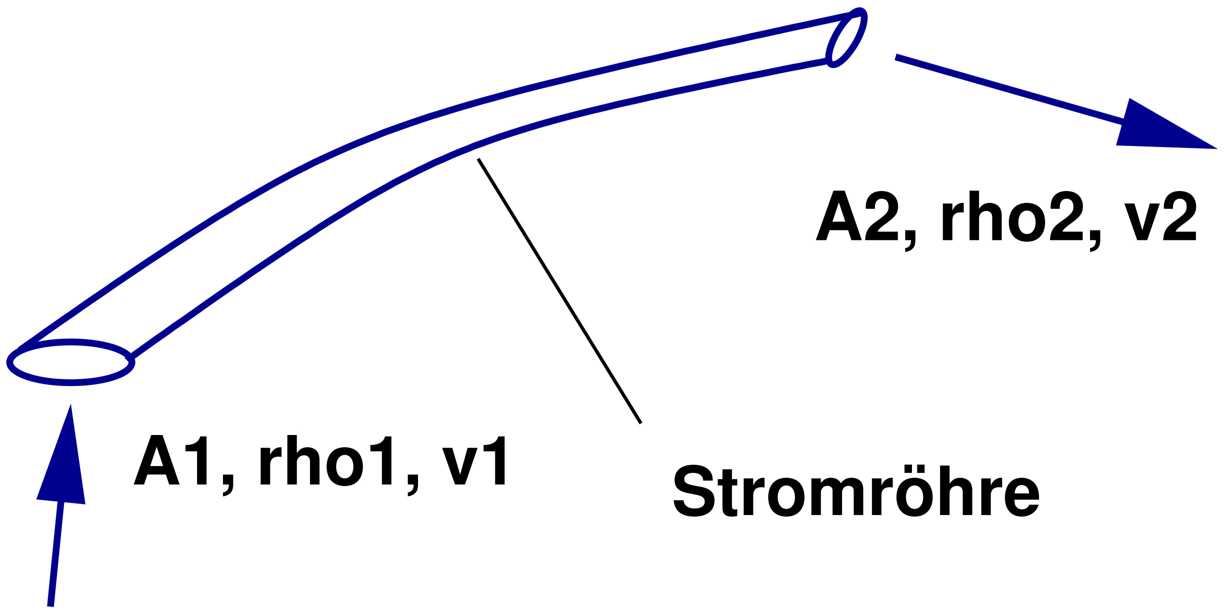
# Hydrodynamics

Continuity equation




Conservation of mass: $$\underbrace{\rho_1 v_1 A_1}_{\dot{m}_1} = \underbrace{\rho_2 v_2 A_2}_{\dot{m}_2}$$ Mass flux

**inkompressible** fluid: ($\rho_1 = \rho_2 = const$)

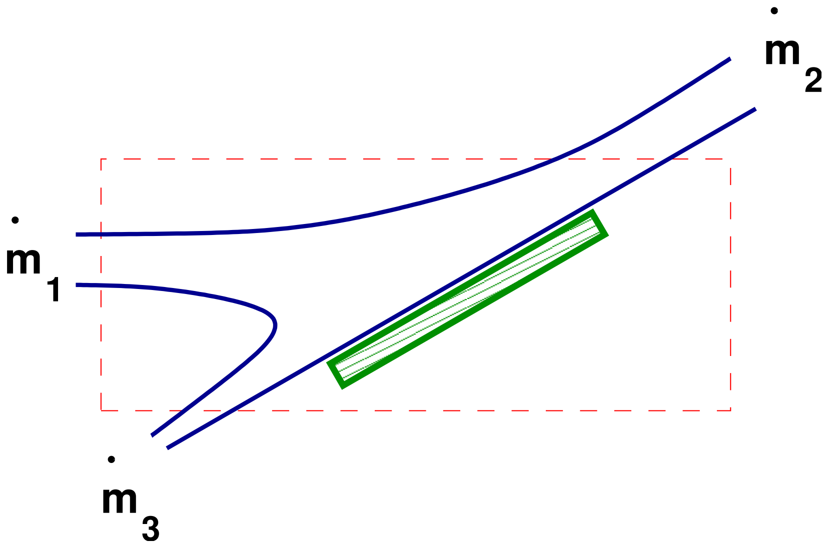
Conservation of volume flux : $$\underbrace{v_1 A_1}_{\dot{Q}_1} = \underbrace{v_2 A_2}_{\dot{Q}_2}$$ Volume flux

