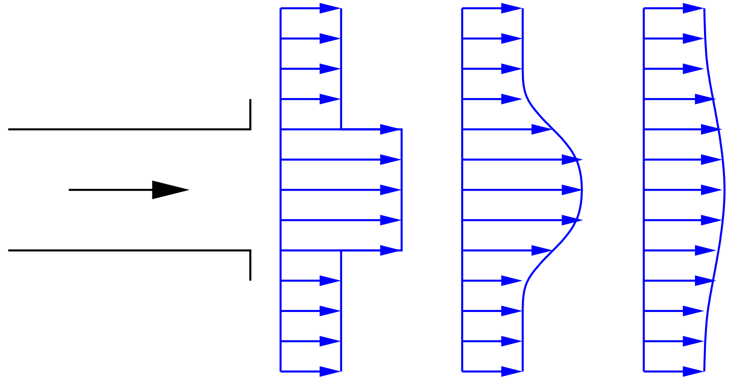
momentum flux for A_∞

$$\left|\frac{d\vec{I}}{dt}\right| = \sqrt{\left(\frac{dI}{dt}\right)_x + \left(\frac{dI}{dt}\right)_y} = |\oint_{A_\infty} \rho \underbrace{\vec{v}}_{v_\infty} \underbrace{(\vec{v}\cdot\vec{n})}_{\leq v_\infty} dA| <$$

$$< \oint_{A_\infty} |\rho\vec{v}\,(\vec{v}\cdot\vec{n})\,dA| \leq \oint_{A_\infty} \rho v_\infty^2 dA = \rho v_\infty^2 A\infty$$

$$v_\infty = \frac{Av}{A_\infty} \rightarrow \left|\frac{d\vec{I}}{dt}\right| < \frac{\rho v^2 A^2}{A_\infty} = \frac{\dot{m}}{\rho A_\infty} \rightarrow 0 \text{ for } A_\infty \rightarrow \infty$$

