Potential theory

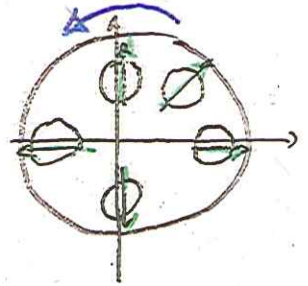
Requirements: frictionless, no rotation
2-dimensional (planar)
incompressible, steady

rotation free: $\vec{\omega} = 0$

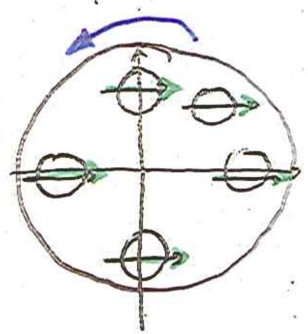
$$\vec{\omega} = \frac{1}{2}\,\mathrm{rot}\,\vec{v} = \frac{1}{2}\nabla \times \vec{v} = \frac{1}{2}\begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix} \leftarrow$$

2-dimensional flow: $\omega_x = \omega_y = 0$

$$\rightarrow \omega_z = \frac{1}{2}(v_x - u_y) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$



rotational flow



rotation free
$\Rightarrow$ potential flow

[Example 13.1] $\mathrm{rot}(\mathrm{grad} f) = \vec{0}$

Potential function

If $\omega_z = 0 \rightarrow$ A function ϕ exists with the property $\vec{v} = \vec{\nabla}\phi \rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{pmatrix}$

Potential

$$\Rightarrow \boxed{u = \frac{\partial \phi}{\partial x}\ ;\ v = \frac{\partial \phi}{\partial y}}$$

$\rightarrow$ Continuity (2-D, steady, incompressible)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \vec{\nabla} \cdot \vec{v} = 0 \Rightarrow \boxed{\nabla^2 \phi = \Delta \phi = 0}$$

$$\Rightarrow \boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0}$$

Laplace equation of the potential function

Linear differential equation

$\Rightarrow$ The principle of superposition is applicable

If ϕ_1, ϕ_2 are solution of the equation, then $C_1 \cdot \phi_1$, $C_2 \phi_2$ and $C_1 \phi_1 + C_2 \phi_2$ are also solutions

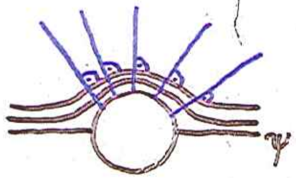
Stream function: $\boxed{u = \frac{\partial \psi}{\partial y}\ ;\ v = -\frac{\partial \psi}{\partial x}}$ (fulfills the Continuity)

$$\omega = 0 \rightarrow \boxed{\nabla^2 \psi = \Delta \psi = 0}$$

$$\underbrace{\phi_x = \psi_y}_{u}\ ;\ \underbrace{\phi_y = -\psi_x}_{v} \rightarrow \phi \text{ and } \psi \text{ are perpendicular}$$

ϕ = const → lines of constant potential

ψ = const → streamlines



ϕ and ψ are used to describe flow fields and flows around bodies.

The contour is presented by a special streamline

⇒ The velocity vector $\vec{v}$ is parallel to the wall

But: Stokes no-slip condition cannot be fulfilled (frictionless + rotation free)

→ Drag forces and viscous stresses cannot be determined

- complex numbers:

$$\boxed{z = x + iy} = r e^{i\varphi} = r(\cos\varphi + i \sin\varphi)$$

$$x = r \cos\varphi \quad y = r \sin\varphi \iff r = \sqrt{x^2 + y^2} \quad \varphi = \arctan\left(\frac{y}{x}\right)$$

- complex velocity

$w = u + iv$

- conjugate complex velocity

$\overline{w} = u - iv$

complex potential function

complex stream function

$$\boxed{F(z) = \int \overline{w}\, dz = \phi(x,y) + i\psi(x,y)}$$

$$\Rightarrow \text{Laplace-equation} \quad \boxed{\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) + i\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = 0}$$

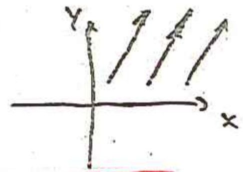
$$\overline{w} = u - iv = \frac{dF}{dz}$$

Singularities:

$\boxed{\text{Parallel flow}}$: $\boxed{F(z) = (u_\infty - i v_\infty) z} = \phi + i\psi$

$\phi = u_\infty x + v_\infty y \qquad \psi = u_\infty y - v_\infty x$

$u = u_\infty \qquad v = v_\infty$

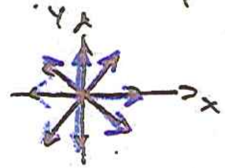


Streamlines

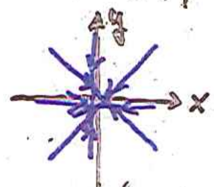
$\boxed{\text{Source, sink}}$: $\boxed{F(z) = \frac{E}{2\pi} \ln z} = \phi + i\psi$

$\phi = \frac{E}{2\pi} \ln r \qquad \psi = \frac{E}{2\pi} \varphi$

$u = \frac{E}{2\pi} \frac{x}{x^2 + y^2} \qquad v = \frac{E}{2\pi} \frac{y}{x^2 + y^2}$



Source

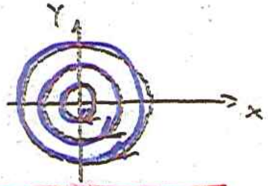


sink

Vortex: $\boxed{F(z) = \frac{\Gamma}{2\pi} i \ln z}$

$\phi = -\frac{\Gamma}{2\pi} \arctan \frac{y}{x}$ $\qquad \psi = \frac{\Gamma}{2\pi} \ln \sqrt{x^2+y^2}$

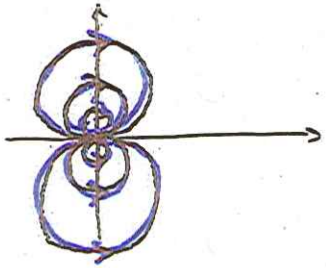
$u = \frac{\Gamma}{2\pi} \frac{y}{x^2+y^2}$ $\qquad v = -\frac{\Gamma}{2\pi} \frac{x}{x^2+y^2}$



Dipole, Doublet $\boxed{F(z) = \frac{m}{z}}$

$\phi = \frac{mx}{x^2+y^2}$ $\qquad \psi = -\frac{my}{x^2+y^2}$

$u = m \frac{y^2-x^2}{(x^2+y^2)^2}$ $\qquad v = -m \frac{2xy}{x^2+y^2}$



Superposition of a source and a sink

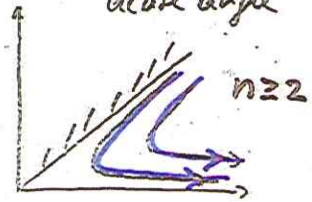
usually: $m = \frac{M}{2\pi}$

$\Rightarrow \boxed{F(z) = \frac{M}{2\pi z}}$

Corner flow $\boxed{F(z) = \frac{a}{n} z^n}$ $\quad n \in \mathbb{R}$, $a \in \mathbb{C}$

