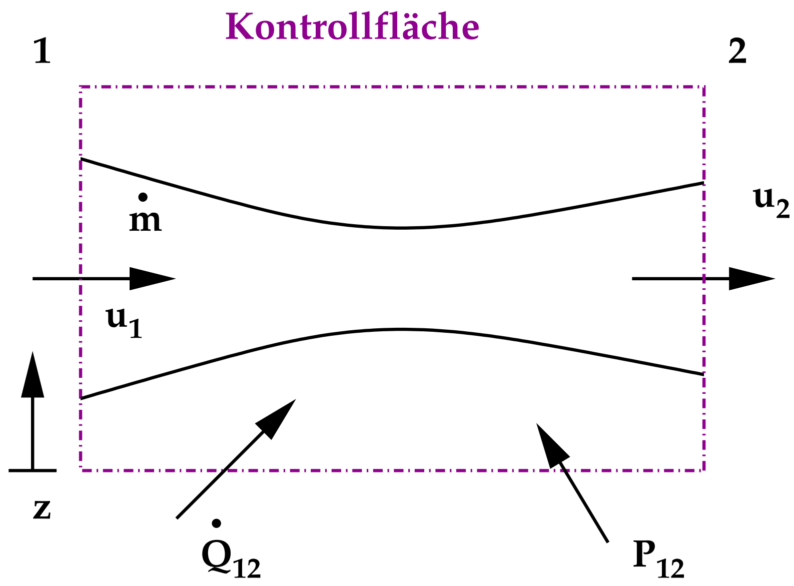
Gasdynamics

1-D compressible, inviscid, stationary, adiabatic flows

$$\boxed{\rho \neq const}$$



1st law of thermodynamics

$$\dot{Q}_{12} + P_{12} = \dot{m}\left[\underbrace{h_2 - h_1}_{\text{Enthalpy}} + \underbrace{\frac{1}{2}(u_2^2 - u_1^2)}_{\text{kin. Energy}} + \underbrace{g(z_2 - z_1)}_{\text{pot. Energy}}\right]$$

Gasdynamics

for $\dot{Q}_{12} + P_{12} = 0$ (i.e.: adiabatic nozzle flow)

potential energy is neglected

$$h_2 - h_1 + \frac{1}{2}(u_2^2 - u_1^2) = 0$$

with $h_0 = h + u^2/2$ (Index 0 for stagnation condition)

$$h_{02} - h_{01} = 0 \; ; \; h_0 = \text{const}$$

for perfect gases $\gamma =$ const. $\gamma (= \kappa) = \frac{c_p}{c_v} (= 1.4,$ for air$)$

c_p: specific heat capacity at constant pressure
c_v: spezific heat capacity at constant volume

$$\rightarrow dh = c_p \, dT$$

Gasdynamics

Energy equation for 1-d adiabatic flow without mechanical energy:

$$c_p T_0 = c_p T + \frac{1}{2} u^2$$

in dimensionless form

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} \frac{u^2}{\gamma R T} \qquad ; c_p = \frac{\gamma}{\gamma - 1} R$$

Speed of sound: $a = \sqrt{\gamma R T}$

Machnaumber: Ma $= u/a$

$$\frac{T_0}{T} = 1 + \frac{1}{2}(\gamma - 1)\,\mathrm{Ma}^2$$

for ideal gases with γ = konst, $\dot{Q} = P = 0$

Example

Determine for an isentrropic flowfield ($\gamma = 1.4$)

a) the critical temperature ratio

b) the critical pressure ratio

c) the limit if the critical Mach number Ma$^\star = u/a^\star$ for Ma $\Rightarrow \infty$!

1. stat. flowfield with $\rho \neq$ const
   perfect gas , $\dot{Q} = P = 0; \ \gamma =$ const.

$$\frac{T_0}{T} = 1 + \frac{1}{2}(\gamma - 1)\frac{u^2}{\gamma RT} = 1 + \frac{1}{2}(\gamma - 1)\text{Ma}^2$$

   isentropic flow: $s =$const.

$$\frac{p}{p_0} = \left(\frac{T}{T_0}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Example

$\Rightarrow$ p and ρ only depend on the initial stagnation state and the Mach number

$$T = T(T_0, \mathrm{Ma}) \quad p = p(p_0, \mathrm{Ma})$$

$$\rho = \rho(\rho_0, \mathrm{Ma}) \quad \rho_0 = \rho_0(p_0, T_0)$$

Definition ”critical value”

$$T^\star = T(\mathrm{Ma} = 1)$$

$$p^\star = p(\mathrm{Ma} = 1)$$

$$\rho^\star = \rho(\mathrm{Ma} = 1)$$

Example

$$\Longrightarrow \frac{T_0}{T^\star} = 1 + \frac{1}{2}(\gamma - 1)c^2 = \frac{\gamma + 1}{2}$$

isentropic: $\rightarrow \dfrac{P_0}{p^\star} = \left(\dfrac{\rho_0}{\rho^\star}\right)^\gamma = \left(\dfrac{\gamma + 1}{2}\right)^{\frac{\gamma}{\gamma - 1}}$

$T^\star, p^\star, \rho^\star$ only depend on T_0, p_0, γ

Definition "critical Mach number"

$$\text{Ma}^\star = \frac{u}{c^\star} = \frac{u}{\sqrt{\gamma R T^\star}} = \frac{u}{c_0}\sqrt{\frac{T}{T^\star}} = \frac{u}{c_0}\sqrt{\frac{\gamma + 1}{2}}$$

Advantage: Only one value in the quotien is variable
$\Rightarrow$ Ma$^\star \sim u$

Example

There is a limit Ma* for Ma $\rightarrow \infty$

From the energy equation:

$$\text{Ma}^2 = \frac{2}{\gamma - 1}(\frac{T_0}{T} - 1) \; ; \; c_p T_0 = c_p T + \frac{1}{2}u^2$$

Expansion into Vacuum: $\Rightarrow u \rightarrow u_{\text{max}} \; ; \; T \rightarrow 0$

Ma $\rightarrow \infty$, while u is finite.

