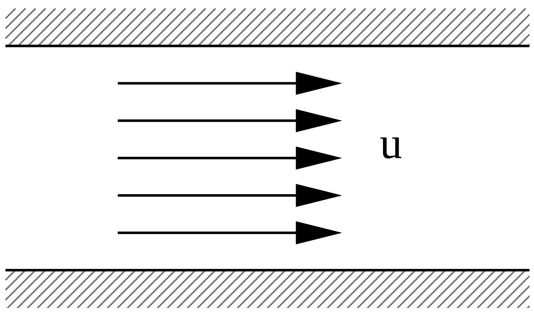
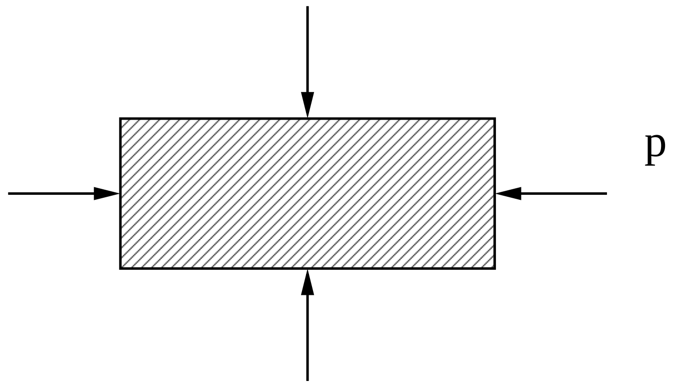
flows with friction

up to now: frictionless flows

$\longrightarrow$ only normal forces $\longrightarrow$ pressure

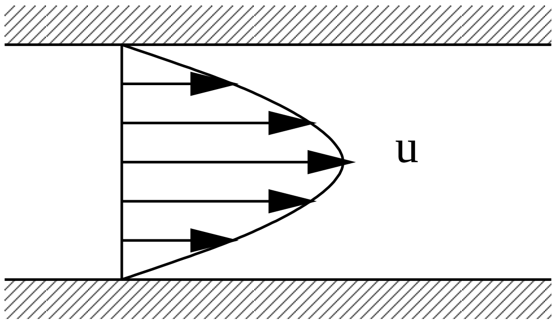


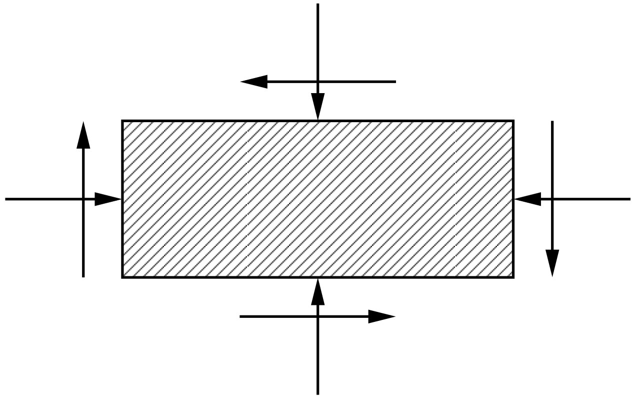


now: flows with friction

$\longrightarrow$ normal- + tangential forces

flows with friction





Friction forces $\vec{F}_R$ react to movements and accelerations
The higher the viscosity the higher the friction force
The tangential forces depend strongly on the velocity gradient
The friction model depends on the fluid
Example: Newtonian fluid

$$\tau = -\eta \frac{du}{dy}$$

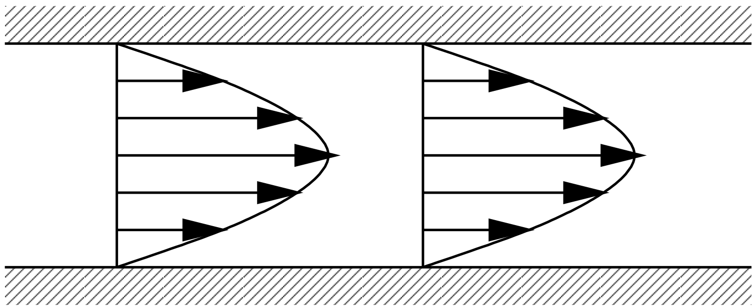
simplifications

- steady flow $\partial/\partial t = 0$
- fully developed flow
- laminare flow
- incompressible flow

fully developed $\longrightarrow$ The velocity profiles is <u>not</u> changing along the axis

$$\longrightarrow \frac{\partial u}{\partial s} = 0; \ \frac{\partial^2 u}{\partial s^2} = 0; \ \frac{\partial v}{\partial s} = 0; \ \frac{\partial^2 v}{\partial s^2} = 0$$

$\longrightarrow$ parallel flow



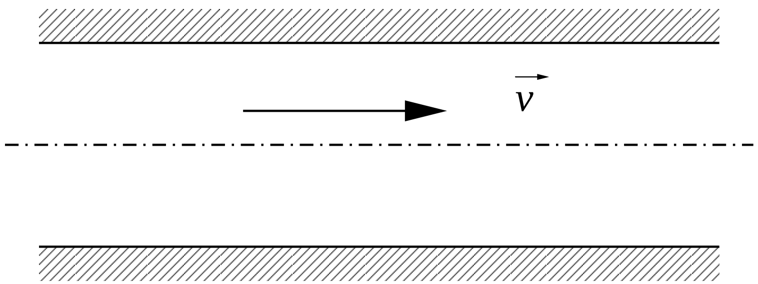
simplifications

Continuity equation for incompressible flows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 : \quad \frac{\partial u}{\partial x} = 0 \longrightarrow \frac{\partial v}{\partial y} = 0$$

$\longrightarrow v = 0$ at the wall / on the axis $\longrightarrow v = 0$ in the entire flow field

Example: flow between parallel walls (pipe, plate)