impossible at the exit

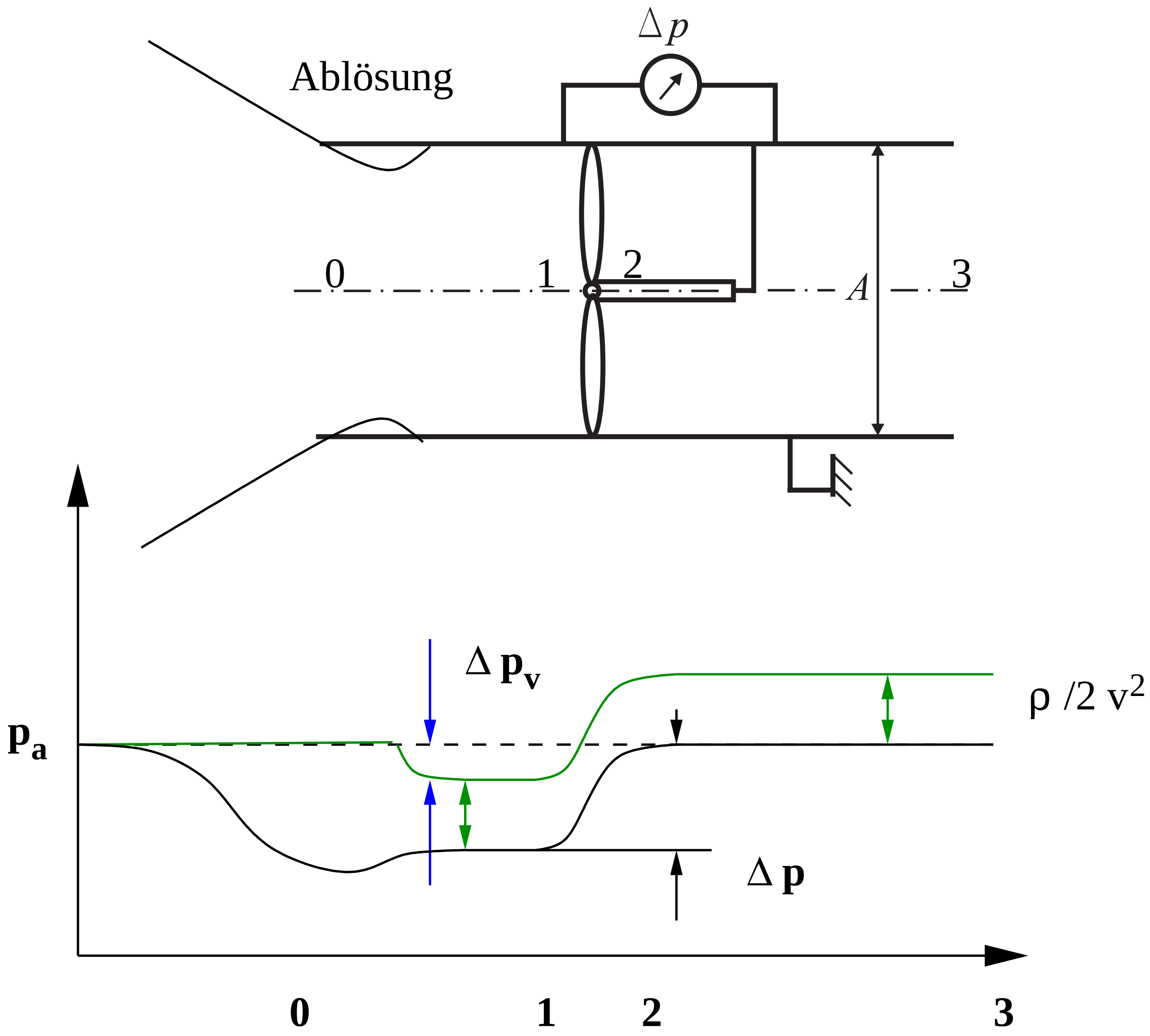


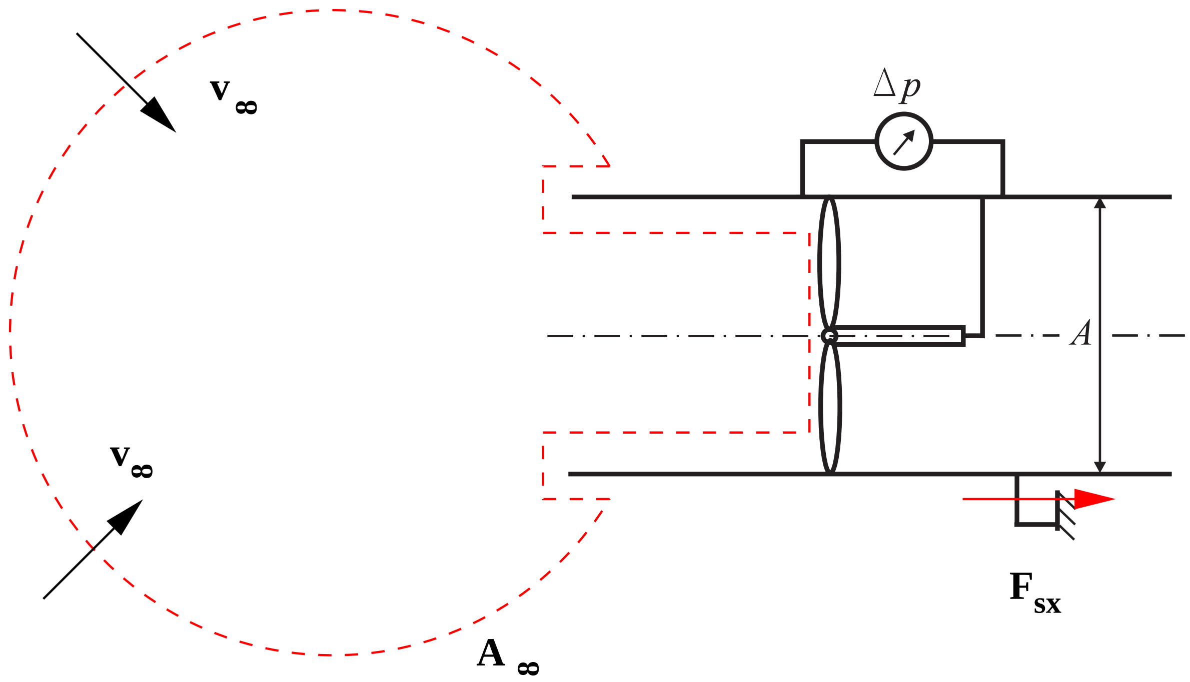
$\longrightarrow$ directed flow

momentum equation

$$\left|\frac{dI}{dt}\right|_x = \rho v^2 A = -p_a \oint_{A_\infty} \vec{n} dA + F_{sx}$$