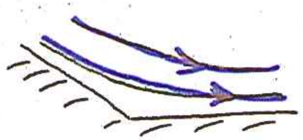
$\phi = \frac{a}{n} r^n \cos(n\varphi)$ $\qquad \psi = \frac{a}{n} r^n \sin(n\varphi)$

acute angle

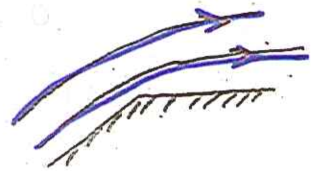


$n \geq 2$

concave



$2 > n > 1$



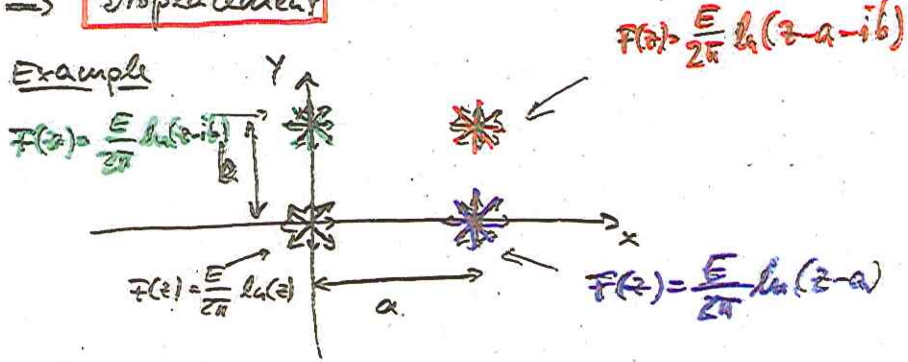
convex

$1 > n > \frac{1}{2}$

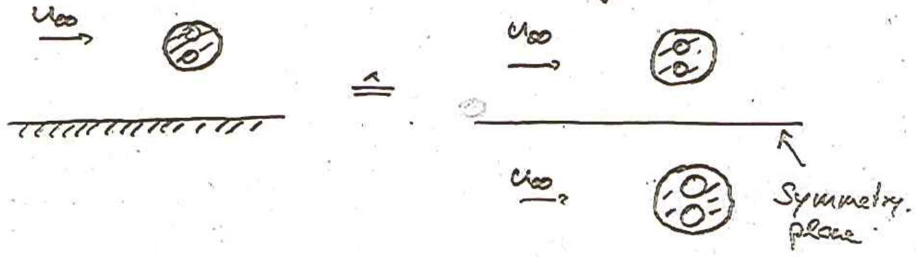
Singularities have their center in the origin of the coordinate system

$\longrightarrow$ **Displacement**

Example

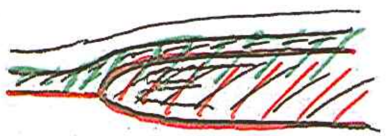


- Simulation of walls by mirroring



- Usually, the contours are presented by the stagnation streamlines
  - locate the stagnation point ($u = v = 0$) : x_S, y_S
  - determine ψ at the stagnation point
  - sketch the streamlines
    $\psi_k(x, y) = \psi_k(x_S, y_S) = \text{const.}$

- Streamlines do not intersect
  → each streamline can represent a body contour. In this case usually $u_\omega \neq 0$



- Bernoulli's equation is valid
  $p_0 = p_\infty + \frac{1}{2}\rho(u_\infty^2 + v_\infty^2) = p + \frac{1}{2}\rho(u^2 + v^2) = \text{const}$
  - Compute $c_p = \dfrac{p - p_{ref}}{\frac{\rho}{2} u_{ref}^2} = \dfrac{\frac{1}{2}\rho u_{ref}^2 - \frac{1}{2}\rho \vec{V}^2}{\frac{\rho}{2} u_{ref}^2} = 1 - \dfrac{\vec{V}^2}{u_{ref}^2}$

14.5 A planar flow is described by the stream function $\psi = (\frac{U}{L})xy$.
In $x_{ref} = 0, y_{ref} = 1\ m$ the pressure is $p_{ref} = 10^5\ N/m^2$.

$$U = 2\ m/s \qquad L = 1\ m \qquad \rho = 10^3\ kg/m^3$$

a) Proof, if the flow has a potential.

Determine

b) the stagnation points, the pressure coefficient, and the lines of constant total velocity
c) the velocity and the pressure at $x_1 = 2m, y_1 = 2m$,
d) the coordinates of a particle at $t = 0.5s$ if it passes at $t = 0$ the point x_1, y_1,
e) the pressure difference between these two points.
f) Sketch the streamlines.

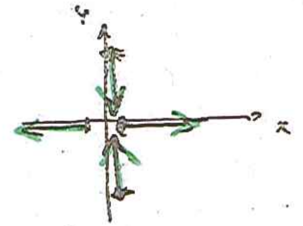
Sketch of the flow field

- stagnation points → stagnation streamline
- asymptotic streamlines for
  $x, y \to \infty$; $x, y \to 0$
- flow direction

Stagnation points : $\vec{v} = \vec{0}$: $u = v = 0$

$u = \frac{U}{L} x$, $v = -\frac{U}{L} y$ → $(x_s, y_s) = (0, 0)$

Additionally: $u = 0$ on the y-axis
$v = 0$ on the x-axis



Stream lines : $\psi = \text{const.}$

$$\psi = \frac{U}{L} xy = \text{const}$$

$$\rightarrow y = \frac{L}{U} \text{const} \cdot \frac{1}{x} = \frac{c}{x} \quad \text{for } x \neq 0$$

$$x = \frac{L}{U} \text{const} \frac{1}{y} = \frac{c}{y} \quad \text{for } y \neq 0$$

→ Hyperbola

Stagnation point stream line

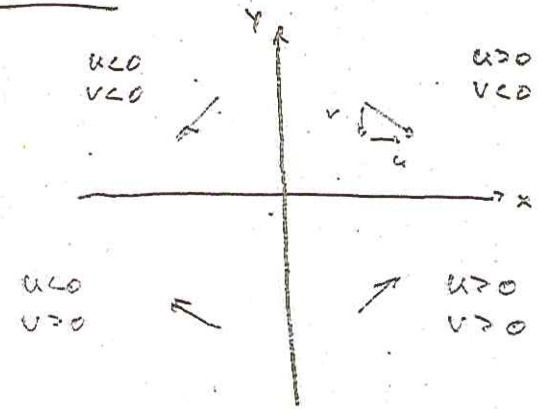
$$\psi_{sp} = \frac{U}{L} x_{sp} y_{sp} \stackrel{!}{=} 0 \quad \Big| \text{ depended on the problem}$$

$\psi = 0$ → $x = 0$ or $y = 0$

→ x-axis and y-axis are stagnation stream lines

flow direction

$u = \frac{U}{L} x$; $v = -\frac{U}{L} y$



a) given: Streamfunction $\psi = \frac{U}{L} xy$

ϕ exists, if $\underline{\vec{\omega} = 0}$

planar flow $\rightarrow$ 2-dimensional $\rightarrow$ $\omega_x = \omega_y = 0$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y} = \frac{U}{L} x \rightarrow \frac{\partial u}{\partial y} = 0 \\ v &= -\frac{\partial \psi}{\partial x} = -\frac{U}{L} y \rightarrow \frac{\partial v}{\partial x} = 0 \end{aligned}\right\} \omega_z = 0 \rightarrow \text{the flow is rotation free and the potential exists}$$

ϕ exists $\rightarrow$ Computation of ϕ

1) $u = \frac{\partial \phi}{\partial x} \rightarrow \phi = \int u\,dx + f_1(y) + C_1$

2) $v = \frac{\partial \phi}{\partial y} \rightarrow \phi = \int v\,dy + f_2(x) + C_2$

1.) $\phi(x,y) = \int \frac{U}{L} x\,dx + f_1(y) + C_1$

2) $\phi(x,y) = \int -\frac{U}{L} y\,dy + f_2(x) + C_2$

1.) $\phi(x,y) = \frac{U}{L}\frac{x^2}{2} + f_1(y) + C_1$

2) $\phi(x,y) = -\frac{U}{L}\frac{y^2}{2} + f_2(x) + C_2$

Comparison of 1) and 2)

$$\frac{U}{L}\frac{x^2}{2} + f_1(y) + C_1 = -\frac{U}{L}\frac{y^2}{2} + f_2(x) + C_2$$

$$f_1(y) = -\frac{U}{L}\frac{y^2}{2} \; ; \; f_2(x) = \frac{U}{L}\frac{x^2}{2} \; ; \; C_1 = C_2 = C$$