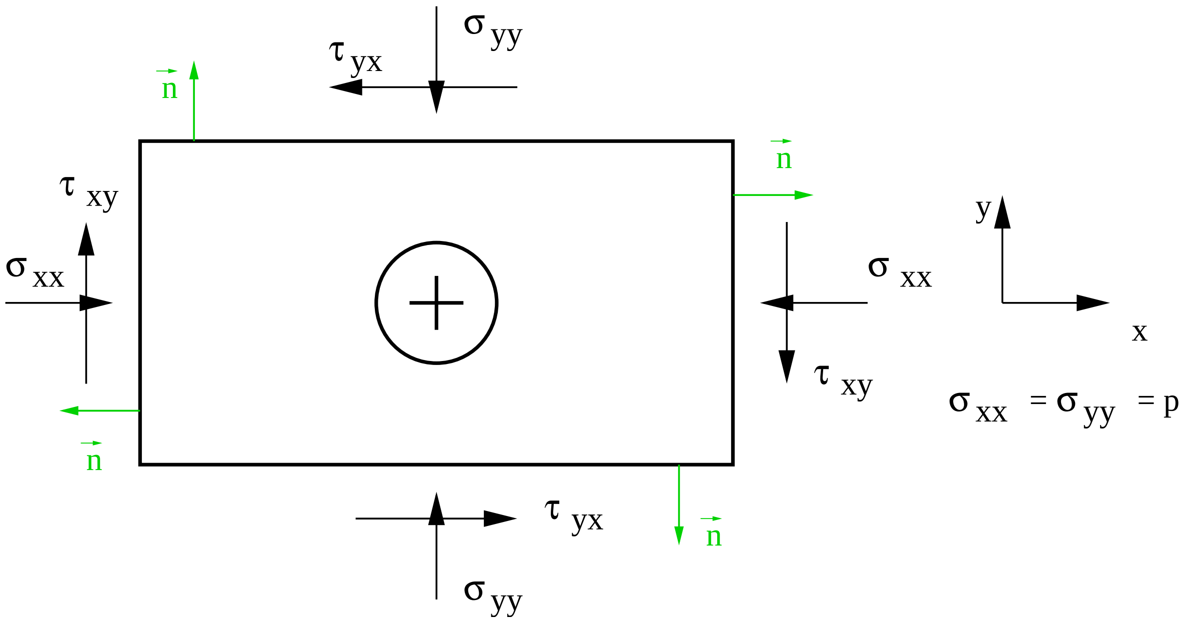


$$\vec{v} = \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$\longrightarrow v = 0$ everywhere

$\longrightarrow$ parallel flow

Equilibrium of forces



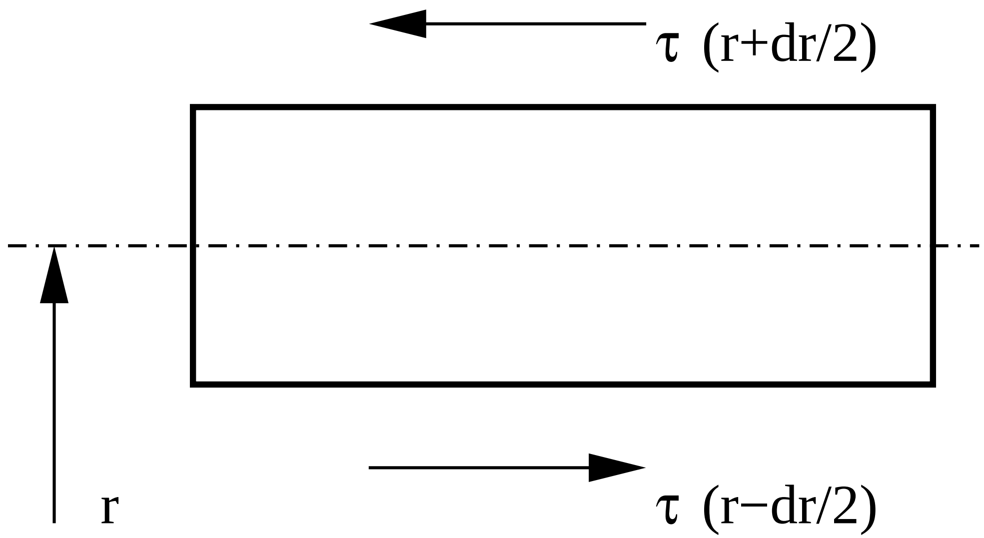
convention of sign

- positive normal stress ($\equiv pressure$), if p is contrary to the normal vecrtor $\vec{n}$
- positive tangential stress points at the coordinate direction, if the normal vector points aginst the coordinate direction

Equilibrium of forces

Remark:

- in rounded cross sections use of cylinder coordinates
- origin in the center of the element
- equilibrium of forces



$$\frac{\partial u}{\partial x} = 0 \rightarrow \frac{dI_x}{dt} = \oint_A \rho \underbrace{v_x}_{const} (\underbrace{\vec{v} \cdot \vec{n}}_{const}) dA; \; (u = v_x = \text{const.}, v = v_y = 0)$$

$$\oint_A C dA = 0 \rightarrow \sum F_A = 0 \longrightarrow$$ equilibrium of forces for infinitesimal elements

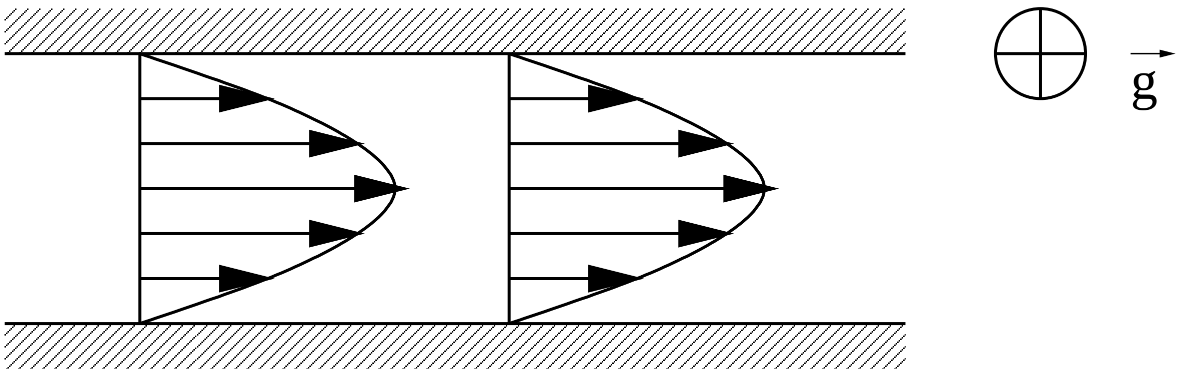
for neglectable volume forces:

$$\int_{CV} \rho \vec{g} dV = 0$$

not accelerated flow

$$\longrightarrow \frac{dI_y}{dt} = \oint_A \rho v_y (\vec{v} \cdot \vec{n}) dA = 0 = \sum F_p$$

$$\longrightarrow \frac{\partial p}{\partial y} = 0$$

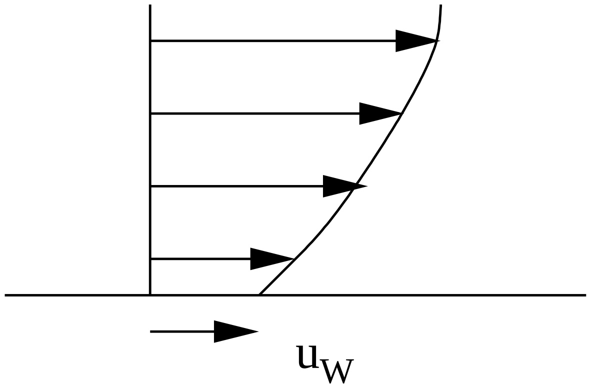


Mechanisms:

- pressure gradient Δp in pipes or between plates
- moving walls u_W (Couette flow, no slip condition)
- Gravitation $\vec{g}$ (oil films with free surface)

boundary conditions

- wall



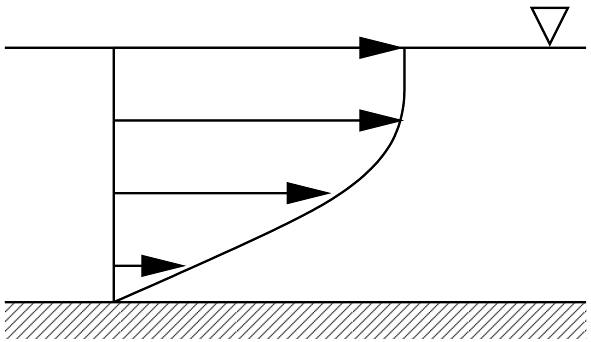
no slip condition

$\rightarrow u = u_W$

$v = 0$

but $\tau \neq 0$ is unknown

- free surface



ambient pressure

$\tau \approx 0$

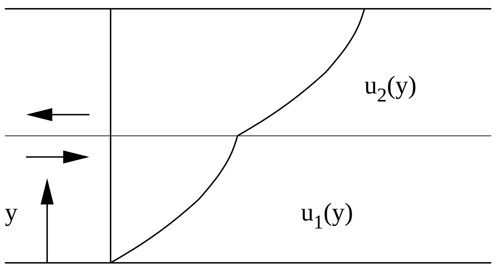
$$\tau = -\eta \frac{du}{dy} = 0$$

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2} = 0$$

friction between air and fluid can be neglected

boundary conditions

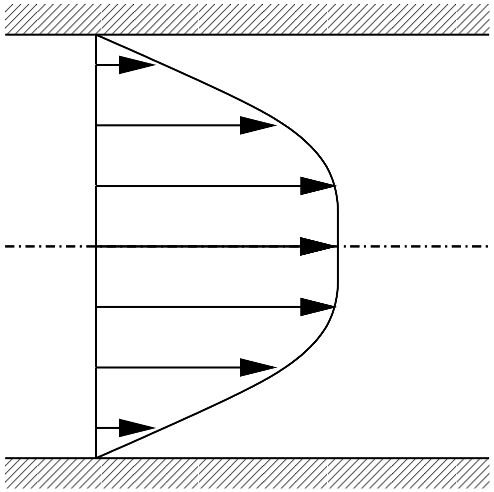
- limiting surface between two fluids



on the contact surface

$$u_1 = u_2$$
$$\tau_1 = \tau_2$$

- symmetry



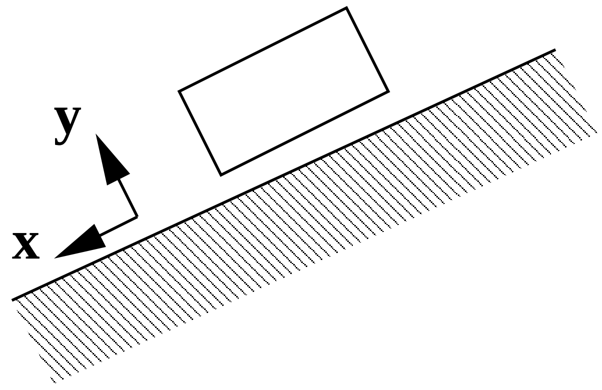
on the axis

$$\tau = 0$$
$$\frac{du}{dy} = 0$$

solution of typical laminar flow problems

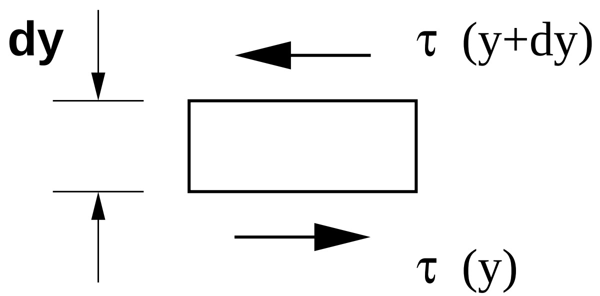
1. choose an applicable coordinate system

   (x along the stream lines)
   sketch an infinitesimal element



2. sketch all forces and stresses

3. formulate the equilibrium of forces in the direction of streamlines



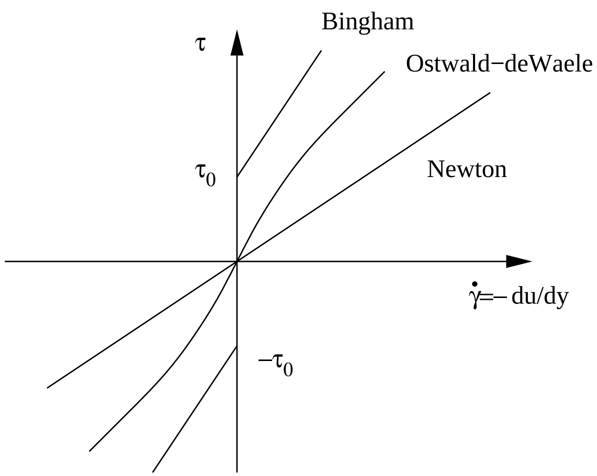
Taylor expansion

$$\tau(y+dy) = \tau(y) + \frac{\partial \tau}{\partial y} dy + \ldots$$

solution of typical laminar flow problems

4. 1st integration of the differential equation
   $\rightarrow$ distribution of the shear stress

5. introduce the model of τ



Bingham: $\tau = -\eta\frac{\partial u}{\partial y} \pm \tau_0$

Ostwald-de Waele: $\tau = -C|\frac{\partial u}{\partial y}|^{n-1}\frac{\partial u}{\partial y}$

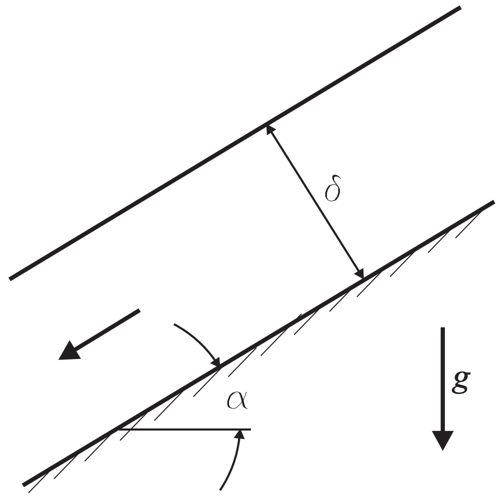
Newton: $\tau = -\eta\frac{\partial u}{\partial y}$

solution of typical laminar flow problems

6. 2nd integration of the differential equation
   $\rightarrow$ distribution of the velocity
7. Use the boundary conditions for the unknown constants

9.1

An oil film of constant thickness and width is flowing on an inclined plane.



$\delta = 3 \cdot 10^{-3}\ m \quad B = 1\ m \quad \alpha = 30^o \quad \rho = 800\ kg/m^3 \quad \eta = 30 \cdot 10^{-3}\ Ns/m^2 \quad g = 10\ \frac{m}{s^2}$

Calculate the volume flux.