Example

Relation between $\mathrm{Ma}^\star$ and Ma

$$\mathrm{Ma}^2 = \frac{u^2}{c^2} = \frac{u^2}{{c^\star}^2}\,\frac{{c^\star}^2}{c_0^2}\,\frac{c_0^2}{c^2} =$$

$$= {\mathrm{Ma}^\star}^2 \frac{T^\star}{T}\frac{T_0}{T} =$$

$$= {\mathrm{Ma}^\star}^2 \frac{T^\star}{T}\left(1 + \frac{1}{2}(\gamma - 1)\mathrm{Ma}^2\right)$$

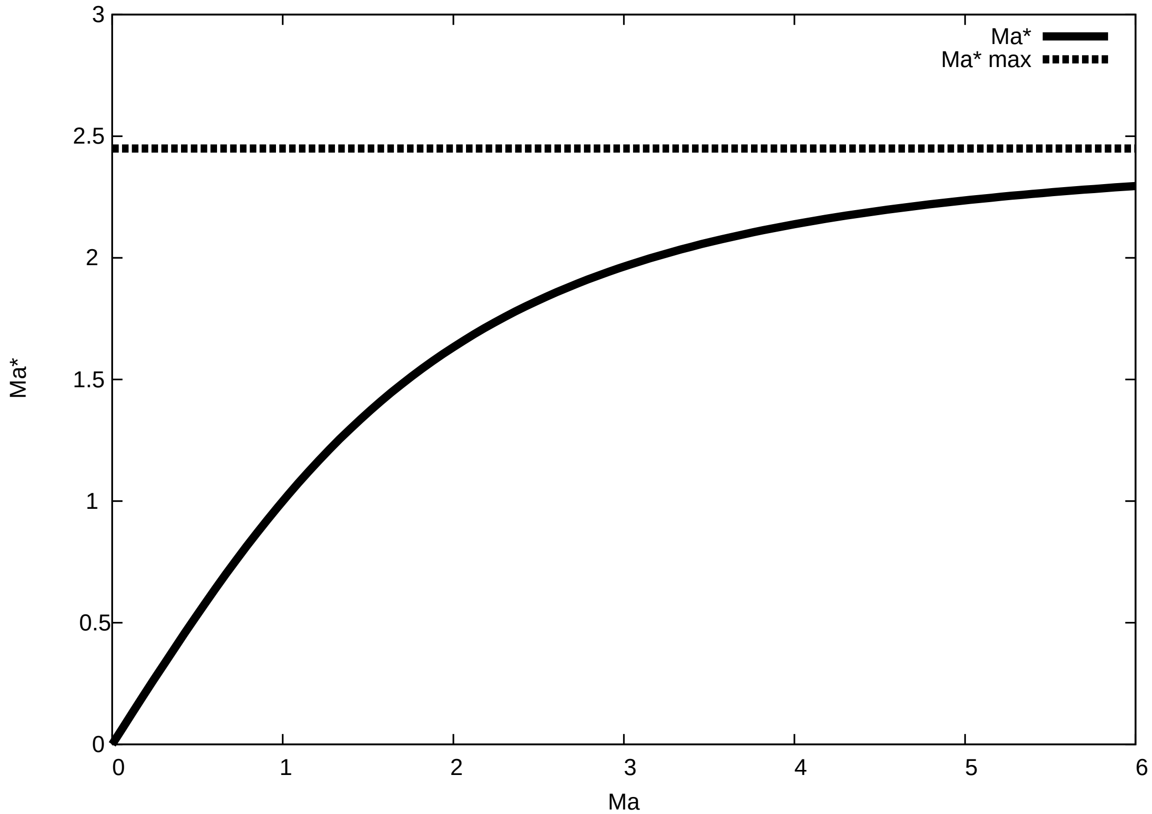
$$= {\mathrm{Ma}^\star}^2 \frac{2}{\gamma + 1}\left(1 + \frac{1}{2}(\gamma - 1)\mathrm{Ma}^2\right)$$

$$\Longrightarrow \mathsf{Ma}^{\star 2} = \frac{(\gamma+1)\mathsf{Ma}^2}{2+(\gamma-1)\mathsf{Ma}^2} = f(\mathsf{Ma},\gamma)$$

Example

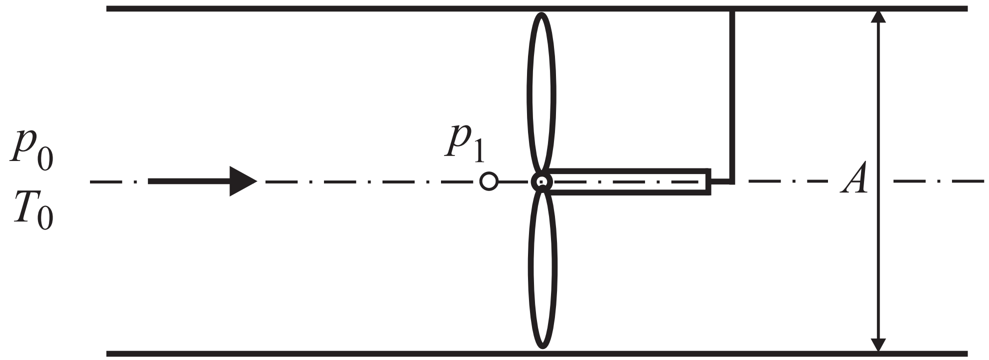
$$\Rightarrow \frac{T}{T_0}(\text{Ma}^\star),\ \frac{p}{p_0}(\text{Ma}^\star),\ \frac{\rho}{\rho_0}(\text{Ma}^\star)$$

Expansion into Vacuum: $T \rightarrow 0$; $\text{Ma} \rightarrow \infty$; $\text{Ma}^\star \rightarrow \sqrt{\frac{\gamma+1}{\gamma-1}}$