$$\longrightarrow F_{sx} = \rho v^2 A = \boxed{2\Delta p \cdot A}$$


Δ p
Ablösung
0
1
2
A
3
Δ p_v
p_a
ρ /2 v^2
Δ p
0
1
2
3

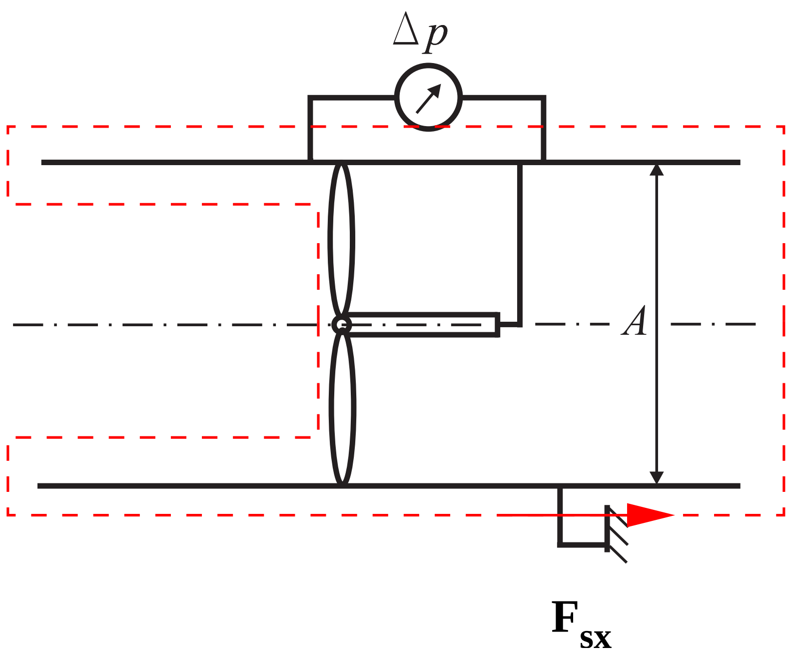


momentum equation

$$\rho v^2 A = p_a A_\infty - (p_a(A_\infty - A) + p_1 A)$$
$$= (p_a - p_1)A = \Delta p A$$
$$v_1 = v = \sqrt{\frac{\Delta p}{\rho}} \longrightarrow \boxed{\dot{Q} = \sqrt{\frac{\Delta p}{\rho}} A}$$

power:

$$P = \Delta p_0 \dot{Q} = \Delta p A \sqrt{\frac{\Delta p}{\rho}}$$



$$\rho v(-v)A + \rho v v A = (p_1 - p_3)A + F_{sx} \longrightarrow \boxed{F_{sx} = \Delta p A}$$

momentum of momentum equation (angular momentum)

2nd Newtonian law + trnasport theorem of Reynolds
$\rightarrow$ momentum of momentum equation (Vector equation)

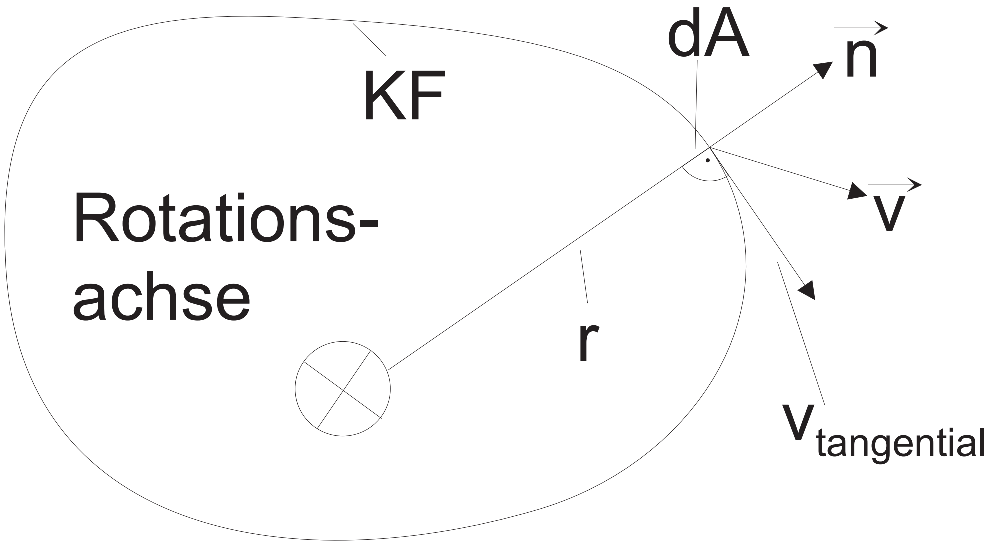
$$\frac{\partial}{\partial t} \int_{KV} (\vec{r} \times \vec{v}) \rho \mathrm{d}V + \int_{KF} (\vec{r} \times \vec{v}) \rho (\vec{v} \cdot \vec{n}) \mathrm{d}A = \sum \vec{M}_a$$