$$\rightarrow \boxed{\phi = \frac{U}{2L}(x^2 - y^2) + C}$$

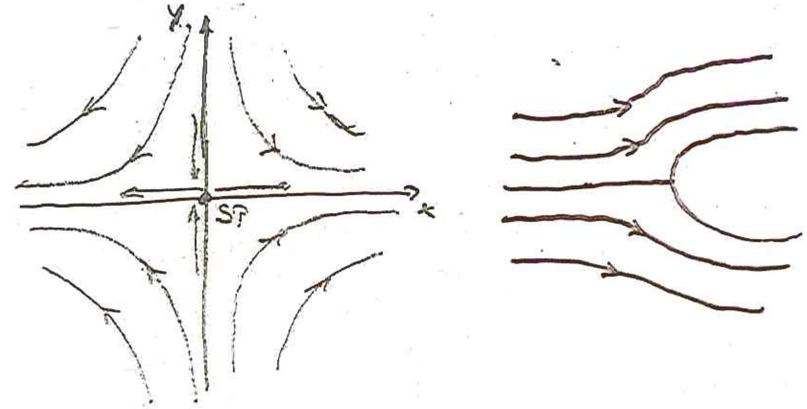
Complex potential $F(z)$

$$\begin{aligned} F(z) &= F(x+iy) = \phi(x,y) + i\psi(x,y) \\ &= \frac{U}{2L}(x^2 - y^2) + i\frac{U}{L}xy \\ &= \frac{U}{2L}(x^2 + 2ixy - y^2) \\ &= \frac{U}{2L} z^2 \end{aligned}$$

pressure coefficient

$$\boxed{c_p = \frac{p-p_{ref}}{\frac{\rho}{2}V_{ref}^2}} = 1 - \frac{\vec{V}^2}{\vec{V}_{ref}^2} = 1 - \frac{u^2+v^2}{u_{ref}^2+v_{ref}^2}$$

$$\left.\begin{aligned} u &= \frac{U}{L}x \\ v &= -\frac{U}{L}y \end{aligned}\right\} \quad c_p = 1 - \frac{x^2+y^2}{x_{ref}^2+y_{ref}^2}$$

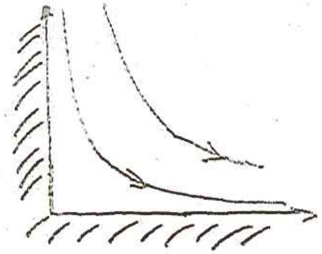


Lines of constant velocity

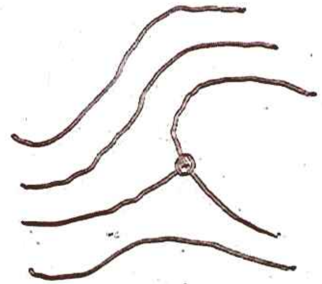
$$|\vec{V}| = \sqrt{u^2+v^2} = const$$

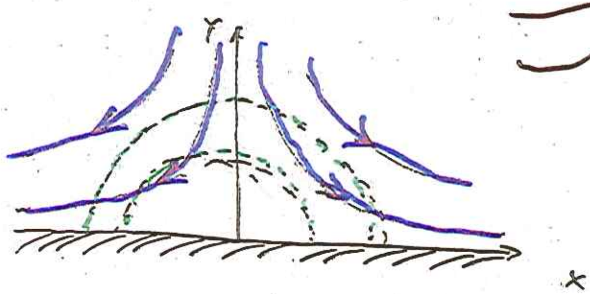
$$= \sqrt{\left(\frac{U}{L}x\right)^2 + \left(-\frac{U}{L}y\right)^2} = \left(\frac{L}{U}|\vec{V}|\right)^2 = x^2+y^2$$

circle with the radius $R = \frac{|\vec{V}|L}{U}$



90° - Corner flow





planar stagnation point flow

$z^{3/2}$

14.4 The complex stream function is given

$$F(z) = \frac{2}{3}\frac{u_\infty}{\sqrt{L}} z^{\frac{3}{2}} + \frac{E}{2\pi}\ln(z)$$

Given: L, u_∞

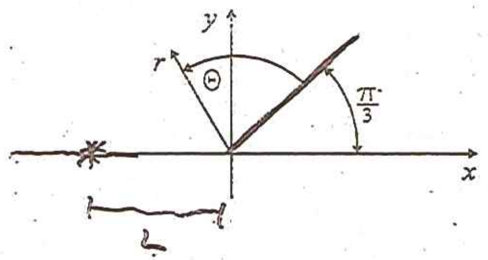
Determine

a) the potential $\phi(r,\theta)$ and the stream function $\psi(r,\theta)$.
b) the components of the velocity v_r, v_θ.
c) the constant E such that a stagnation point is at $(x=-L, y=0)$.
d) the equation that describes the contour $r_k(\theta)$.

Hints:

$$z = x' + iy' = re^{i\theta}$$

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$



$$e^{i\varphi} = \cos\varphi + i\cdot\sin\varphi$$

$3/2\,\theta$

1

a) $$F(z) = \frac{2}{3}\frac{u_\infty}{\sqrt{L}} z^{2/3} + \frac{E}{2\pi}\ln z$$

$$= \frac{2}{3}\frac{u_\infty}{\sqrt{L}} r^{3/2} e^{i\frac{3}{2}\theta} + \frac{E}{2\pi}\left(\ln r + \ln \underbrace{e^{i\theta}}_{i\theta}\right)$$

$$\phi(r,\theta) = \mathrm{Re}(F(z)) = \frac{2}{3}\frac{u_\infty}{\sqrt{L}} r^{3/2}\cos\left(\frac{3}{2}\theta\right) + \frac{E}{2\pi}\ln r$$

$$\psi(r,\theta) = \mathrm{Im}(F(z)) = \frac{2}{3}\frac{u_\infty}{\sqrt{L}} r^{3/2}\sin\left(\frac{3}{2}\theta\right) + \frac{E}{2\pi}\theta$$

b) $$v_r(r,\theta) = \frac{\partial\phi}{\partial r} = u_\infty\sqrt{\frac{r}{L}}\cos\left(\frac{3}{2}\theta\right) + \frac{E}{2\pi r}$$

$$v_\theta(r,\theta) = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -u_\infty\sqrt{\frac{r}{L}}\sin\left(\frac{3}{2}\theta\right)$$

c) stagnation point at $(x=-L, y=0) \rightarrow (r=L;\ \theta=\frac{2}{3}\pi)$

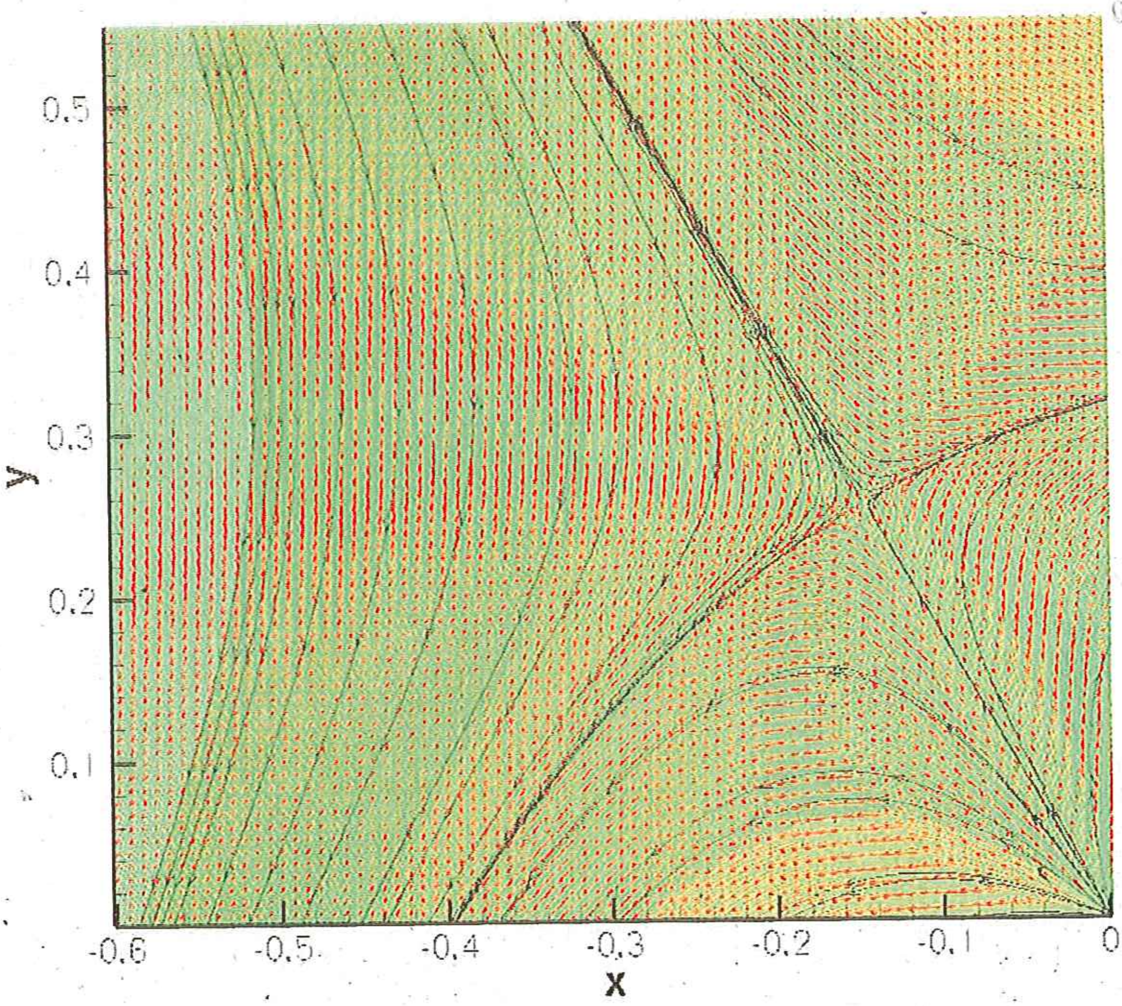
$$v_\theta = -u_\infty\sqrt{\frac{L}{L}}\underbrace{\sin\pi}_{=0} \overset{!}{=} 0$$