Example

pipe flow: $A = const$

closed stream tube

water jet

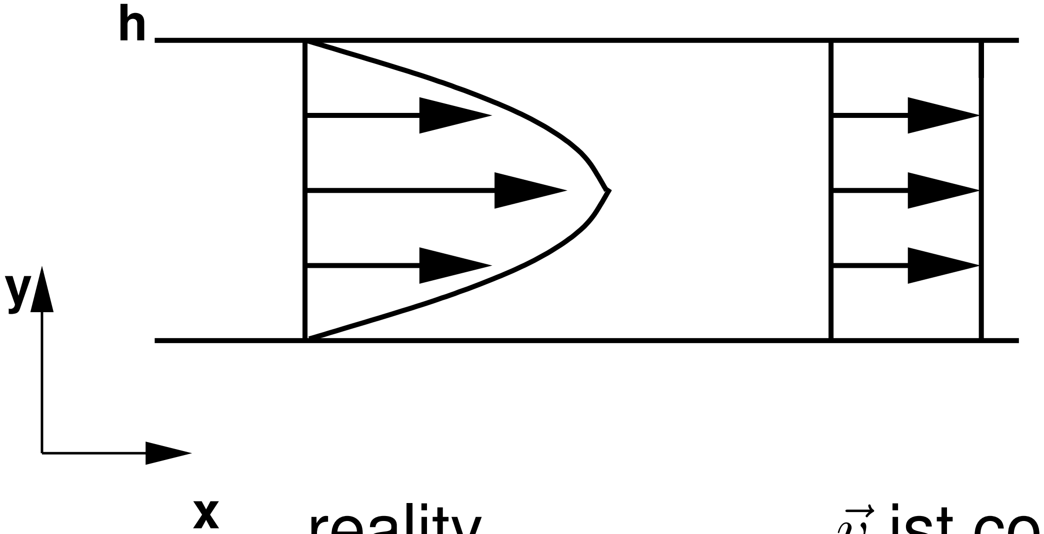



closed control volume

$\dot{m}_1 = \dot{m}_2 + \dot{m}_3$

# Continuity

**Important: In the 1-dimensional continuit equation $\vec{v}$ is an average value of the velocity. In reality $\vec{v}$ is not constant due to friction, vortices, $\ldots$!**




| reality | $\vec{v}$ ist constant |
|---|---|
| $\vec{v} = \vec{v}(y)$ | in the continuit equation |

mass flux must be constant

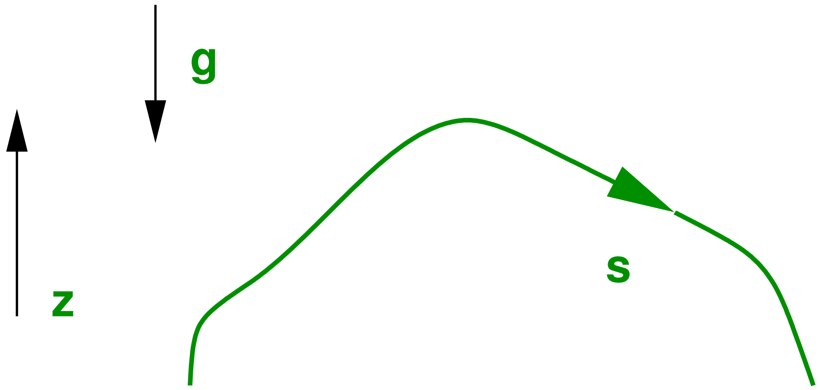
$$\longrightarrow \int \rho v(y)\, dy \; = \; \rho \bar{v} h$$

# Bernoulli

2.Newtonian law:
mass × acceleration = sum of outer forces

$$m \cdot \frac{d\vec{v}}{dt} = \Sigma F_a$$

Equation of motion for an infinitesimal element along one streamline




pressure, friction

$$\rho \frac{d\vec{v}}{dt} = -\frac{\partial p}{\partial s} - \rho g \frac{dz}{ds} - R'$$

inertia, gravitation

# Bernoulli

along a streamline: $v = v(s,t)$

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + \frac{\partial \vec{v}}{\partial s} ds$$

$$\longrightarrow \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{ds}{dt}\frac{\partial \vec{v}}{\partial s} = \frac{\partial \vec{v}}{\partial t} + v\frac{\partial \vec{v}}{\partial s}$$