9.1

oil film of constant thickness and width

$\rightarrow$ fully developed flow

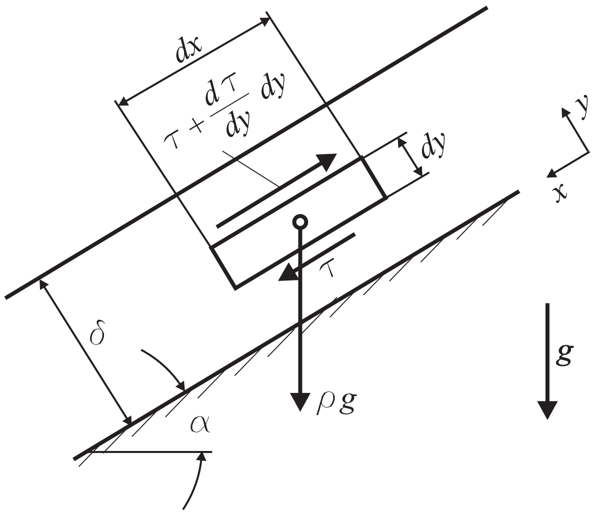
$\rightarrow dI_x/dt = 0 \rightarrow$ equilibrium of forces

$\rightarrow \partial u/\partial x = 0 \rightarrow u = u(y),\ u(y=0) = 0$ no slip condition

conti: $\dfrac{du}{dx} + \dfrac{dv}{dy} = 0 \wedge v(y=0) = 0 \rightarrow v \equiv 0$ anywhere

$$\dot{Q} = \int_A \vec{v} \cdot \vec{n} dA = \int_0^\delta u(y) B dy$$

equilibrium of forces for an infinitesimal element



$$\frac{dI_x}{dt} = 0 = pBdy - (p + \frac{\partial p}{\partial x}dx)Bdy +$$

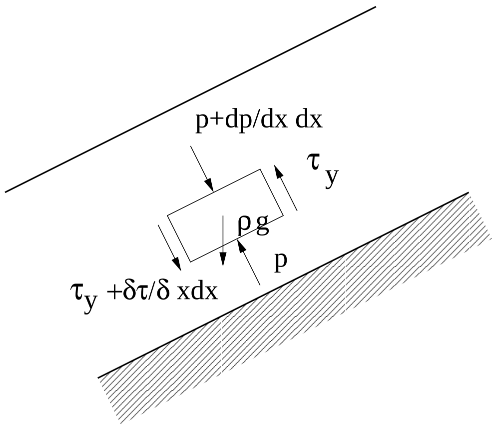
$$+\tau Bdx - (\tau + \frac{\partial \tau}{\partial y}dy)Bdx +$$

$$+\rho g \, \sin\alpha \, Bdxdy \; =$$

$$= \; -\frac{\partial p}{\partial x}\,dxdyB \; - \frac{\partial \tau}{\partial y}\,dyBdx \; + \rho g \sin\alpha \, Bdxdy \; = \; 0$$

$\partial p/\partial x =$?

momentum equation: equilibrium of forces in y-direction



fully developed flow $\rightarrow v = \text{const} = 0$

$$\rightarrow \tau_y = -\eta \frac{\partial v}{\partial x} = 0$$

$$\rightarrow \frac{\partial \tau_y}{\partial x} = 0$$

9.1

$$0 = \frac{dI_y}{dt} = p\,Bdx - \,(p + \frac{\partial p}{\partial y}dy)Bdx \,-\, \rho g\, \cos\alpha\, Bdxdy$$

$\rightarrow \quad \dfrac{\partial p}{\partial y} = \rho g \, \cos\alpha \, = \text{const} \neq f(y)$

$\rightarrow \quad p(x,y) = \rho g y \, \cos\alpha + C(x)$

R.B.: $p(x, y = \delta) = p_a = \text{const}$

$$\longrightarrow \; C(x) \neq f(x)$$

$$\longrightarrow p \neq f(x) \longrightarrow \boxed{\frac{\partial p}{\partial x} = 0}$$ for free surfaces

$$\frac{\partial \tau}{\partial y} = \rho g \sin\alpha = \frac{d\tau}{dy}$$

1st Integration: $\tau(y) = \rho g \sin\alpha y + C_1$

B.C.: $\tau(y = \delta) = 0 \longrightarrow C_1 = -\rho g \delta \sin\alpha$

Newtonian fluid: $\tau = -\eta \frac{du}{dy} \longrightarrow \frac{du}{dy} = -\frac{\tau}{\eta} = \frac{\rho g \sin\alpha}{\eta}(\delta - y)$

9.1

2nd Integration: $u(y) = \frac{\rho g \sin\alpha}{\eta}(\delta y - \frac{1}{2}y^2 + C_2)$

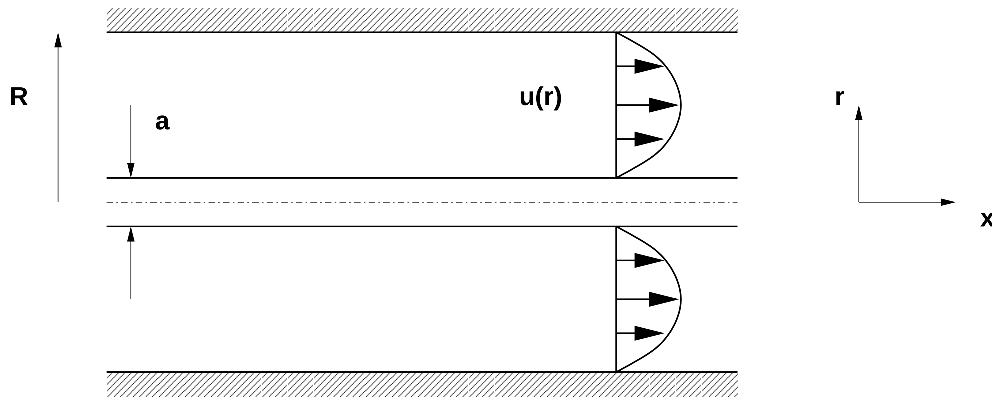
B.C.: $u(y=0) = 0 \longrightarrow C_2 = 0$

$$\boxed{u(y) = \frac{\rho g \sin\alpha}{\eta}(\delta y - \frac{1}{2}y^2)}$$

$$\dot{Q} = \int_0^\delta u(y) B dy = B \left[\frac{\rho g \sin\alpha}{\eta}\left(\frac{\delta}{2}y^2 - \frac{1}{6}y^3\right)\right]_0^\delta = 1.2 \cdot 10^{-3}\frac{m^3}{s}$$

Example

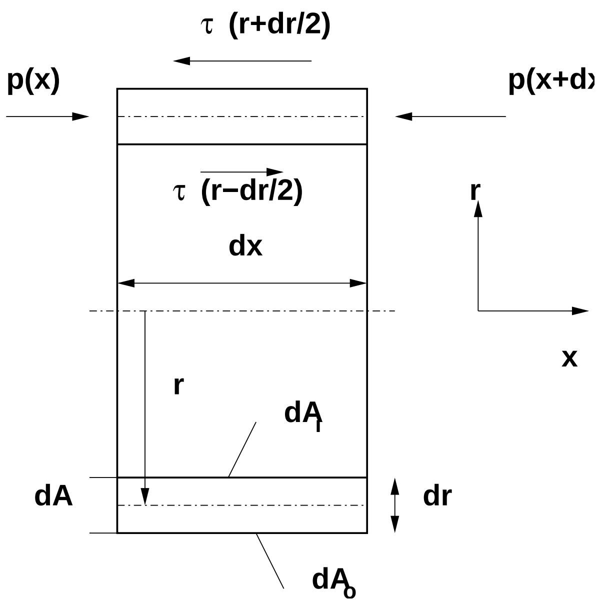
- fully developed flow of a Newtonian fluid between 2 caoaxial cylinders
- given: $R,\ a,\ \eta,\ \dfrac{dp}{dx}$



a) Derive the differential equation for the shear stress distribution $\tau(r)$ and the velocity $u(r)$. Integrate the equations.