Moments of pressure forces: $\vec{M}_p = - \int_{KF} p(\vec{r} \times \vec{n}) \mathrm{d}A$
Moments of volume forces: $\vec{M}_g = \int_{KF} (\vec{r} \times \rho \vec{g}) \mathrm{d}V$
Moments of friction forces: $\vec{M}_R = - \int_{KF} \vec{r} \times (\overline{\overline{\sigma}}' \, \vec{n}) \mathrm{d}A$
Moments of outer forces: $\vec{M}_s = \vec{r}_s \times \vec{F}_s$

steady flow: $\frac{\partial}{\partial t} \int_{KV} (\vec{r} \times \vec{v}) \rho \mathrm{d}V = 0$

rotation around an axis (instead of a point) $\rightarrow$ scalar equation

momentum of momentum equation (angular momentum)



$$M = \int_{KF} v_{tangential} \cdot |\vec{r}| \cdot \rho(\vec{v} \cdot \vec{n}) \mathrm{d}A$$

usually: steady, fixed rotation axis, rotational symmetrical

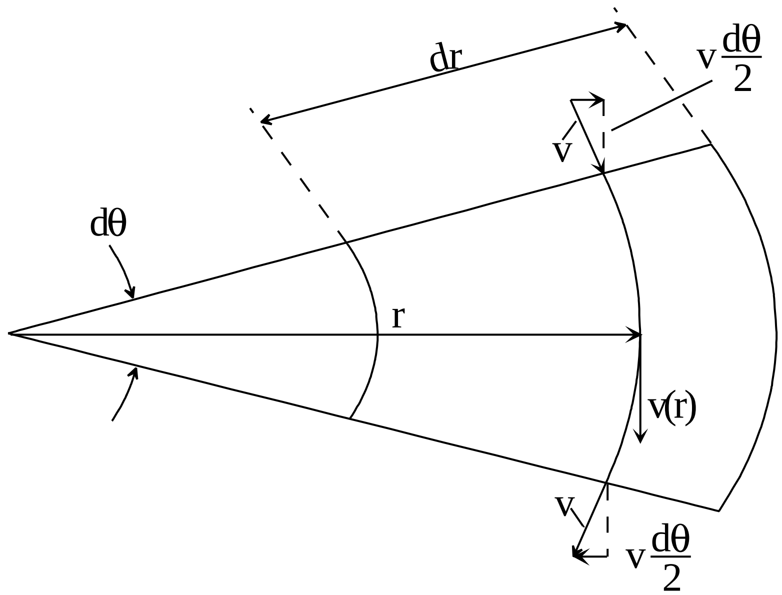
$\rightarrow$ quasi 1 dimensional

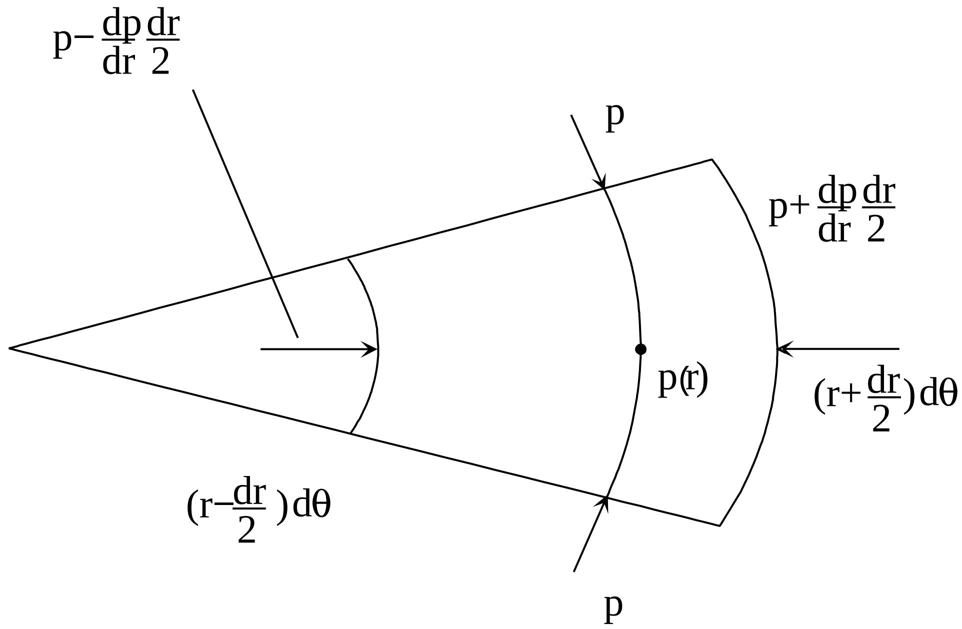
momentum of momentum equation (angular momentum)