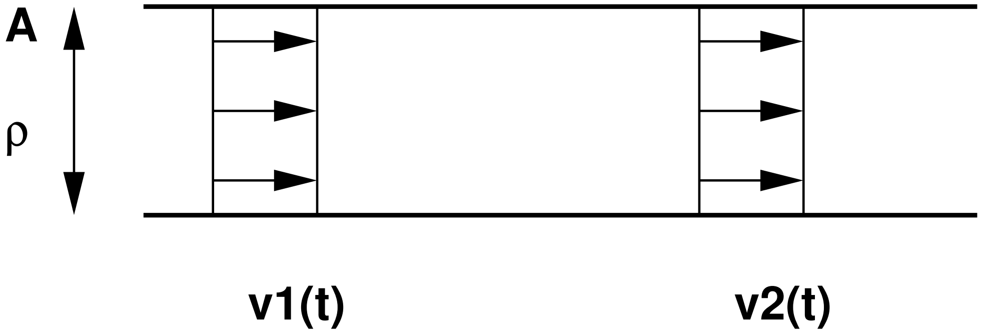
total (substantial) acceleration of a particle

lokal acceleration

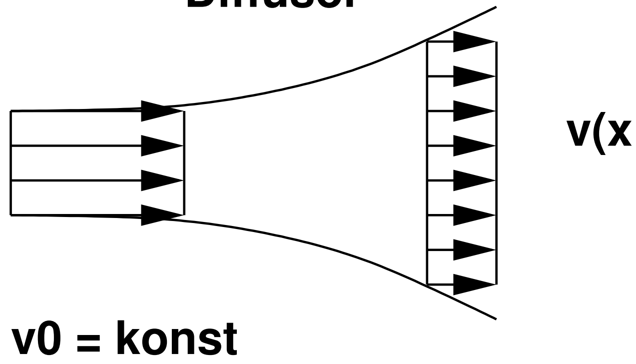
convective acceleration

# example

**Rohrströmung**




**Diffusor**




$$A, \rho = konst$$

$$\longrightarrow v_1(t) = v_2(t)$$

only local acceleration

only convective acceleration

# example

assumptions:

- incompressible $(\rho = const)$
- frictionless $(R` = 0)$
- steady $\frac{\partial}{\partial t} = 0$
- constant gravitation $(\vec{g} = const)$

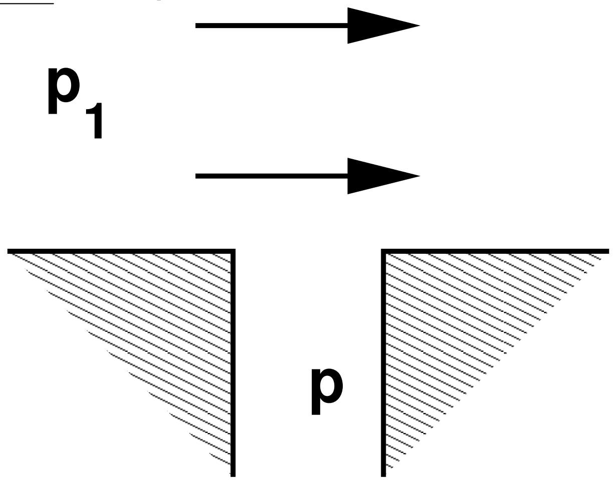
$$\rho \left[ \underbrace{\frac{\partial v}{\partial t}}_{=0} + v \frac{\partial v}{\partial s} \right] = -\frac{\partial p}{\partial s} - \rho g \frac{dz}{ds} \underbrace{-R`}_{=0}$$

$$f(s) \longrightarrow \frac{\partial}{\partial s} = \frac{d}{ds}$$

$$\frac{1}{2} \rho \frac{dv^2}{ds} = -\frac{dp}{ds} - \rho g \frac{dz}{ds} \longrightarrow \frac{\rho}{2} v^2 + p + \rho g z = const$$

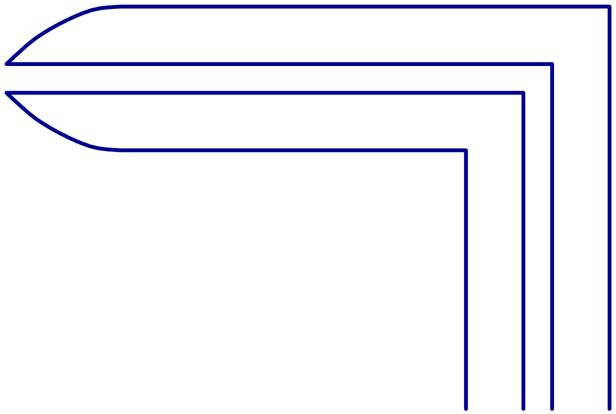
# pressure measurement

static pressure: $p$ (Index: $1,\ 2,\ a,\ \infty$)