$$v_r = u_\infty\sqrt{\frac{L}{L}}\underbrace{\cos\pi}_{-1} + \frac{E}{2\pi L} \overset{!}{=} 0 \rightarrow \boxed{E = 2\pi u_\infty L}$$

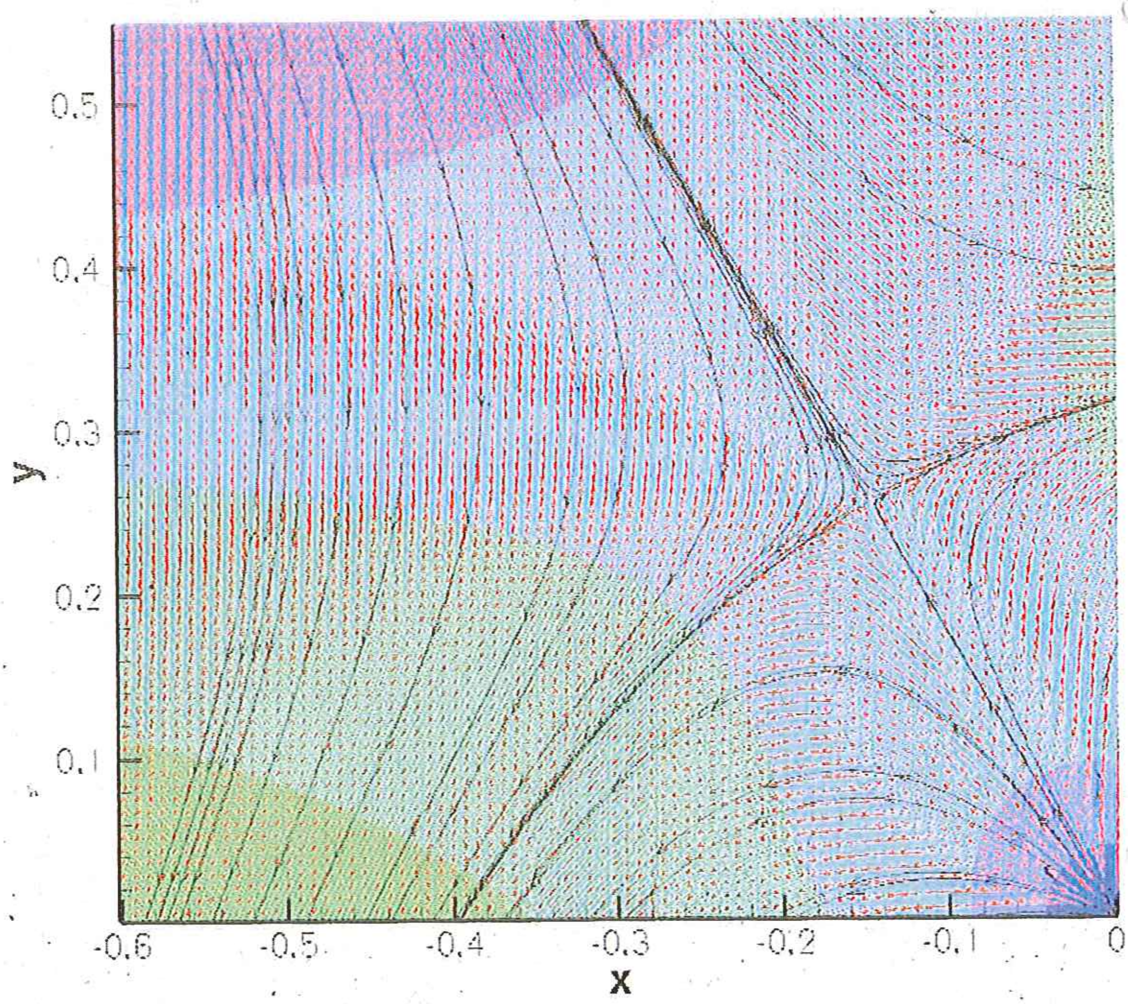
d) $$\psi_{sp} = \frac{E}{2\pi}\cdot\frac{2}{3}\pi = \frac{E}{3} = \boxed{\frac{2}{3}\pi u_\infty L = \psi_k}$$

$$\frac{2}{3}\pi u_\infty L = \frac{2}{3}\frac{u_\infty}{\sqrt{L}} r^{3/2}\sin\left(\frac{3}{2}\theta\right) + u_\infty L\theta$$

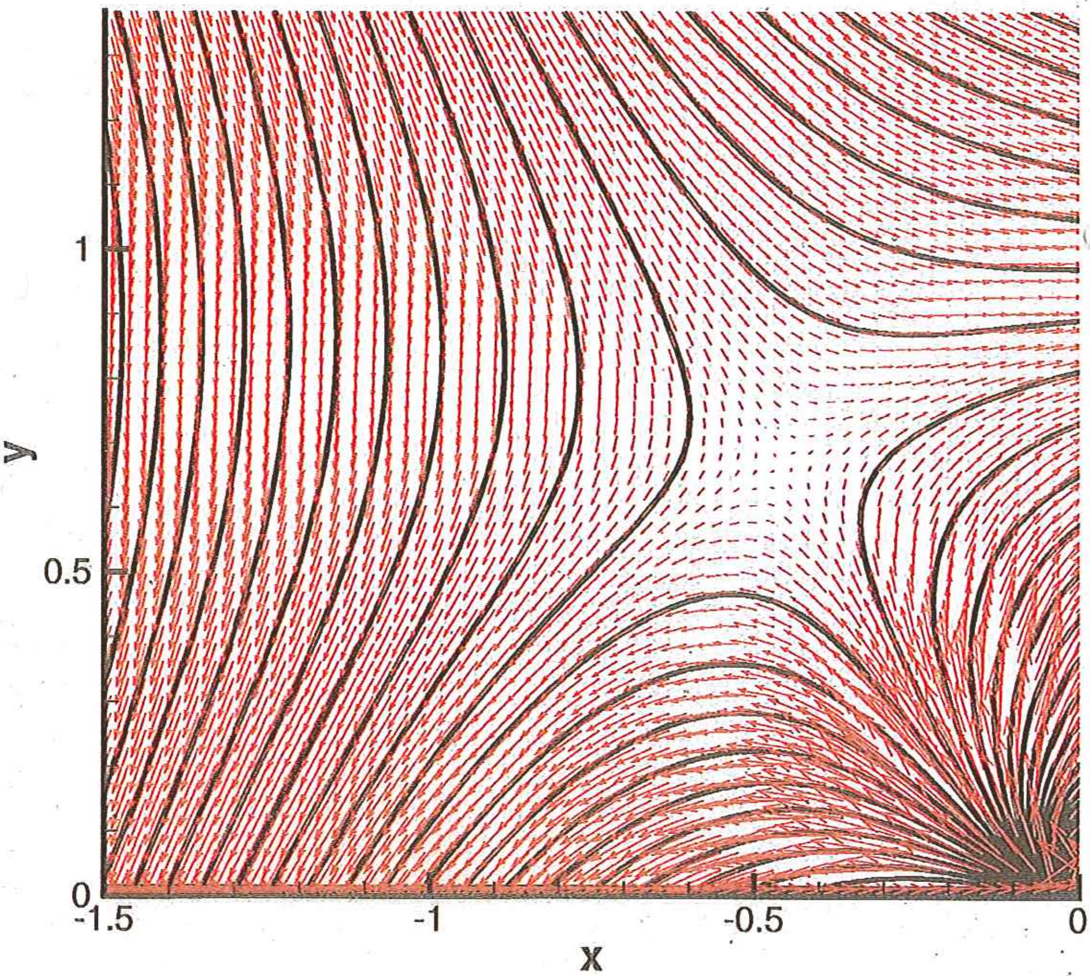
$$\rightarrow \boxed{r_k(\theta) = L\cdot\left(\frac{\pi - \frac{3}{2}\theta}{\sin\frac{3}{2}\theta}\right)^{2/3}}$$