b) What is the velocity of the inner cylinder $u_{c,i}$ for the case that the flow does not impose any force on it in x-direction?

a) $\tau(r), u(r)$?



$$dA = 2\pi r dr$$

$$dA_i = 2\pi(r - \frac{dr}{2})dx$$

$$dA_o = 2\pi(r + \frac{dr}{2})dx$$

$$\tau(r \pm \frac{dr}{2}) = \tau(r) \pm \frac{1}{2}\frac{\partial \tau}{\partial r}dr + \ldots$$

a) $\tau(r), u(r)$?

equilibrium of forces: $\sum F_x = 0$

$$0 = pdA - (p + \frac{\partial p}{\partial x}dx)dA + (\tau - \frac{1}{2}\frac{\partial \tau}{\partial r}dr)dA_i - (\tau + \frac{1}{2}\frac{\partial \tau}{\partial r}dr)dA_o$$

$$= -\frac{\partial p}{\partial x}dx2\pi rdr + (\tau - \frac{1}{2}\frac{\partial \tau}{\partial r}dr)(2\pi(r - \frac{dr}{2})dx) - (\tau + \frac{1}{2}\frac{\partial \tau}{\partial r}dr)(2\pi(r + \frac{dr}{2})dx)$$

a) $\tau(r), u(r)$?

$$= -\frac{\partial p}{\partial x}dx2\pi r dr \; +2\pi dx\left[\underbrace{\tau r} - \tau\frac{dr}{2} - \frac{1}{2}\frac{\partial \tau}{\partial r}rdr + \underbrace{\frac{1}{4}\frac{\partial \tau}{\partial r}drdr}\right.$$

$$\left. - (\underbrace{\tau r} + \tau\frac{dr}{2} + \frac{1}{2}\frac{\partial \tau}{\partial r}rdr + \underbrace{\frac{1}{4}\frac{\partial \tau}{\partial r}drdr})\right]$$

$$= -\frac{\partial p}{\partial x}2\pi r dr dx \; -\tau dr 2\pi dx - \frac{\partial \tau}{\partial r}drr2\pi dx$$

$$\longrightarrow -\frac{\partial p}{\partial x} - \frac{\tau}{r} - \frac{\partial \tau}{\partial r} = \boxed{-\frac{dp}{dx} - \frac{1}{r}\frac{d(\tau r)}{dr} = 0}$$

Example

Newtonian fluid: $\tau = -\eta \dfrac{du}{dr}$

$$\frac{dp}{dx} - \frac{\eta}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0$$

b) $u_{c,i} = ?$

Boundary conditions:

- $u(r = R) = 0$, no slip condition

  Friction: $F_R(r = a) = 0 \rightarrow \tau(r = a) = 0 : (F_R = \tau A) \rightarrow \frac{du}{dr}|_{r=a} = 0$

- fully developed flow: $\rightarrow \dfrac{\partial p}{\partial x} \neq f(r) \rightarrow \dfrac{r}{\eta}\dfrac{dp}{dx} = \dfrac{d}{dr}\left(r\dfrac{du}{dr}\right)$

Example

1st Integration: $$\frac{1}{2\eta}\frac{dp}{dx}r^2 + C_1 = r\frac{du}{dr}$$

B. C..: $$\frac{du}{dr}|_{r=a} = 0 \rightarrow C_1 = -\frac{a^2}{2\eta}\frac{dp}{dx}$$

$$r\frac{du}{dr} = \frac{1}{2\eta}\frac{dp}{dx}(r^2 - a^2) \rightarrow \frac{du}{dr} = \frac{1}{2\eta}\frac{dp}{dx}(r - \frac{a^2}{r})$$

2nd Integration: $$u(r) = \frac{1}{2\eta}\frac{dp}{dx}(\frac{1}{2}r^2 - a^2 \ln r) + C_2$$

Example

B. C.: $u(r = R) = 0$

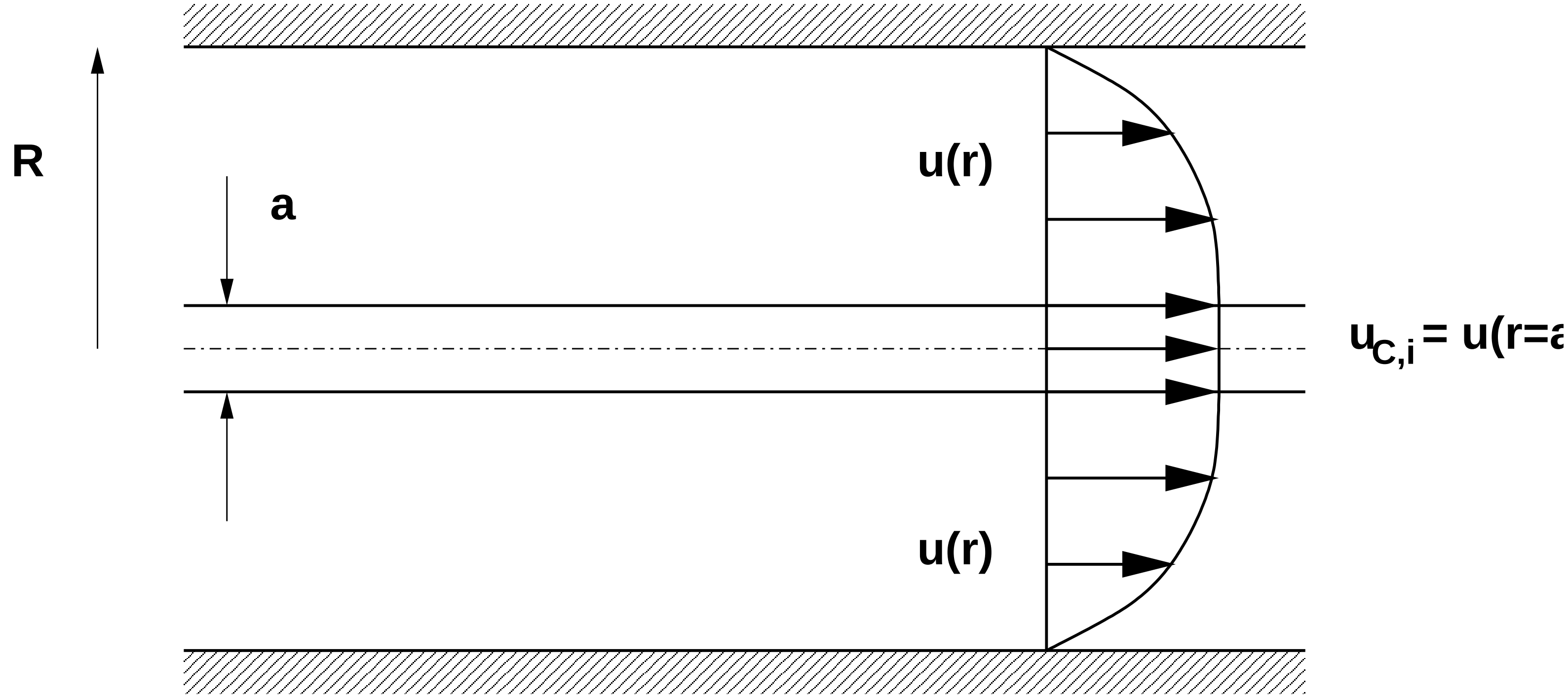
$$\rightarrow C_2 = -\frac{1}{2\eta}\frac{dp}{dx}\left(\frac{1}{2}R^2 - a^2 \ln R\right)$$