Total pressure (pitot tube): $p_0,\ p_{01},\ p_{02},\ p_t$

$$p_0 = p + \frac{1}{2}\rho v^2 + \rho g h$$

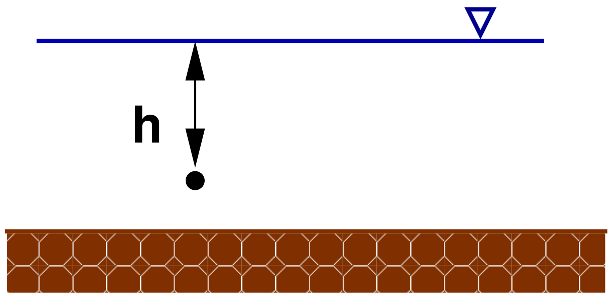


at constant height $\Delta h = 0$

$$\longrightarrow\ p_0 = p + \frac{1}{2}\rho v^2$$

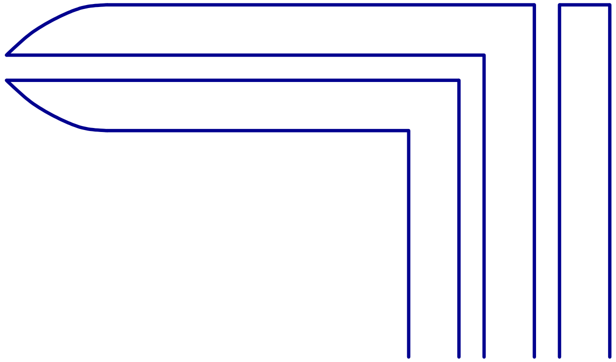
# pressure measurement

potential pressure: $p_{pot} = \rho g h$



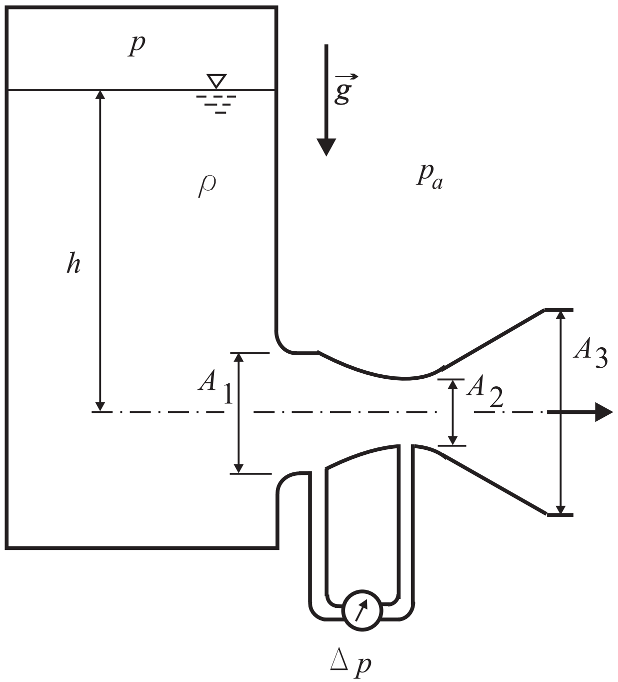

dynamical pressure: $p_{dyn} = \frac{1}{2}\rho v^2$

kinetic energy is converted, when the flow is decelerated to $\vec{v} = 0$



# 6.4

Water flows from a large pressurized tank into the open air. The pressure difference $\Delta p$ is measured between $A_1$ and $A_2$.




$$A_1 = 0,3\ m^2, \qquad A_2 = 0,1\ m^2,$$
$$A_3 = 0,2\ m^2, \qquad h = 1\ m,$$
$$\rho = 10^3\ kg/m^3, \qquad p_a = 10^5\ N/m^2,$$
$$\Delta p = 0,64 \cdot 10^5\ N/m^2 \qquad g = 10\ m/s^2$$