0.5
0.4
0.3
0.2
0.1
y
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
x

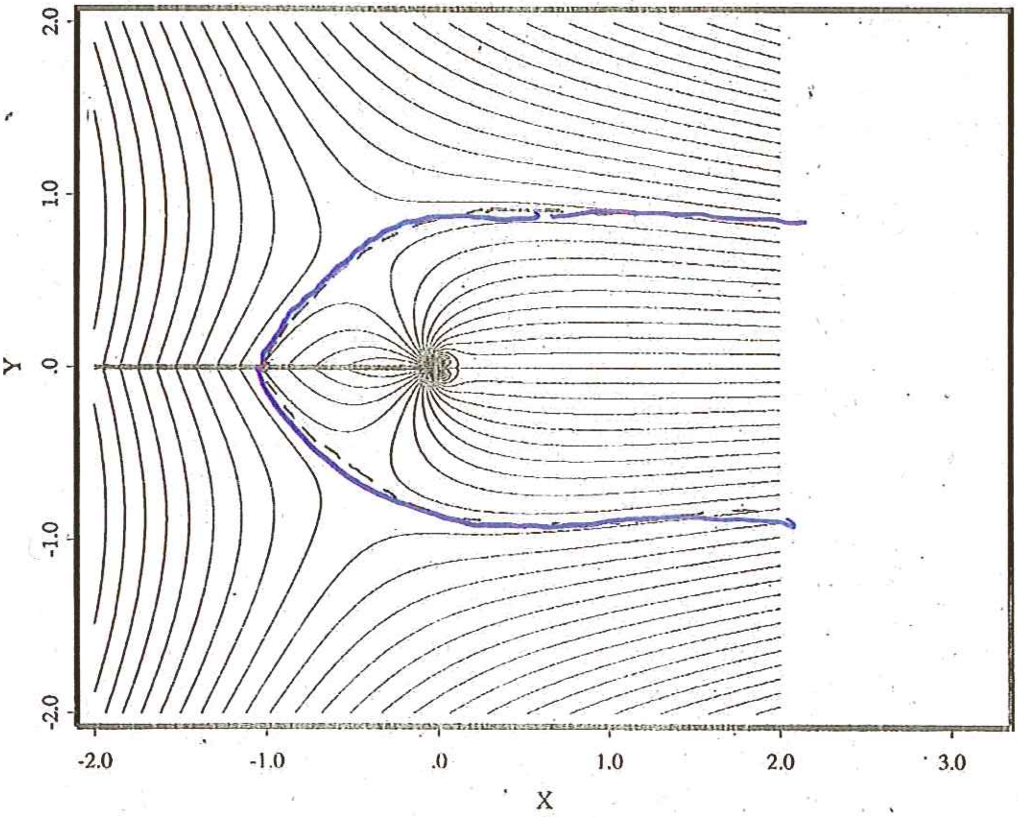


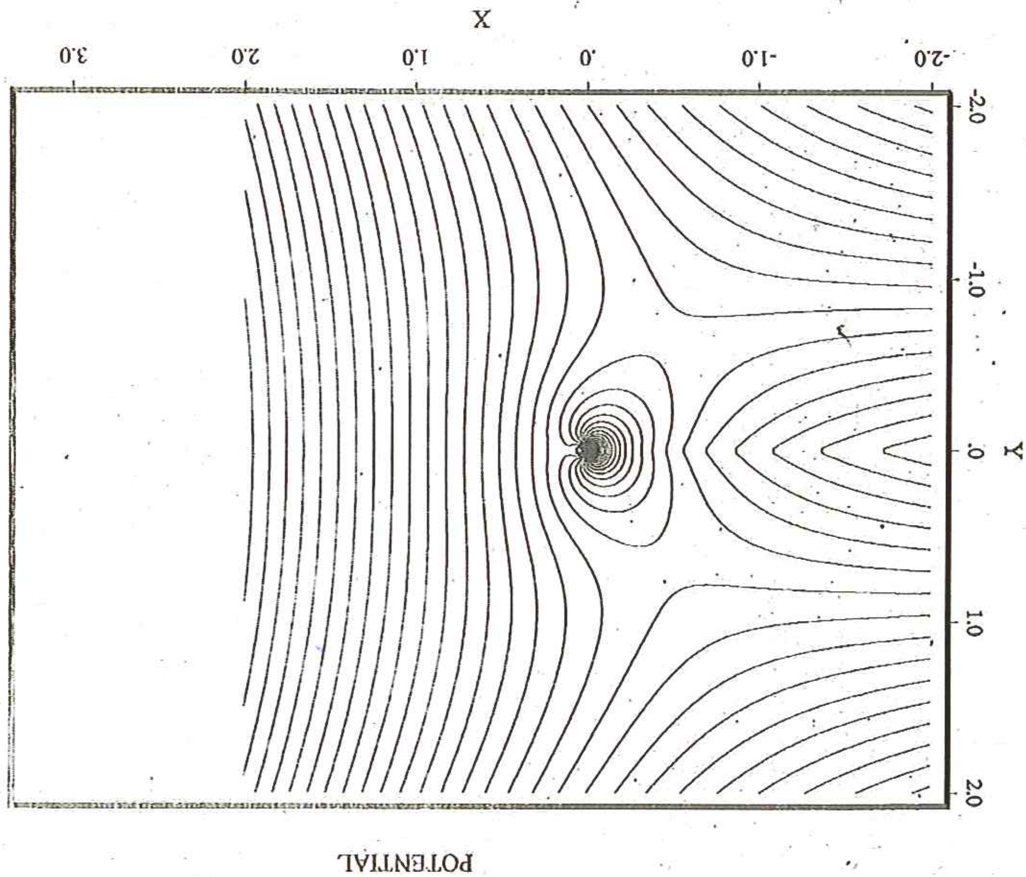
0.5
0.4
0.3
y
0.2
0.1
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0
x