$$\rightarrow u(r) = \frac{1}{2\eta}\frac{dp}{dx}\left[\frac{1}{2}\left(r^2 - R^2\right) - a^2 \ln r + a^2 \ln R\right]$$

$$= \frac{1}{2\eta}\frac{dp}{dx}\left(\frac{r^2 - R^2}{2} + a^2 \ln \frac{R}{a}\right)$$

$$u_{C,i} = u(a) = \frac{1}{2\eta}\frac{dp}{dx}\left(\frac{a^2 - R^2}{2} + a^2 \ln \frac{R}{a}\right)$$

Example



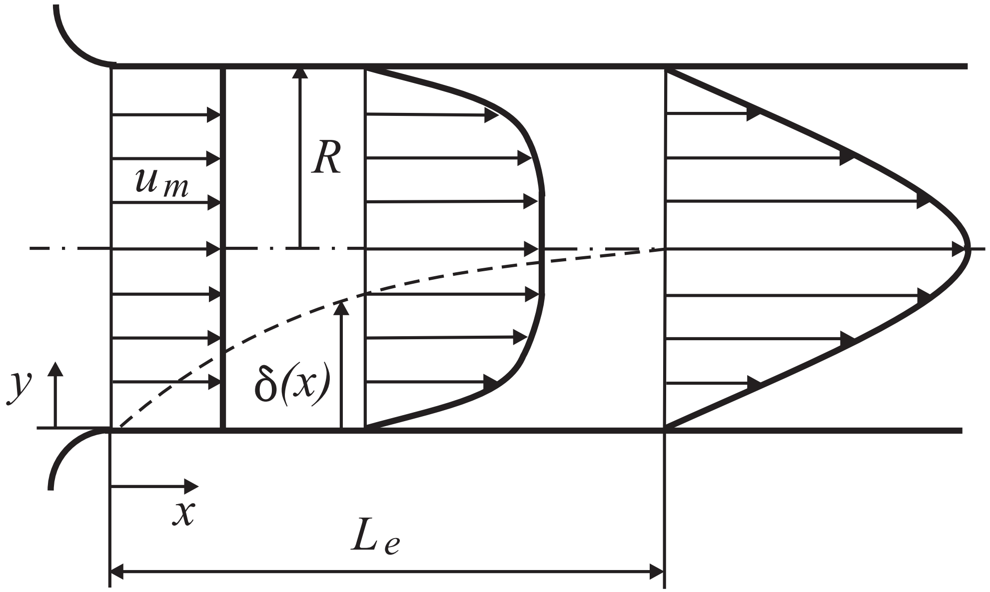
9.6

The velocity distribution of a laminar pipe flow can be described in the inlet with the following approximation

$$\frac{u}{u_m} = \frac{f\left(\frac{y}{\delta}\right)}{1 - \frac{2}{3}\frac{\delta}{R} + \frac{1}{6}\left(\frac{\delta}{R}\right)^2}$$

$$f\left(\frac{y}{\delta}\right) = \begin{cases} 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 & 0 \leq y \leq \delta(x) \\ 1 & \delta(x) \leq y \leq R \end{cases}$$

9.6



Given: $u_m, \ R, \ \rho, \ \eta$

Determine in the inlet cross section, at the end of the inlet section and for $\delta/R = 0,5$

a) the momentum flux,
b) the wall shear stress.

9.6

a)

$$r = R - y$$

$$\dot{I} = \frac{dI_x}{dt} = \int_o^R \rho u^2 2\pi r dr = 2\rho u_m^2 \pi R^2 \int_0^1 \left(\frac{u}{u_m}\right)^2 \frac{r}{R} d\left(\frac{r}{R}\right)$$

$$\delta = 0 : \quad \frac{u}{u_m} = 1$$

$$\Longrightarrow \quad \dot{I} = \rho u_m^2 \pi R^2$$

$$\delta = R : \quad \frac{u}{u_m} = 2\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$\Longrightarrow \quad \dot{I} = 1,33 \rho u_m^2 \pi R^2$$

9.6

$$\delta = \frac{R}{2}: \quad \frac{u}{u_m} = \begin{cases} \frac{96}{17}\frac{r}{R}\left(1 - \frac{r}{R}\right) & \frac{R}{2} \leq r \leq R \\ \frac{24}{17} & 0 \leq r \leq \frac{R}{2} \end{cases}$$

$$\Longrightarrow \dot{I} = 2\rho u_m^2 \pi R^2 \left\{ \int_0^{0,5} \left(\frac{24}{17}\right)^2 \frac{r}{R} d\left(\frac{r}{R}\right) \right.$$

$$\left. + \int_{0,5}^{1} \left[\frac{96}{17}\frac{r}{R}\left(1 - \frac{r}{R}\right)\right]^2 \frac{r}{R} d\left(\frac{r}{R}\right) \right\} = 1,196\rho u_m^2 \pi R^2$$

$$2\rho u_m^2 \pi R^2 \left\{ \frac{576}{289} \quad \frac{1}{2}\left(\frac{r}{R}\right)^2 \Big|_0^{0.5} + \frac{9216}{289} \int_{0.5}^{1} \left(\frac{r^3}{R^3} - 2\frac{r^4}{R^4} + \frac{r^5}{R^5}\right) d\frac{r}{R} \right\}$$

9.6

$$= 2\rho u_m^2 \pi R^2 \left\{ \frac{576}{289} \quad \frac{1}{2}\left(\frac{r}{R}\right)^2 \Bigg|_0^{0.5} + \frac{9216}{289}\left(\frac{1}{4}\frac{r^4}{R^4} - \frac{2}{5}\frac{r^5}{R^5} + \frac{1}{6}\frac{r^6}{R^6}\right)_0^{0.5} \right\}$$

b)

$$\tau_w = \eta u_m \frac{2/\delta}{1 - \frac{2}{3}\frac{\delta}{R} + \frac{1}{6}\left(\frac{\delta}{R}\right)^2}$$

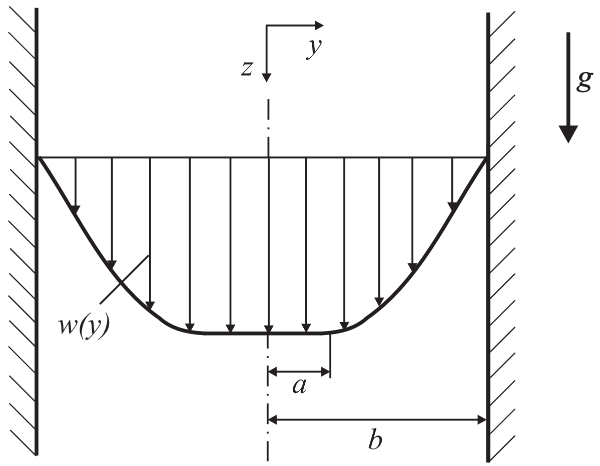
$$\delta \longrightarrow 0 : \quad \tau_w \longrightarrow \infty$$

$$\delta = R : \quad \tau_w = 4\frac{\eta \cdot u_m}{R}$$

$$\delta = \frac{R}{2} : \quad \tau_w = 5,65\frac{\eta \cdot u_m}{R}$$

9.8

A Bingham fluid is flowing between two infinite parallel plates under the influence of gravity.