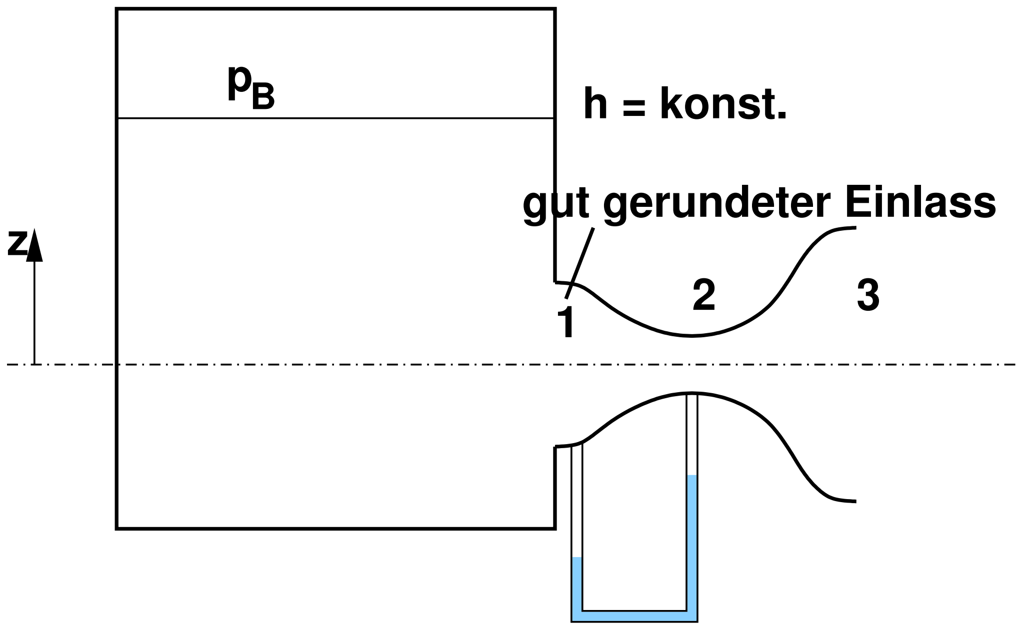
Compute the

a) velocities $v_1,\ v_2,\ v_3$,

b) pressures $p_1,\ p_2,\ p_3$ and the pressure $p$ above the water surface!