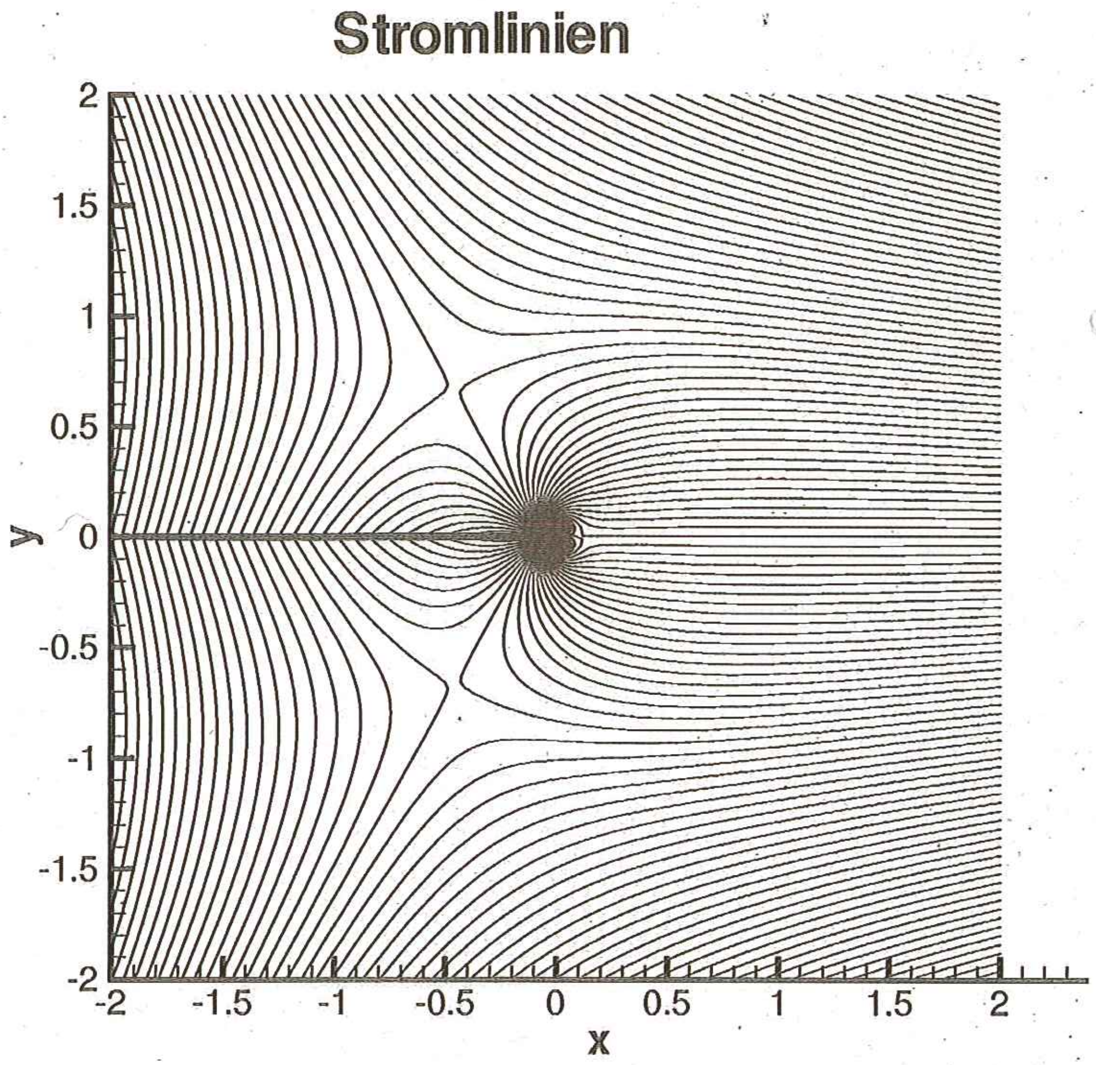
y
1
0.5
0
-1.5
-1
-0.5
0
x

STROMFUNKTION

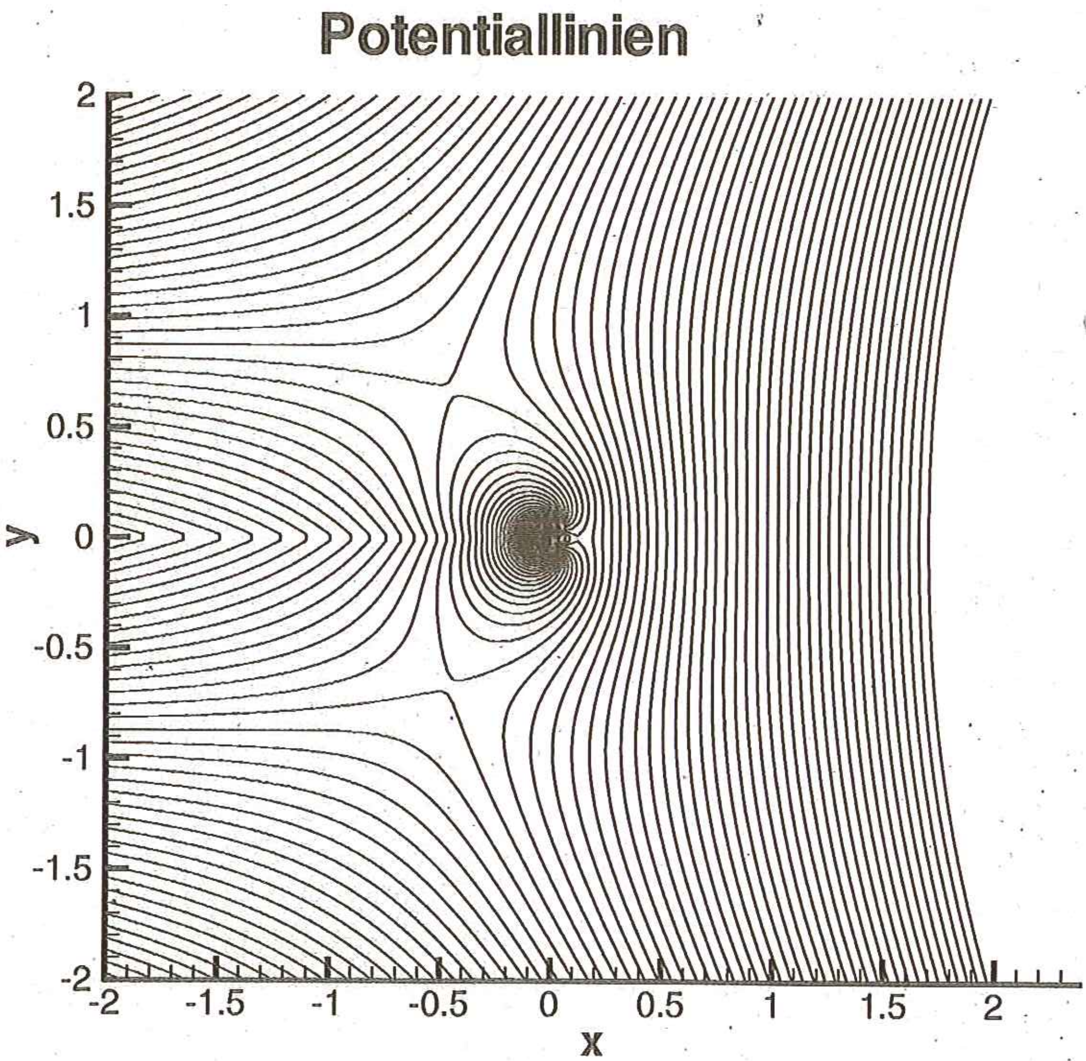




POTENTIAL

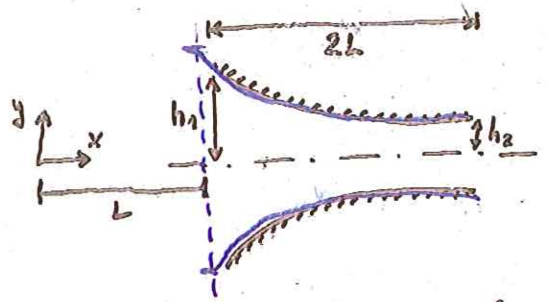


Stromlinien
y
x
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


Potentiallinien
2
1.5
1
0.5
y 0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

The stream function for a flow of an incompressible fluid through a planar nozzle is given: $\psi(x,y) = \frac{y}{h(x)} \cdot u_\infty \cdot L$

given: $u_\infty, L, B, h_1 = L, h_2 = \frac{1}{3}L$



a) Determine the shape $h(x)$ of the nozzle such that the flow can be described by the potential theory.

b) Determine the velocity distribution $u(x,y)$ and $v(x,y)$ and the volume flux.