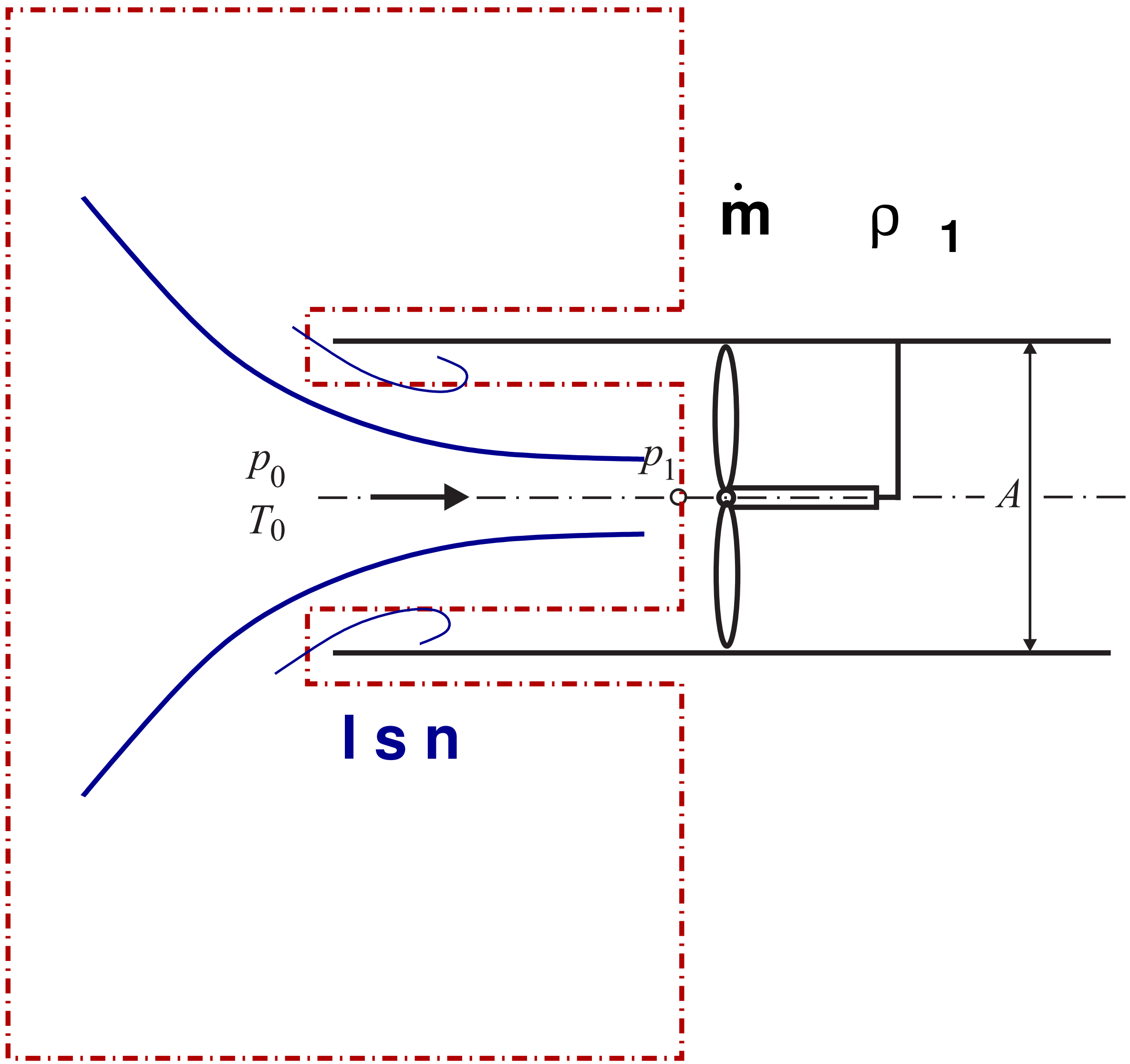
19.5

A turbine engine sucks air from the atmosphere. Immediately before the compressor the pressure is p_1.



$p_0 = 10^5\ N/m^2 \quad T_0 = 287\ K \quad p_1 = 0.74 \cdot 10^5\ N/m^2 \quad A = 9 \cdot 10^3\ mm^2$
$R = 287\ Nm/(kgK) \quad \gamma = 1.4$

Compute the mass flux flowing through the engine!



ṁ
ρ 1
p0
T0
p1
A
l s n

Separation $\rightarrow$ Losses $\rightarrow$ p_0 not constant

$\dot{Q} + P_t = 0 \Rightarrow h_0 = konst.$

separation $\rightarrow$ no isentropic flow

$\Longrightarrow$ irreversible change from kinetic to inner energy

Remark: For isentropic flow momentum and energy equation are equivalent

1-D momentum equation

$$\rho v \frac{dv}{dx} = -\frac{dp}{dx} \Rightarrow v\,dv = -\frac{1}{\rho}\,dp$$

Definition of entropy: $\boxed{T\,ds = dh - \frac{dp}{\rho}}$ (2nd law of thermodynamics)

19.5

Isentropic flow: $dh = \frac{dp}{\rho}$

$$\Rightarrow \; d\left(\frac{v^2}{2}\right) \; = \; -dh \Rightarrow d\left(h + \frac{v^2}{2}\right) = 0$$

$$\Longrightarrow \; h + \frac{v^2}{2} \; = \; h_0 \; = \; konst$$

For isentropic flows the momentum equation gives no new information against the energy equation