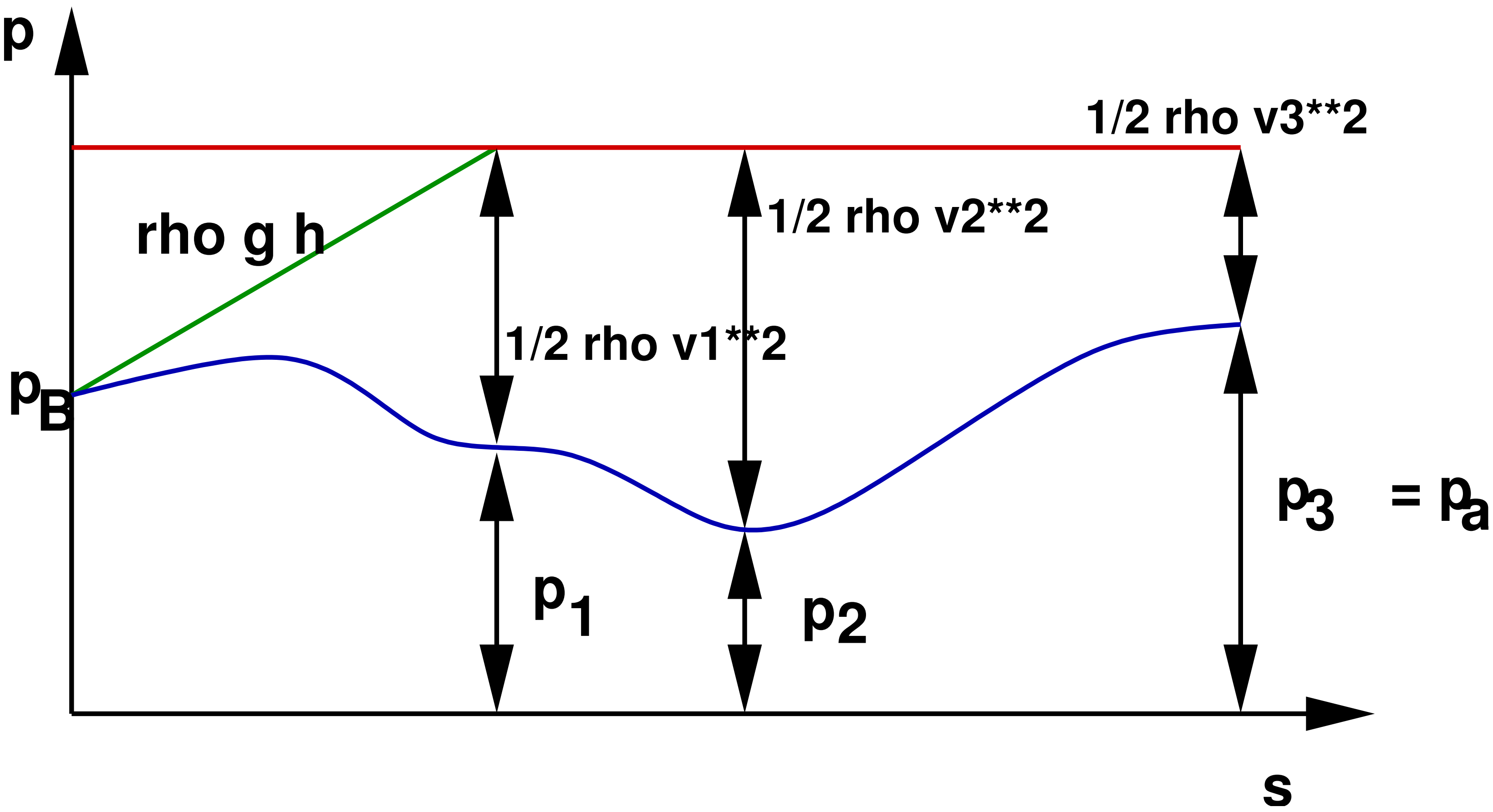
# 6.4

## pressure tank with nozzle




Venturidüse

conservation of total energy along a streamline → qualitatively




Bernoulli: $p_0 = p_b + \rho g h = p_i + \frac{1}{2}\rho v_i^2$

# 6.4

continuity (mass balance): $\Longrightarrow = \dot{m} = \rho \dot{Q} = const.$

$$\rho = const \Longrightarrow v_1 A_1 = v_2 A_2 = v_3 A_3 \Longrightarrow A \downarrow \Longrightarrow v \uparrow \Longrightarrow p \downarrow$$

a) measured $\Delta p = p_1 - p_2$ Bernoulli: $p_1 + \frac{\rho}{2} v_1^2 = p_2 + \frac{\rho}{2} v_2^2$

$$\Longrightarrow \Delta p = p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2) > 0$$

$$v_1 = v_2 \frac{A_2}{A_1} \rightarrow \Delta p = \frac{\rho}{2}\left[1 - \frac{A_2^2}{A_1^2}\right] v_2^2 \longrightarrow v_2 = \sqrt{\frac{2}{\rho} \frac{\Delta p}{\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}} = 12 \frac{m}{s}$$

$$v_1 = v_2 \frac{A_2}{A_1} = 4 \frac{m}{s} \qquad v_3 = v_2 \frac{A_2}{A_3} = 6 \frac{m}{s}$$

## The Venturi-nozzle is used to measure mass- and volume fluxes!

$$\dot{Q} = vA = v_2 A_2$$

prinziple:

- measurement of $\Delta p$
- computation of $v_2$
- computation of volume- and massflux

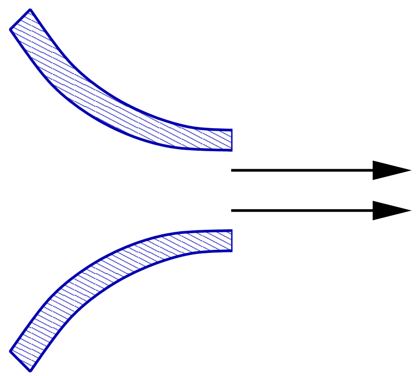
b) determination of pressures $p_B, p_1, \ldots, p_3$

$p_0$ represents the energy that can be converted into kinetic energy

$$p_0 = p_B + \rho g h = p_1 + \frac{\rho}{2} v_1^2 = p_2 + \frac{\rho}{2} v_2^2 = p_3 + \frac{\rho}{2} v_3^2$$