---

a) prerequisite: $\omega \overset{!}{=} 0 \Rightarrow \nabla^2 \psi \overset{!}{=} 0$ with $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \overset{!}{=} 0 \qquad \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} \overset{!}{=} 0 = \frac{\partial}{\partial x}\left(y u_\infty L \frac{\partial}{\partial x}\left(\frac{1}{h(x)}\right)\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\left(-y u_\infty L \cdot \frac{h'(x)}{h^2(x)}\right) = 0$$

inner derivative

$$\Rightarrow \frac{d}{dx}\left(\frac{h'(x)}{h^2(x)}\right) = 0$$

$\Rightarrow$ integrate twice: $-\frac{1}{h(x)} = C_1 x + C_2$

B.C.: $x = L, h = L \Rightarrow -\frac{1}{L} = C_1 \cdot L + C_2$

$x = 3L, h = \frac{1}{3}L \Rightarrow -\frac{3}{L} = 3C_1 \cdot L + C_2$

$\Big\} \; C_2 = 0, \; C_1 = -\frac{1}{L^2}$

$$\Rightarrow \boxed{h(x) = \frac{L^2}{x}}$$

b) $u = \frac{\partial \psi}{\partial y}$; $v = -\frac{\partial \psi}{\partial x}$ with $\psi = \frac{u_\infty}{L} xy$

$$\Rightarrow \boxed{u = u_\infty \frac{x}{L}} \text{ and } \boxed{v = -u_\infty \, y/L}$$

$$\dot{Q} = \psi(y=h) - \psi(y=-h)\Big|_{x=L} \Rightarrow \dot{Q} = (h_1 + h_1) B \cdot u_\infty = \boxed{2 u_\infty L B}$$

Rotating cylinder: → Force in lateral direction

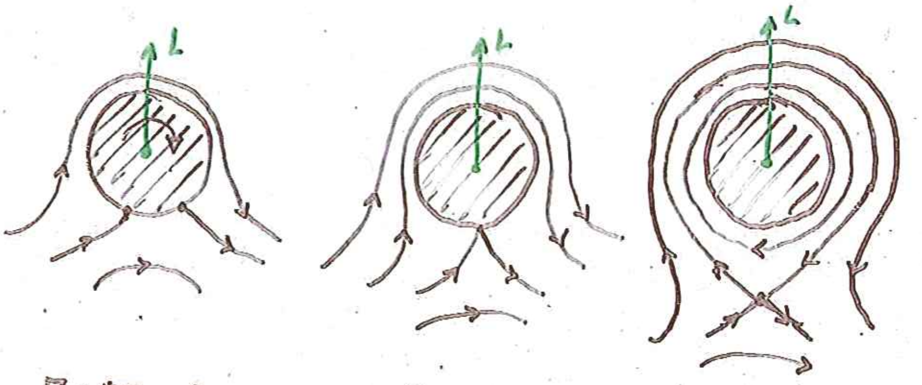
like the lift force on an airfoil

⇒ potential theory can be used to describe airfoil flows

cylinder: $F(z) = u_\infty z + \frac{M}{2\pi z}$

rotating cylinder: $F(z) = u_\infty z + \frac{M}{2\pi z} + i\frac{\Gamma}{2\pi}\ln z$

with $M = 2\pi R^2 u_\infty$



effect of Magnus

$\Gamma < 4\pi u_\infty R$ $\quad$ $\Gamma = 4\pi u_\infty R$ $\quad$ $\Gamma > 4\pi u_\infty R$

→ the stronger Γ, the faster the velocities on the upper surface → pressure drop

⇒ pressure difference ⇒ lift force : $L = \rho u_\infty \Gamma$

hint: $2\cdot\sin\varphi\cdot\cos\varphi = \sin 2\varphi$ and $-\sin 2\varphi = \sin 2(\pi-\varphi) = \sin 2\varphi'$

$$\Rightarrow c_{p_K} = \frac{\sin 2\varphi'}{\varphi'} - \left(\frac{\sin\varphi'}{\varphi'}\right)^2$$

d) . isobars: $c_p = \text{const}$

$$c_p = -\frac{h}{\pi}\cdot\frac{2x+h/\pi}{x^2+y^2} \Rightarrow x^2+y^2 = -2\frac{h}{\pi}\frac{x}{c_p} - \left(\frac{h}{\pi}\right)^2\cdot\frac{1}{c_p} + \left(\frac{h}{\pi}\right)^2\cdot\frac{1}{c_p^2} - \left(\frac{h}{\pi}\right)^2\cdot\frac{1}{c_p^2}$$

$$\Rightarrow x^2 + 2\frac{h}{\pi}\frac{x}{c_p} + \left(\frac{h}{\pi}\right)^2\cdot\frac{1}{c_p^2} + y^2 = -\frac{h^2}{\pi^2}\cdot\frac{1}{c_p} + \frac{h^2}{\pi^2}\cdot\frac{1}{c_p^2}$$

$$\Rightarrow \left(x+\frac{h}{\pi\cdot c_p}\right)^2 + y^2 = \underbrace{(1-c_p)\left(\frac{h}{\pi c_p}\right)^2}_{R^2}$$

$\Rightarrow$ circles around $\left(-\frac{h}{\pi c_p}, 0\right)$ with radius $\frac{h}{\pi c_p}\cdot\sqrt{1-c_p}$

Potential theory

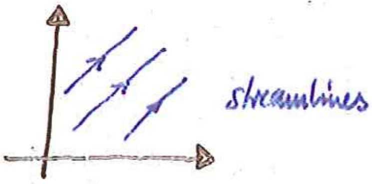
$$F(z) = \int \bar{w}(z)\,dz = \phi(x,y) + i\,\psi(x,y)$$

ϕ: potential function ψ: stream function

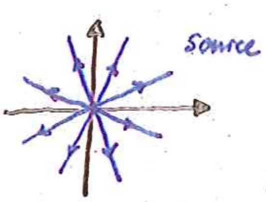
$$\rightarrow \bar{w} = \frac{dF}{dz} = u - iv$$

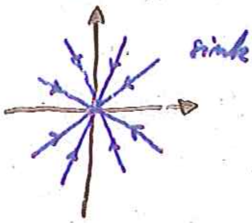
$$z = x + iy = r \cdot e^{i\varphi} = r(\cos\varphi + i \cdot \sin\varphi)$$

Parallel flow: $F(z) = (u_\infty - i v_\infty) \cdot z \longrightarrow \phi = u_\infty x + v_\infty y \;;\; \psi = u_\infty y - v_\infty x$



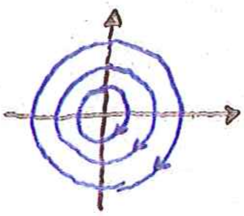
Source, sink: $F(z) = \frac{E}{2\pi} \ln z \longrightarrow \phi = \frac{E}{2\pi} \ln r \;;\; \psi = \frac{E}{2\pi} \varphi$





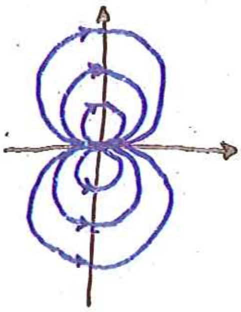
Vortex: $F(z) = \frac{\Gamma}{2\pi} i \cdot \ln z \longrightarrow \phi = -\frac{\Gamma}{2\pi} \arctan\frac{y}{x} \;;\; \psi = \frac{\Gamma}{2\pi} \ln\sqrt{x^2+y^2}$

$$\phi = -\frac{\Gamma}{2\pi} \varphi \qquad \psi = \frac{\Gamma}{2\pi} \ln r$$



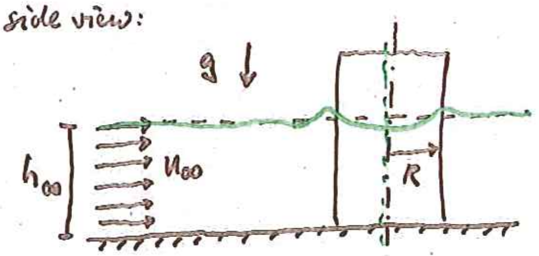
Dipole: $F(z) = \frac{M}{2\pi z} \longrightarrow \phi = \frac{M}{2\pi} \frac{x}{x^2+y^2} \;;\; \psi = \frac{-M}{2\pi} \cdot \frac{y}{x^2+y^2}$

$\rightarrow$ superposition of sink and source



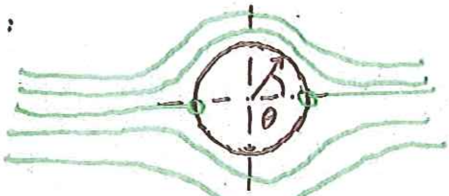
Water is flowing against a bridge pier with the velocity u_∞. The pier has a circular cross-section. Far away from the pier the water level is h_∞.

side view:



$u_\infty = 1\,m/s$
$h_\infty = 6\,m$
$R = 2\,m$
$\rho = 10^3\,kg/m^3$
$g = 10\,m/s^2$

top view:



a) Determine the water level at the pier as a function of θ.

b) Calculate the water level in the stagnation points.

c) Calculate the minimum water level.