assumption: pressure and velocity only depend on the radius

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial \theta} = 0$$

control surface: segment of a circle





momentum of momentum equation (angular momentum)

$$-\rho v v \frac{d\theta}{2} dr - \rho v v \frac{d\theta}{2} dr = (p - \frac{dp}{dr}\frac{dr}{2})(r - \frac{dr}{2})d\theta - (p + \frac{dp}{dr}\frac{dr}{2})(r + \frac{dr}{2})d\theta +$$
$$+2 \cdot pdr sin\frac{d\theta}{2}$$

$$\Rightarrow \quad -2\rho v^2 \frac{d\theta}{2} dr = prd\theta - p\frac{dr}{2}d\theta - \frac{dp}{dr}\frac{dr}{2}rd\theta + \frac{dp}{dr}\frac{dr}{2}\frac{dr}{2}d\theta$$
$$- prd\theta - p\frac{dr}{2}d\theta - \frac{dp}{dr}\frac{dr}{2}rd\theta - \frac{dp}{dr}\frac{dr}{2}\frac{dr}{2}d\theta + 2pdr\frac{d\theta}{2}$$
$$\Rightarrow \quad \rho v^2 d\theta dr = \frac{dp}{dr} dr \; rd\theta$$

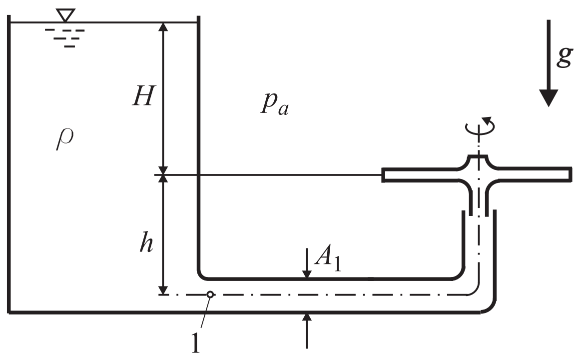
$$\Rightarrow \quad \frac{dp}{dr} = \rho\frac{v^2}{r} \qquad \text{mit } \omega = \frac{v}{r} \Rightarrow \frac{dp}{dr} = \rho\omega^2 r$$

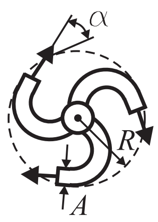
7.11

A sprinkler with three arms is supplied by a large tank and rotates with the angular velocity $\omega = const.$. The angle between the outflowing jets and the circumferential direction is α.