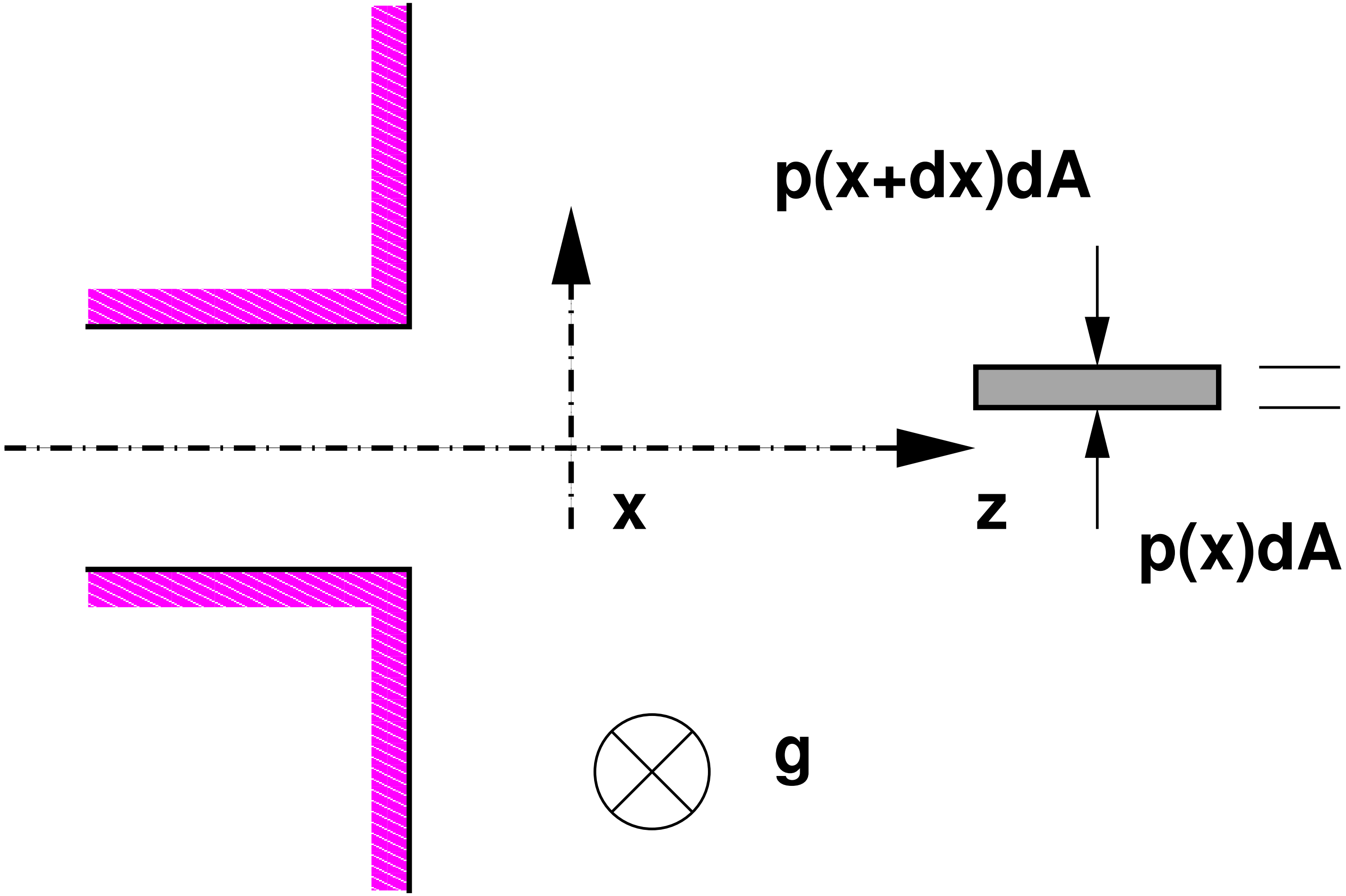
If we know one pressure, we can compute the other values by using Bernoulli's equation

$p_3$ in the exit cross section



**Assumption:** parallel streamlines at the sharp edged exit

# equation of motion for an element




equation of motion in $x$-direction for a moving control volume $dAdx$ (includes always the same particlesl)

# equation of motion for an element

$$m\frac{du}{dt} = \ddot{x}\rho dAdx = p(x)dA - p(x+dx)dA$$

$$\longrightarrow \ddot{x}\rho dAdx = p(x)dA - \left(p + \frac{\partial p}{\partial x}dx\right)dA$$

$$\longrightarrow \rho\ddot{x} = -\frac{\partial p}{\partial x}$$

Assumption: parallel stream lines

$\longrightarrow \dot{x} = 0$ velocity $u = \frac{dx}{dt} = \dot{x}$

$\longrightarrow$ necessary condition: $\ddot{x} = 0 \longrightarrow \frac{\partial p}{\partial x} = 0$