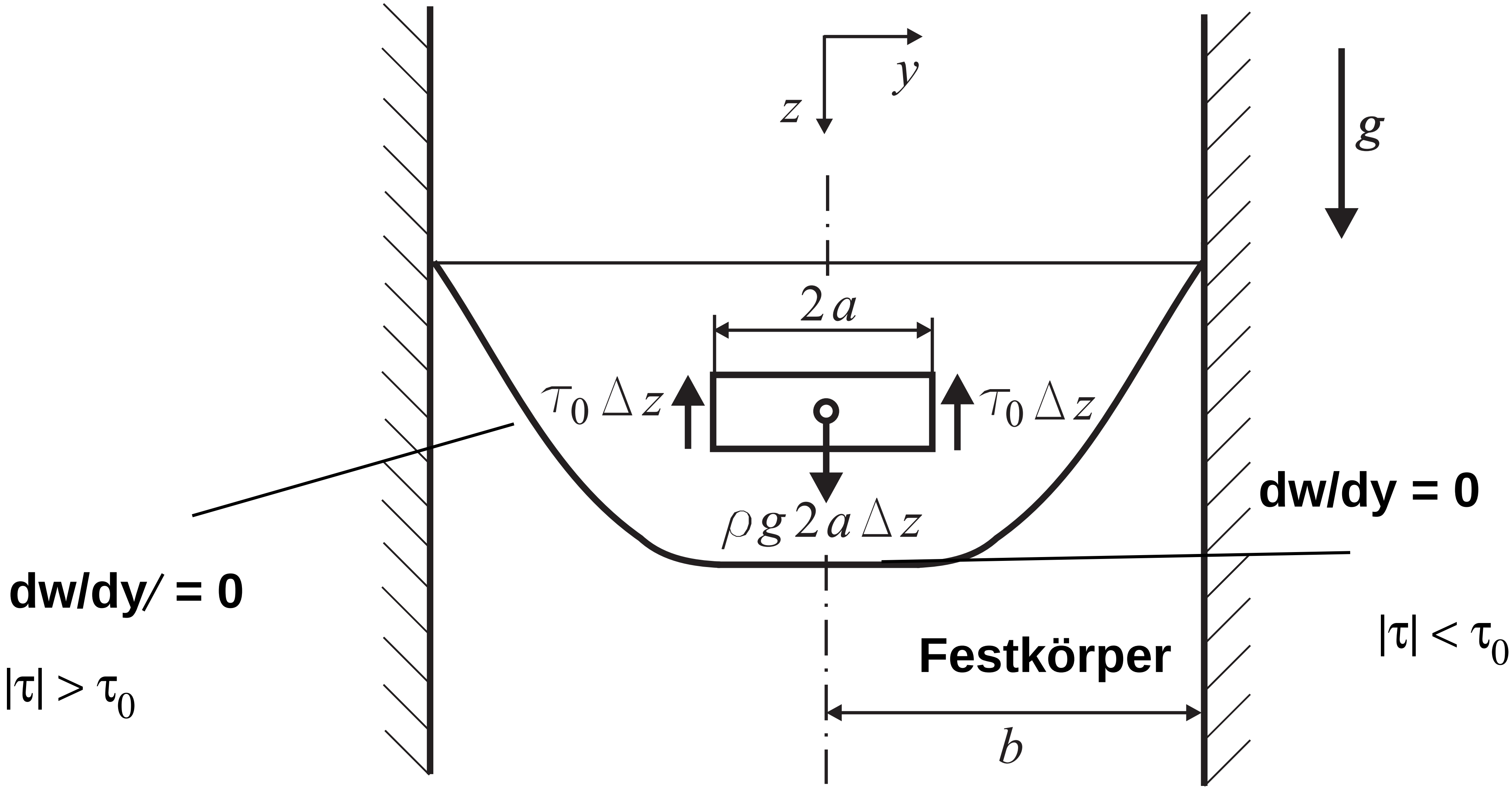
Given: $b,\ \rho,\ \eta,\ \tau_0,\ g,\ dp/dz = 0$

Assume a fully developed flow and determine

a) the distance a,
b) the velocity distribution.

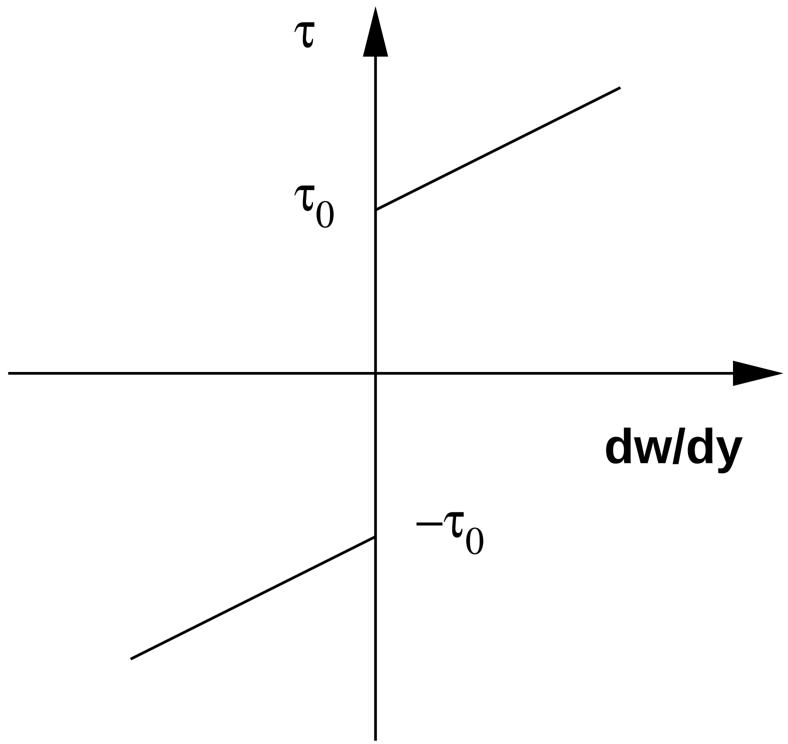
9.8

a=?