$$H = 10\ m, R = 0,5\ m, h = 1\ m,$$

$$A = 0,5 \cdot 10^{-4}\ m^2, A_1 = 1,5.10^{-4}\ m^2, \alpha = 30°$$





$$p_a = 10^5\ N/m^2, \rho = 10^3\ kg/m^3, g = 10\ m/s^2,$$

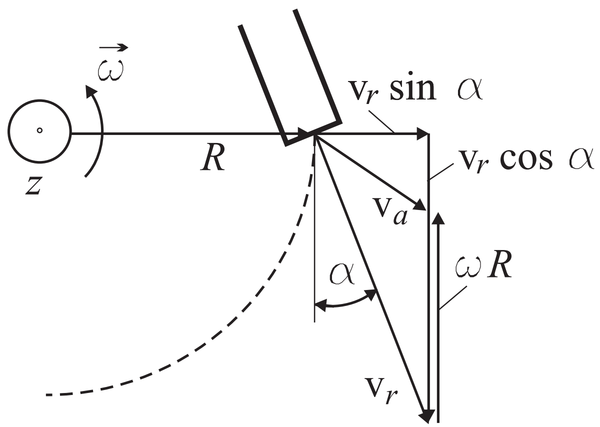
$$\omega = 15\ s^{-1}$$

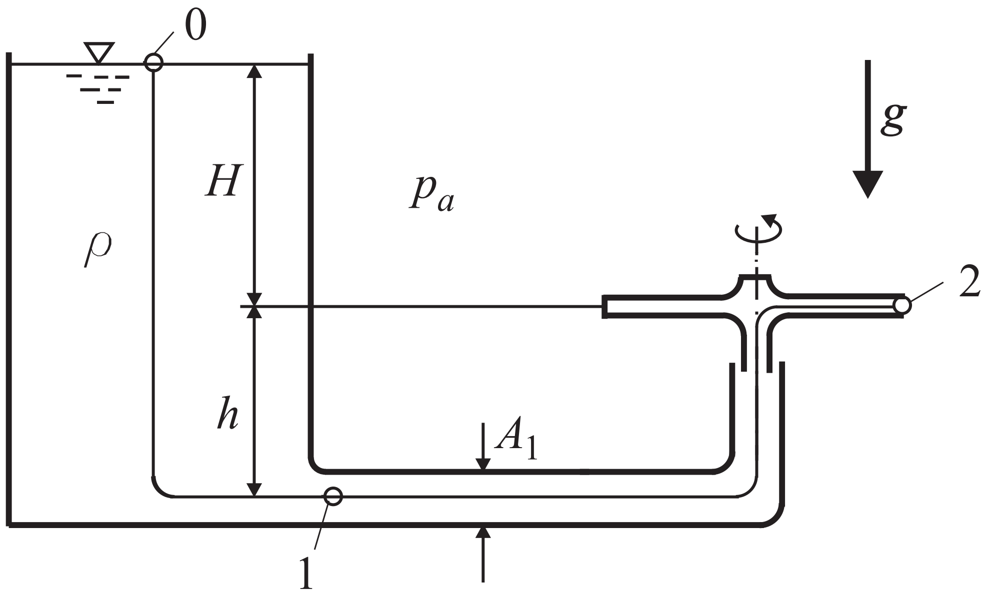
Determine

a) the relative exit velocity, b) the torque and the volume flux,
c) the pressure p_1, d) the maximum torque.

7.11

a)





$\vec{v}_a = v_{absolut}$ $\quad$ $\vec{v}_r = v_{relativ}$ $\quad$ $\vec{\omega} \cdot R = v_{\text{vehicle}}$ $\quad$ $\vec{v}_a = \vec{v}_r + \vec{\omega} R$

v_r using Bernoulli with an additional term

$$0 \quad \rightarrow 2: \quad p_a + \rho g H = p_a + \frac{\rho}{2} v_r^2 - \int_{s0}^{s2} \rho(\vec{b} \cdot d\vec{s}) = p_a + \frac{\rho}{2} v_r^2 - \rho \left(\frac{w^2 R^2}{2} \right)$$

$$\Longrightarrow v_r = \sqrt{2gH + w^2 R^2} = 16 \frac{m}{s}$$

b) steady flow in a moving coordinate system

$$\Longrightarrow \int_{KF} (\vec{r} \times \vec{v}) \rho \vec{v} \cdot \vec{n} dA = \Sigma \vec{M} = \Sigma \left(\vec{r} \times \vec{F} \right)_{KV}$$

$$\Longrightarrow \vec{M} = \int_{KF} \left(\vec{r} \times \vec{v}_a \right) \rho \vec{v}_r \cdot \vec{n} dA$$

$$|\vec{r} \times \vec{v}_a| = |R(\omega R - v_r \cos \alpha)|$$

momentum for three arms:

$$\Longrightarrow M = 3 \rho v_r A R \left(\omega R - v_r \cos \alpha \right) = -6.8 \ Nm$$

the momentum $\vec{M}$ operates in the direction of $\vec{\omega}$. $\vec{F}$ and $\vec{v}$ are directed contrarily.