Momentum in x-direction

$$\frac{dI_x}{dt} = \rho_1 v_1^2 A - 0 = (p_0 - p_1)A$$

$$\rho_1 v_1^2 = p_0 - p_1 \qquad \text{momentum}$$

$$c_p T_0 = c_p T_1 + \tfrac{1}{2} v_1^2 \quad \text{energy}$$

unknown: ρ_1, v_1

express the unknows using the known values (p_1, p_0)

$$\rho_1 = \frac{p_1}{RT_1} \qquad v_1 = \mathrm{Ma}_1 \sqrt{\gamma R T_1}$$

new unknown: T_1, Ma_1

19.5

momentum: $$p_0 - p_1 = \rho_1 v_1^2 = \frac{\rho_1 v_1^2}{p_1} p_1 = \frac{v_1^2}{\gamma \frac{p_1}{\rho_1}} \gamma p_1 =$$

$$= \gamma\, p_1\, \mathrm{Ma}_1^2 \quad \Longrightarrow \quad \mathrm{Ma}_1^2 = \frac{1}{\gamma}\left(\frac{p_0}{p_1} - 1\right)$$

energy: $$c_p T_0 = c_p T_1 + \frac{1}{2} v_1^2\,; \qquad c_p = \frac{\gamma}{\gamma - 1} R$$

$$\gamma R T_0 = \gamma R T_1 + \frac{\gamma - 1}{2} v_1^2 \quad \Longrightarrow \quad \frac{T_0}{T_1} = 1 + \frac{1}{2}(\gamma - 1)\mathrm{Ma}_1^2$$

$$\Longrightarrow T_1 = \frac{T_0}{1 + \frac{1}{2}(\gamma - 1)\mathrm{Ma}_1^2}$$

$$\dot{m} = \rho_1 v_1 A = \frac{p_1}{RT_0} \sqrt{\gamma R T_0} \sqrt{\frac{1}{\gamma}\left(\frac{p_0}{p_1} - 1\right)} \sqrt{1 + \frac{\gamma - 1}{2}\left(\frac{p_0}{p_1} - 1\right)} A$$

19.5

$$\dot{m} = f(T_0, p_0, \rho_0) = 1.41 \frac{\text{kg}}{\text{s}}$$

Machnumber in the pipe $\text{Ma}_1^2 = \frac{1}{\gamma}\left(\frac{p_0}{p_1} - 1\right) = 0.5$. (In a lossfree flow the pressure ratio would be the value of a pipe Machnumber of Ma = 0.67.)

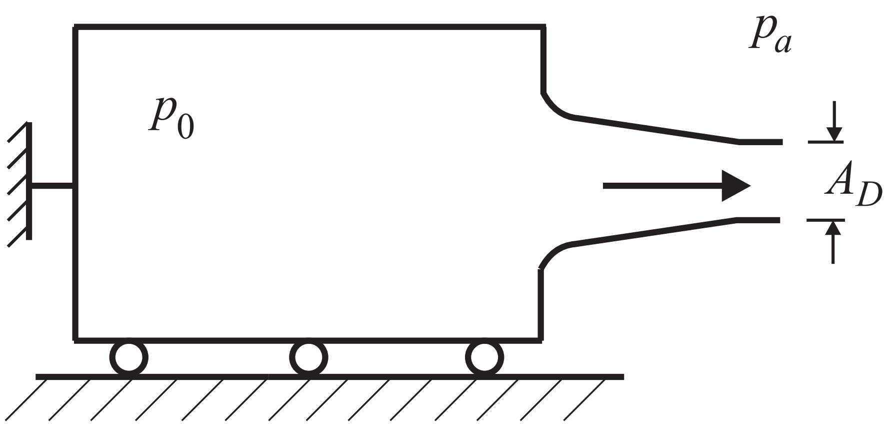
compared to problem 7.8):

incompressible flow mit $\rho = \rho_0 = \dfrac{p_0}{RT_0}$

$$\dot{m} = \rho v A = \sqrt{\rho \Delta p} A = 1.6 \frac{\text{kg}}{\text{s}}$$

19.3

Air is flowing isentropically ($\gamma = 1.4$) from a large and frictionless supported container through a well rounded nozzle into the open air.

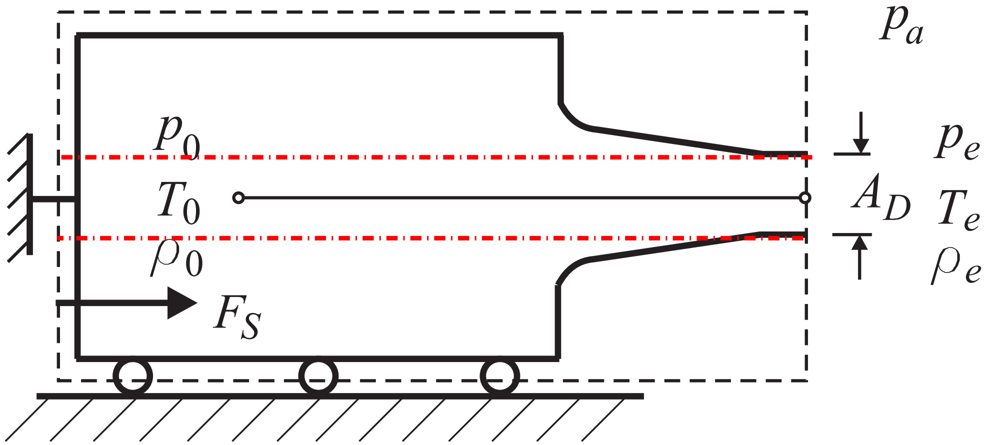


a) Determine the dimensionless thrust $F_s/p_0 A_D$ for the pressure ratios $p_a/p_0 = 1;\ 0.6;\ 0.2;\ 0$!
b) What are these values for an incompressible fluid?

a) Outflow from a large container (perfect gas)

Forces from momentum equation

$$\frac{d\vec{I}}{dt} = \int\limits_{\tau} \frac{\partial}{\partial t}(\rho\vec{v})\, d\tau + \int\limits_{A} \rho\vec{v}(\vec{v}\cdot\vec{n})\, dA = \vec{F}_p + \underbrace{\vec{F}_R}_{=0} + \vec{F}_s + \underbrace{\vec{F}_{vol}}_{=0}$$



$$\frac{dI_x}{dt} = \rho_e v_e^2 A_D = F_s + (p_a - p_e)A_D$$

19.3

$$\frac{F_s}{p_0 A_D} = \frac{\rho_e v_e^2}{p_0} + \frac{p_e - p_a}{p_0}$$

what is: ρ_e, v_e, p_e

expand with γp_e

$$\frac{F_s}{p_0 A_D} = \frac{\gamma \rho_e v_e^2}{\gamma p_e} \frac{p_e}{p_0} + \frac{p_e}{p_0} - \frac{p_a}{p_0} =$$

$$= \frac{p_e}{p_0} \gamma \, \mathrm{Ma}_e^2 + \frac{p_e}{p_0} - \frac{p_a}{p_0}$$