---

circular cylinder: Dipole + parallel flow

$$\Rightarrow F(z) = u_\infty \cdot z + \frac{M}{2\pi z} \qquad \text{with } z = r(\cos\theta + i\cdot\sin\theta)$$

$$\Rightarrow \phi = \left(u_\infty + \frac{M}{2\pi r^2}\right) r\cdot\cos\theta$$

$$\psi = \left(u_\infty - \frac{M}{2\pi r^2}\right) r\cdot\sin\theta$$

$$\boxed{\text{with } \frac{1}{z} = \frac{1}{r(\cos\theta + i\cdot\sin\theta)}\cdot\frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta - i\cdot\sin\theta)} = \frac{r(\cos\theta - i\cdot\sin\theta)}{r^2}}$$

$$\Rightarrow v_r = \frac{\partial\phi}{\partial r} = \left(u_\infty - \frac{M}{2\pi r^2}\right)\cdot\cos\theta$$

$$\Rightarrow v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\left(u_\infty + \frac{M}{2\pi r^2}\right)\sin\theta$$

symmetric flow: stagnation points at $\theta = 0$ and $\theta = \pi$ and $r = R$.

$$\Rightarrow v_r = v_\theta \overset{!}{=} 0 \Rightarrow u_\infty - \frac{M}{2\pi R^2} = 0 \Rightarrow \boxed{M = u_\infty\cdot 2\pi\cdot R^2}$$

$$\Rightarrow v_\theta = -\left(u_\infty + \frac{R^2 u_\infty}{r^2}\right)\cdot\sin\theta$$

On the surface of the cylinder: $v_r = 0$ and $v_\theta = -2u_\infty\cdot\sin\theta$

Bernoulli: $$p_a + \rho g h_\infty + \frac{\rho}{2}u_\infty^2 = p_a + \rho g h(\theta) + \frac{\rho}{2}\vec{v}^2$$

$$\rightarrow \vec{v}^2 = v_\theta^2\big|_{surf} = 4u_\infty^2\cdot\sin^2\theta$$

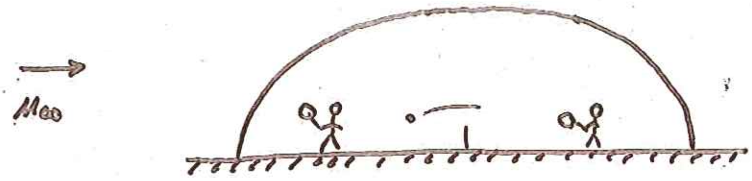
$$\Rightarrow h(\theta) = h_\infty + \frac{u_\infty^2}{2g}(1 - 4\sin^2\theta)$$

$h(0,\pi) =$
$= h_\infty + \frac{u_\infty^2}{2g} = 6.05\,m$

$\theta_{min} = \frac{\pi}{2}, \frac{3\pi}{2}$

$h(\frac{\pi}{2}, \frac{3\pi}{2}) =$
$= h_\infty - \frac{3u_\infty^2}{2g} = 5.85\,m$

The flow around a tennis sports hall should be described by potential theory.



a) What potential function should be used to describe the problem?

b) Determine the velocities $u(x,y)$ and $v(x,y)$ using the complex conjugate velocity $\bar{w}$.

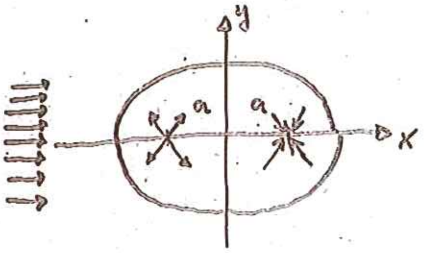
c) Determine the stagnation points and the contour streamline.

d) Sketch the streamlines.

e) What 2 simplifications are used in potential theory according to real flows?

---

a) Superposition of parallel flow, source and sink:



$$F(z) = u_\infty z + \frac{E}{2\pi}\ln(z+a) - \frac{E}{2\pi}\ln(z-a)$$

b) $$\bar{w} = \frac{dF}{dz} = u - iv = u_\infty + \frac{E}{2\pi}\left(\frac{1}{z+a} - \frac{1}{z-a}\right)$$

$$= u_\infty + \frac{E}{2\pi}\left(\frac{x+a-iy}{(x+a)^2+y^2} - \frac{x-a-iy}{(x-a)^2+y^2}\right)$$

$$\Rightarrow \boxed{u = u_\infty + \frac{E}{2\pi}\left(\frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}\right)}$$

$$\boxed{v = \frac{E}{2\pi}\left(\frac{y}{(x+a)^2+y^2} - \frac{y}{(x-a)^2+y^2}\right)}$$

c) stagnation point is part of the contour stream line

s.p. $u = v = 0 \longrightarrow$ symmetry: $\boxed{y_S = 0} \rightarrow v(x, y_S) = 0$

$$u = 0 = u_\infty + \frac{E}{2\pi}\left(\frac{x_S+a}{(x_S+a)^2+\cancel{y_S^2}_{\,0}} - \frac{x_S-a}{(x_S-a)^2+\cancel{y_S^2}_{\,0}}\right) = u_\infty + \frac{E}{2\pi}\left(\frac{1}{x_S+a} - \frac{1}{x_S-a}\right)$$

$$= u_\infty - \frac{E}{2\pi}\left(\frac{2\cdot a}{x_S^2-a^2}\right) \qquad \Rightarrow \boxed{x_S = \pm\sqrt{\frac{aE}{\pi u_\infty} + a^2}}$$

contour stream line: $\psi(x,y) = \psi_k = \mathrm{Im}(F(z))$

$$F(z) = \phi + i\psi$$

$$= u_\infty (x+iy) + \frac{E}{2\pi}\left[\ln r_1 e^{i\varphi_1} - \ln r_2 e^{i\varphi_2}\right]$$

with: $r_1 = \sqrt{(x+a)^2+y^2}$; $r_2 = \sqrt{(x-a)^2+y^2}$

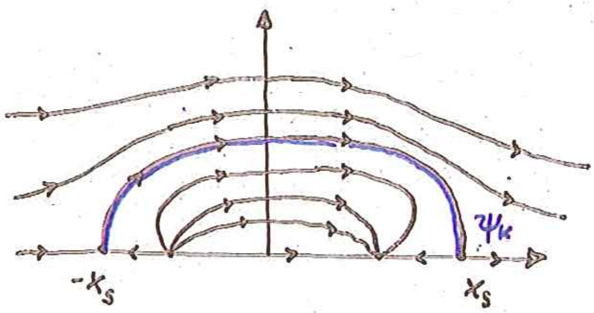
$\varphi_1 = \arctan(\frac{y}{x+a})$; $\varphi_2 = \arctan(y/_{x-a})$

$$\psi(x,y) = u_\infty y + \frac{E}{2\pi}\cdot\left(\arctan\frac{y}{x+a} - \arctan\frac{y}{x-a}\right)$$

$\psi_K = \psi_{sp} \Rightarrow \psi_{sp}(x_s, y_s) = 0$ since $y_s = 0$

$$\Rightarrow 0 = u_\infty y + \frac{E}{2\pi}\cdot\left(\arctan\frac{y}{x+a} - \arctan y/_{x-a}\right)$$

d) sketch:



e) simplifications:
- contour is equal to streamline → no-slip condition not fullfilled
- rotation free flow is assumed