$$\dot{Q} = 3 v_r A = 2.4 \cdot 10^{-3} \ \frac{m^3}{s}$$

7.11

c) Bernoulli 0 $\longrightarrow$ 1:

$$p_a + \rho g(h+H) = p_1 + \frac{\rho}{2} v_1^2$$

$$v_1 = \frac{\dot{Q}}{A_1}$$

$$\Longrightarrow p_1 = p_a + \rho g(h+H) - \frac{\rho}{2}\frac{\dot{Q}^2}{A_1^2} = 0.82 \cdot 10^5 \, \frac{N}{m^2}$$

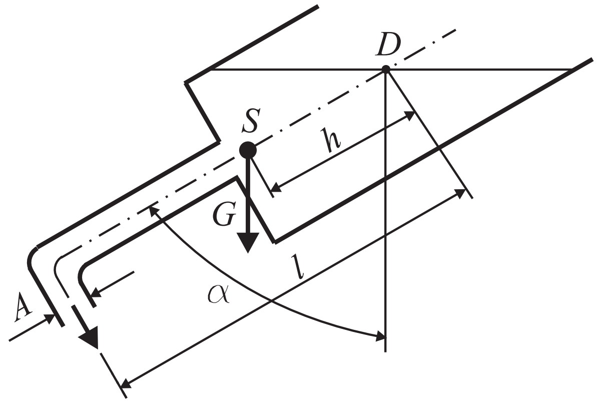
d)

$$\frac{dM}{d\omega} = 0 \quad \Longrightarrow \text{Maximum for } \omega = 0 \quad \Longrightarrow \quad v_r = \sqrt{2gH}$$

$$\Longrightarrow M = 3\rho\sqrt{2gH}\,AR\left(-\sqrt{2gH}\cos\alpha\right) = -6\rho g H\,AR\cos\alpha = 13\ Nm$$

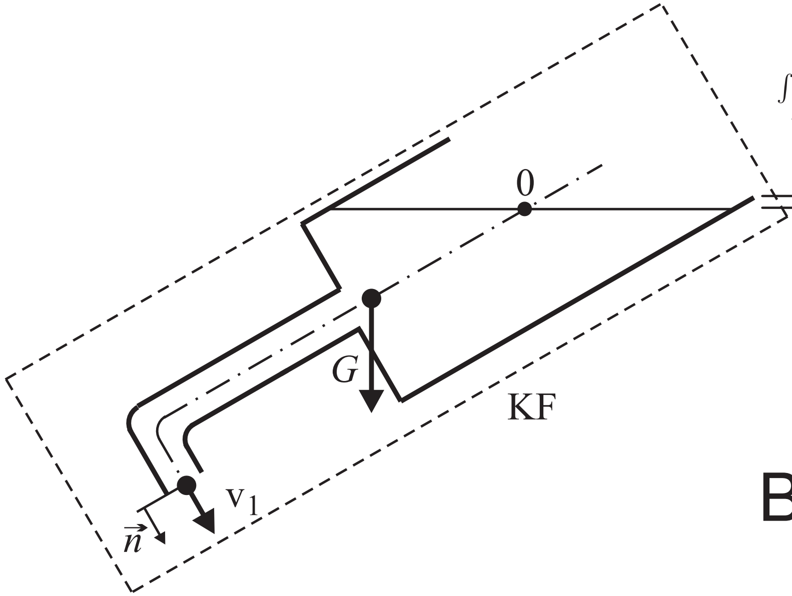
7.12

A tank with the weight G is fixed in a rotatable bearing in D. Its drain-pipe has a $90°$-bend. The center of gravity of the system has the distance h to point D. What is the angle α between the pipe-axis and the vertical axis, if the water flows without friction?



Given: $G,\ l,\ h,\ A,\ \rho$

Hint: The tank is such large that the water surface is not moving.



momentum of momentum:

$$\int_{KF} \rho(\vec{r} \times \vec{v})\vec{v} \cdot \vec{n} dA = \Sigma \vec{M}$$

$$\Longrightarrow \rho l v_1^2 A = G h \sin\alpha$$

Bernoulli $0 \longrightarrow 1$: $\rho g l \cos\alpha = \frac{\rho}{2} v_1^2$

$$\Longrightarrow v_1 = \sqrt{2 g l \cos\alpha}$$

$$\Longrightarrow \rho l 2 g l \cos\alpha = G h \sin\alpha$$

$$\Longrightarrow G h \tan\alpha = \rho l 2 g l$$

$$\Longrightarrow \alpha = \arctan\left(\frac{\rho l 2 g l}{G h}\right)$$