Energie equation: $c_p T_0 = c_p T + \frac{1}{2} v^2 \Rightarrow \frac{T_0}{T} = 1 + \frac{1}{2}(\gamma - 1)\mathrm{Ma}^2$

$$\Longrightarrow \mathsf{Ma}^2 = \frac{2}{\gamma - 1}\left(\frac{T_0}{T} - 1\right) = \frac{2}{\gamma - 1}\left[\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

19.3

$$\Longrightarrow \quad \frac{F_s}{p_0 A_D} = \frac{p_e}{p_0}\frac{2\gamma}{\gamma-1}\left[\left(\frac{p_0}{p_e}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] + \left(\frac{p_e}{p_0} - \frac{p_a}{p_0}\right)$$

Pressure distribution $\frac{p_e}{p_0}$ =??

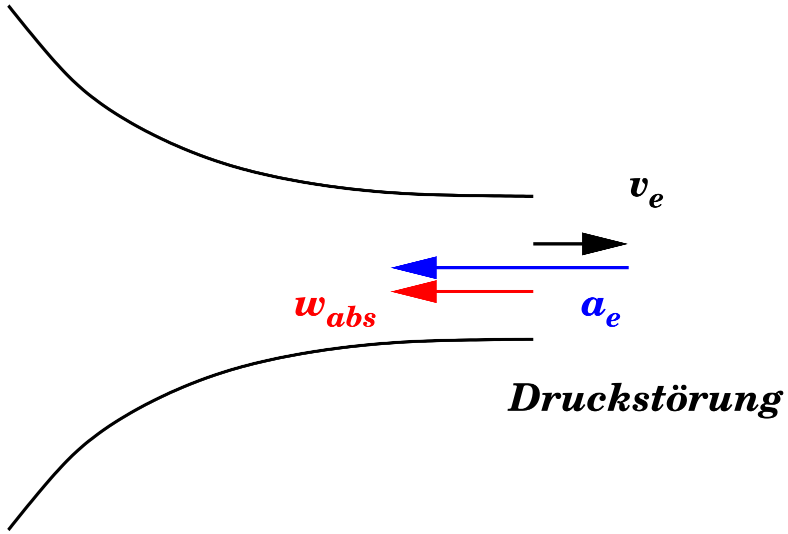
2 possibilities:

$v_e < a_e$ subsonic flow

$v_e = a_e$ sonic flow

$v_e > a_e$ supersonic is impossible, since the exit cross section is the smallest cross section.

1) $v_e \; < \; a_e \;\; \Longrightarrow$ The influence of the ambient pressure porpagates upstream($w_{\text{abs}} < 0$)

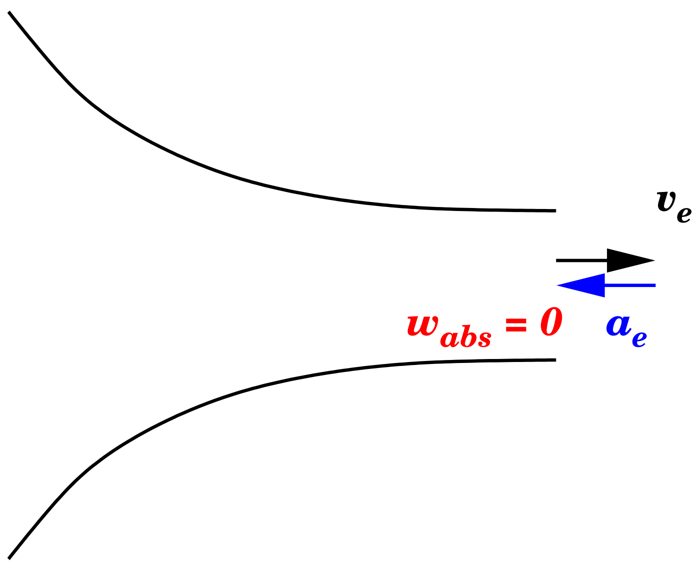


$\Rightarrow$ ambient pressure is impressed

für $p_a \; > \; p^\star \; : \; p_e \; = \; p_a$

$$\mathsf{Ma}_e^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_0}{p_a}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

2) $v_e = a_e \implies$ The flow in the nozzle is not influenced by the downstream boundary condition ($w_{\text{abs}} = 0$)
the pressure decrease in the environent is not felt in the nozzle.



$\Rightarrow$ ambient pressure is not felt

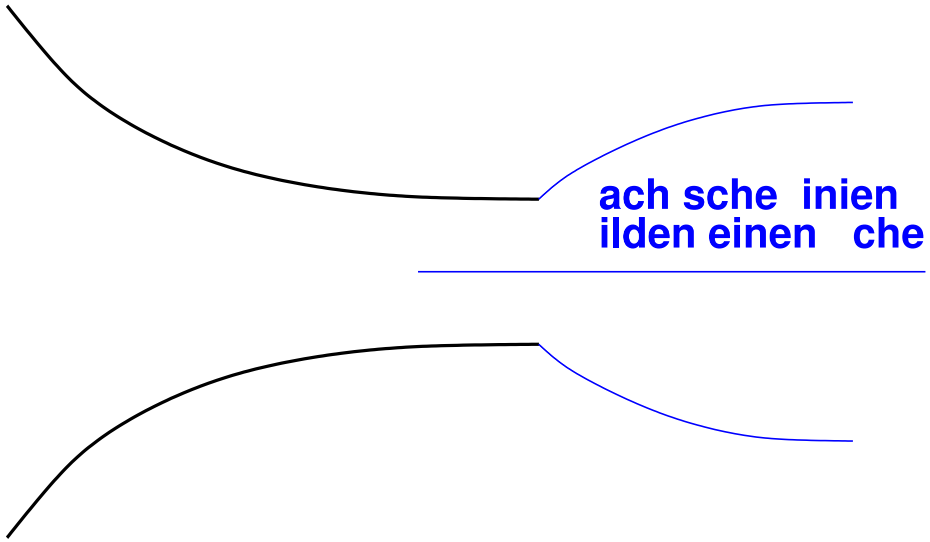
for $p_a \leq p^\star \; : \; p_e = p^\star$