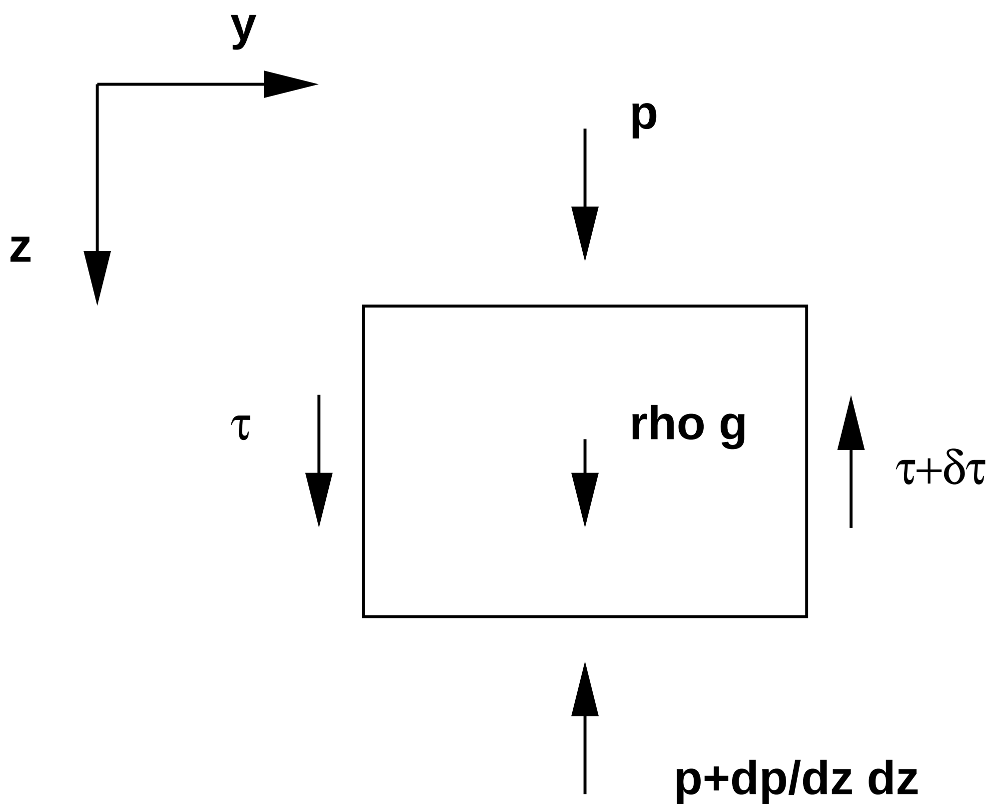
y
z
g
2a
$\tau_0 \Delta z$
$\tau_0 \Delta z$
$\rho g 2a \Delta z$
dw/dy = 0
dw/dy/ = 0
|τ| < τ0
|τ| > τ0
Festkörper
b

Bingham fluid: $\tau = -\eta \frac{\partial w}{\partial y} \pm \tau_0$



- To make the fluid flow τ has to be larger than τ_0
- in symmetry planes the fluid behaves like a solid body

infinitesimal element



developed flow: $\rightarrow \frac{\partial w}{\partial z} = 0 \; ; \; \frac{\partial}{\partial z} = 0$

9.8

equilibrium of forces:

$$\tau B dz - \left(\tau + \frac{\partial \tau}{\partial y} dy\right) B dz$$

$$+ p B dy - \left(p + \frac{\partial p}{\partial z} dz\right) B dy + \rho g B dy dz = 0$$

$$\rightarrow -\frac{\partial \tau}{\partial y} dy B dz + \rho g B dy dz = 0$$

$$\boxed{\frac{\partial \tau}{\partial y} = \rho g = \frac{d\tau}{dy}}$$

Integration: $d\tau \rho g dy \rightarrow \tau = \rho g y + C_1(z)$

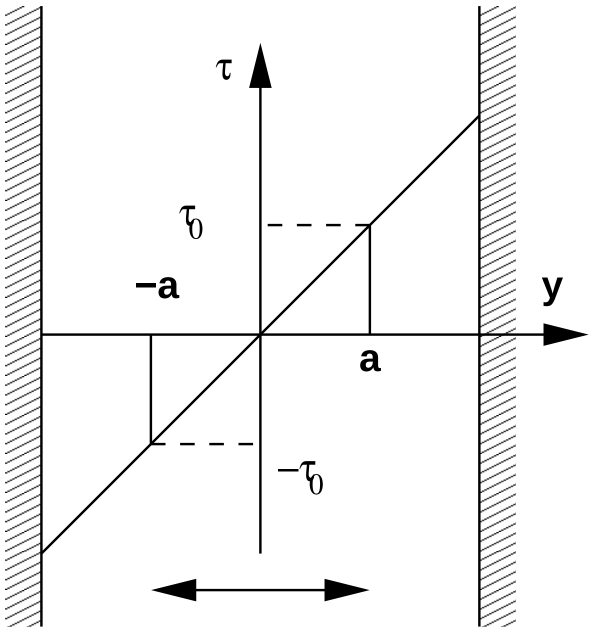
9.8

Boundary condition for $C_1(z)$

Symmetrie: $\tau(y=0)=0 \rightarrow C_1(z)=0$

$$\boxed{\tau(y)=\rho g y}$$ (independent from the fluid)

9.8



Festkörper

$\tau(y)$: straight line

$$y = |a| \rightarrow |\tau| = \tau_0$$

$$\rightarrow \ \tau(y = a) = \rho g a = \tau_0$$

$$\rightarrow \ a = \frac{\tau_0}{\rho g}$$

9.8

b) velocity distribution $w(y)$

inner region: solid body: $|y| \leq a \rightarrow \dfrac{dw}{dy} = 0$

$$\rightarrow w(y) = const.$$

flow for $|y| > a \rightarrow \tau = -\frac{dw}{dy} \pm \tau_0$

(developed flow: $\dfrac{\partial w}{\partial z} = 0$)

$$\frac{dw}{dy} = -\frac{\tau \pm \tau_0}{\eta}$$

Symmetry: $w(y) = w(-y)$

9.8

$$y > 0 : \tau(y) = \rho g y \rightarrow \frac{dw}{dy} = -\frac{\rho g y}{\eta}(+/-)\frac{\tau_0}{\eta}$$

sign: $\frac{dw}{dy} < 0$ for $y > a$

$\frac{dw}{dy} = 0$ for $y \leq a$

Integration:

$$w(y) = \frac{1}{\eta}(\tau_0 y - \frac{1}{2}\rho g y^2) + C_2$$

Boundary condition: no slip condition at the wall

$\rightarrow y = b$: $w = 0$

$$\rightarrow C_2 = \frac{\rho g b^2}{2\eta} - \frac{\tau_0 b}{\eta}$$

$$w(y) = \frac{\rho g}{2\eta}(b^2 - y^2) - \frac{\tau_0}{\eta}(b - y) \qquad y > a$$

$$w(y) = \frac{\rho g}{2\eta}(b^2 - a^2) - \frac{\tau_0}{\eta}(b - a) \qquad y \leq a$$