$$\text{Ma}_e = 1 \qquad T_e = T^\star$$

19.3

In the throat section (A_D) the Mach number is not larger than 1

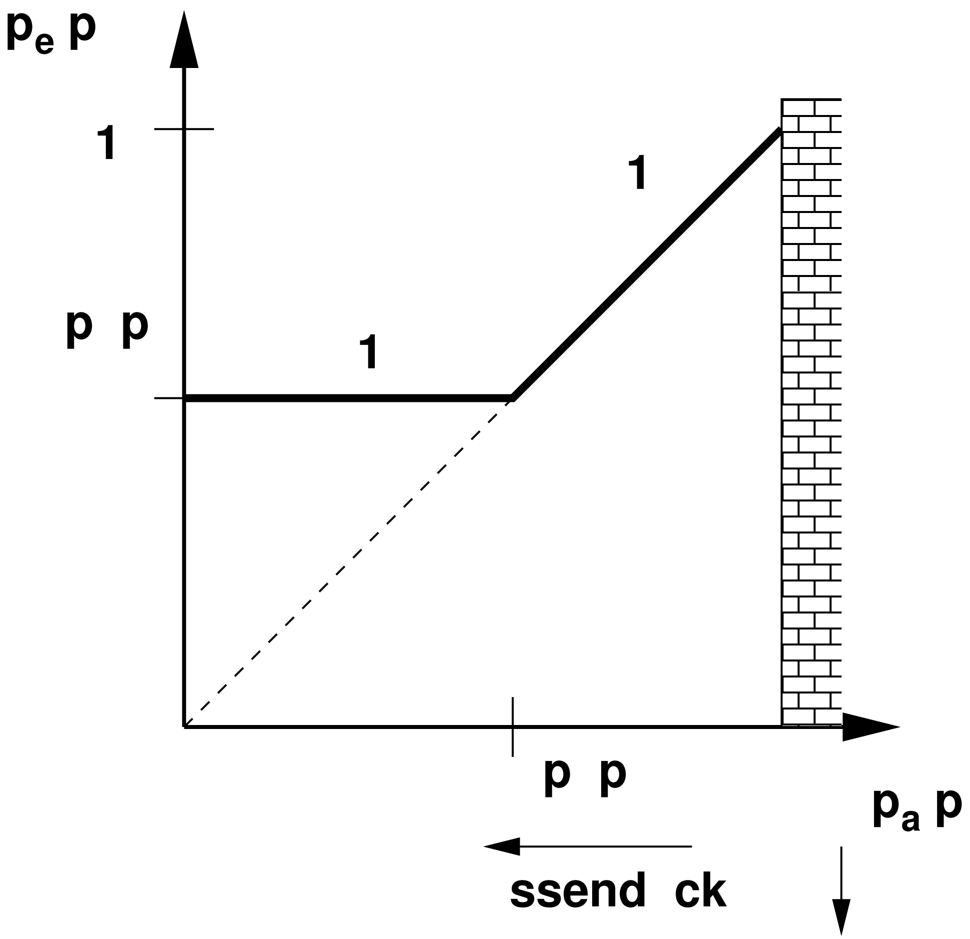
The compensation of the pressure difference via Expansions fans



Expansion fans are without losses

The flow is two- or three dimensional

Sketch



Conclusion

$$p_a \geq p^\star : \qquad F_s = (p_e - p_a)A_D + \rho_e v_e^2 A_D = \gamma p_e \underbrace{\text{Ma}_e^2}_{f\left(\frac{p_e}{p_0}\right)}$$

$$p_a < p^\star : \qquad F_s = (p^\star - p_a)A_D + \underbrace{\rho^\star {v^\star}^2 A_D}_{\text{const. for} p_0 = \text{const.}}$$

further increase is defined by the pressure difference $p_e - p_0$.

b) inkompressible fluid

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s \rightarrow \infty$$

since $d\rho = 0 \Rightarrow w_{\text{abs}} \rightarrow \infty$

The external pressure is impressed independently of v_e.

Bernoulli $\rho =$ const.

$$\frac{1}{2}\rho\left(v_e^2 - \underbrace{v_0^2}_{=0}\right) = (p_0 - p_e) \Longrightarrow \rho v_e^2 = 2(p_0 - p_e)$$

19.3

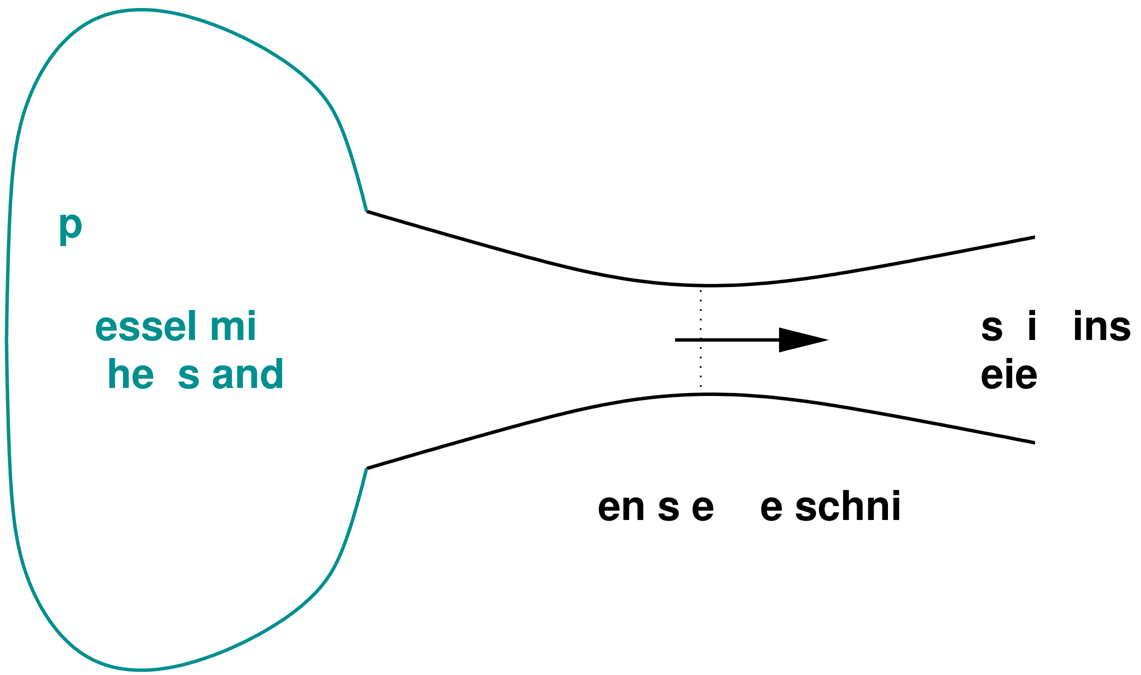
$$\frac{F_s}{p_0 A_D} = \frac{\rho_e v_e^2}{p_0} + \underbrace{\frac{p_e - p_a}{p_0}}_{=0}$$

$$= 2\frac{p_0 - p_a}{p_0} = 2\left(1 - \frac{p_a}{p_0}\right)$$

$$> \left(\frac{F_s}{p_0 A_D}\right)_{\text{compressibel}}$$

since $\dot{m}_{\text{ink}} > \dot{m}_{\text{kompr}}$

Shocks

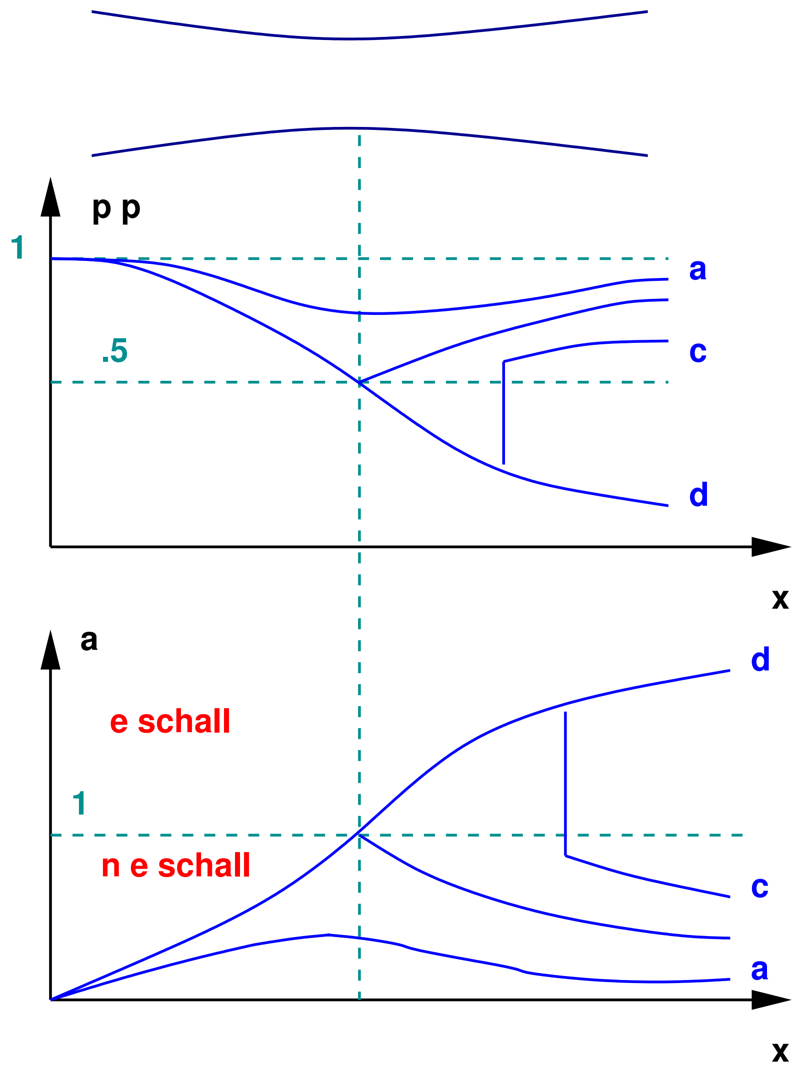


Cross section velocity relation:

$$\frac{du}{u} = \frac{1}{1 - \mathrm{Ma}^2} \frac{dA}{A} \text{ with Ma} = 1 \quad \Rightarrow \quad dA = 0$$

$\rightarrow$ $\mathrm{Ma} = 1$ can only occur in the throat cross section

Shocks



Shocks

a) subsonic flow

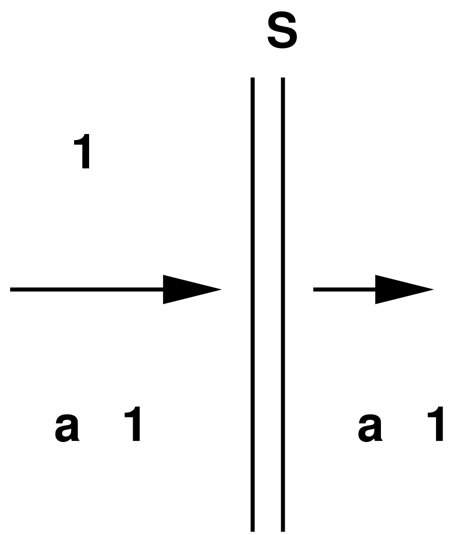
b) sonic flow in the smallest section, then deceleration

c) supersonic flow behind the throat, no information upstream
$\rightarrow$ p is decreased and then increases across the shock

d) isentropic supersonic flow

Shocks

Computation across the normal shock



Entropy increase : $s_2 > s_1$

$\rightarrow$ isentropic equation cannot be used

Shocks

Continuity, momentum, energy equations

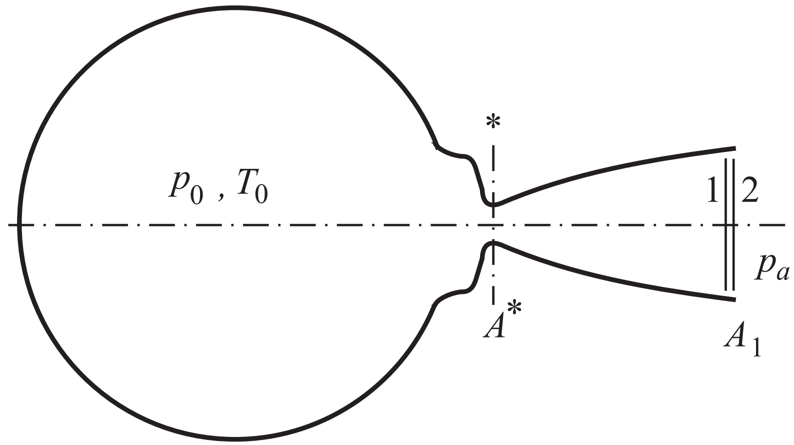
$$\mathrm{Ma}_1^\star = \frac{1}{\mathrm{Ma}_2^\star}$$

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(\mathrm{Ma}_1^2 - 1\right)$$

The pressure ratio increases with the Machnumber.

19.6

Air is flowing from a large reservoir through a well rounded nozzle into the open air. At the exit cross section A_1 a normal shock develops.



a) Compute the mass flux

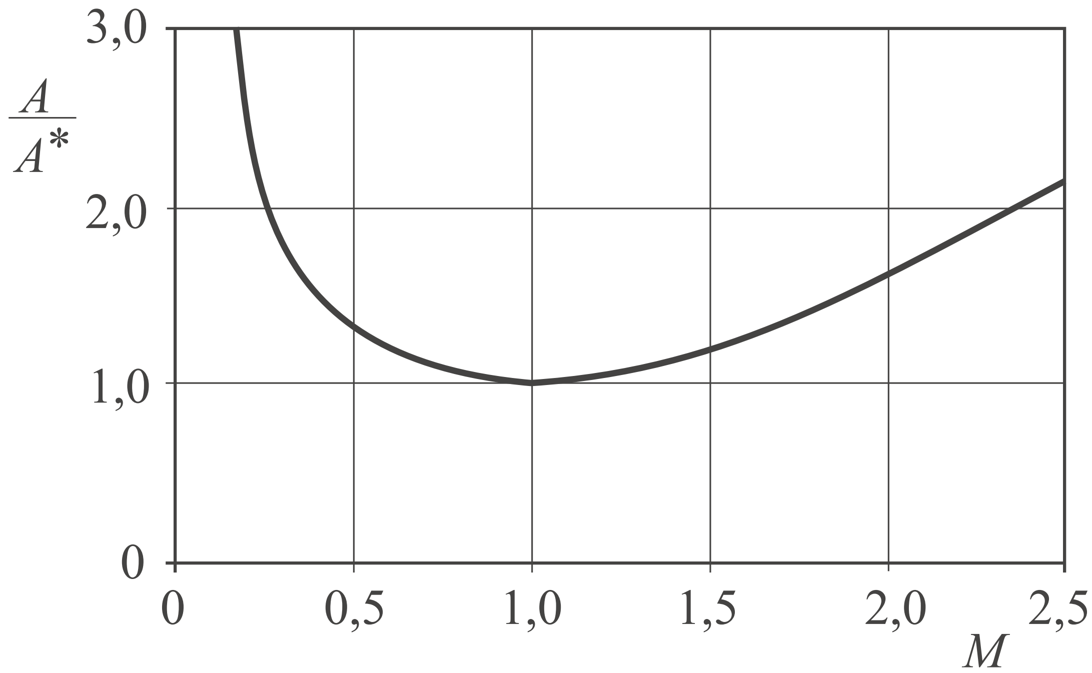
b) Sketch the distribution of the static pressure along the nozzle axis.

19.6

$A_1 = 0.018\ m^2 \quad T_0 = 287\ K \quad A^* = 0.01\ m^2 \quad p_a = 10^5\ \frac{N}{m^2}$

$R = 287\ \frac{J}{kg\ K}$

Hint: $\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)$



19.6

$$\dot{m} = \rho_1 u_1 A_1 = \frac{\rho_1}{\rho_0} \frac{p_0}{RT_0} \frac{u_1}{\sqrt{\gamma R T_1}} \sqrt{\gamma R T_0} \sqrt{\frac{T_1}{T_0}} A_1$$

$$M_1 > 1, \quad M_2 < 1; \quad p_2 = p_a$$

$0 \rightarrow 1$ isentropic flow

$$\frac{A^*}{A_1} = \frac{1}{1.8} \quad \text{(from diagram)} \quad \rightarrow \quad M_1 = 2$$

$$p_1 = \frac{p_2}{1 + \frac{2\gamma}{\gamma+1}(M_1^2 - 1)} = 2.22 \cdot 10^4 \; N/m^2$$

$$T_1 = \frac{T_0}{1 + \frac{\gamma - 1}{2} M_1^2}$$

19.6

$$p_0 = p_1 \left(\frac{T_0}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = p_1 \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\frac{\gamma}{\gamma-1}} = 1.74 \cdot 10^5 \; N/m^2$$

$$\dot{m} = \left(\frac{1}{1 + \frac{\gamma - 1}{2} M_1^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} M_1 \frac{p_0}{\sqrt{RT_0}} \sqrt{\gamma} A_1 = 4.43 \; kg/s$$

19.6

b)