$\Longrightarrow$ the pressure in the exit cross-section is function of $y$

flow into air: $\frac{dp}{dy} = -\rho g$

Neglect the potential energy $\longrightarrow$ $p_{exit} = p_{ambience} = const.$

# 6.4

$p_3 = p_a$

Remark:

Bernoulli: $0 \longrightarrow 3$

$$p_B + \rho g h = p_a + \frac{1}{2}\rho v_3^2$$

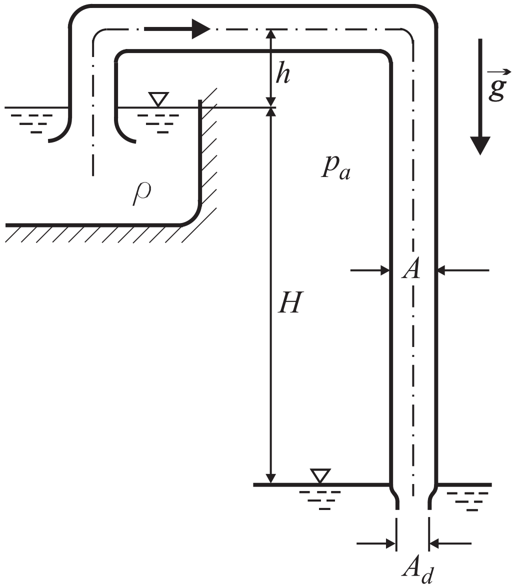
$$\longrightarrow v_3 = \sqrt{\frac{2}{\rho}\left(p_B - p_a + \rho g h\right)}$$

open tank $p_B = p_a$

$\longrightarrow \boxed{v_3 = \sqrt{2gh} \neq f(A_3)}$ theorem of Torricelli (15.Okt. 1608 - 25.Okt. 1647)

# 6.5

Two large basins located one upon the other are connected with a duct.



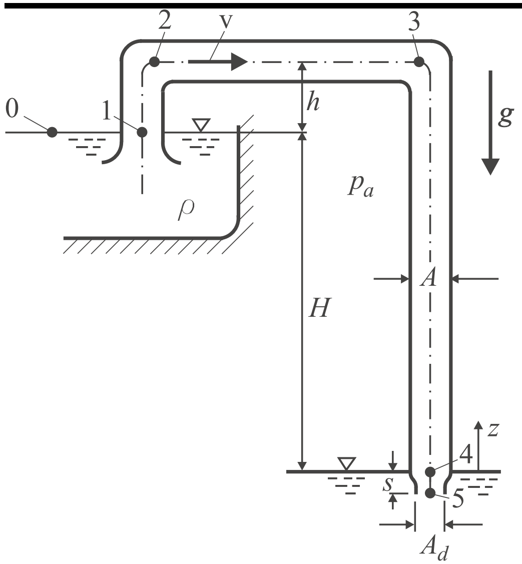

$$A = 1\ m^2, \quad A_d = 0,1\ m^2, \quad h = 5\ m, \quad H = 80\ m,$$

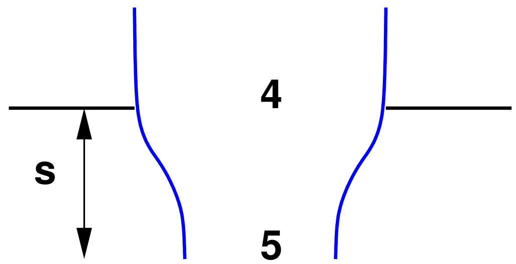
$$p_a = 10^5\ N/m^2, \quad \rho = 10^3\ kg/m^3, \quad g = 10\ m/s^2$$

a) Determine the volume rate!
b) Outline the distribution of static pressure in the duct!
c) At what exit cross section bubbles are produced, when the vapour pressure is

$p_D = 0,025 \cdot 10^5\ N/m^2$?

# 6.5




incompressible, frictionless, steady

$\longrightarrow$ Bernoulli

a) volume flux: $\dot{Q} = vA = v_5 A_5$

Bernoulli: $0 \longrightarrow 5$

$$p_a + \rho g H = p_5 + \rho g(-s) + \frac{\rho}{2} v_5^2$$

computation of $p_5$:

straight parallel streamlines

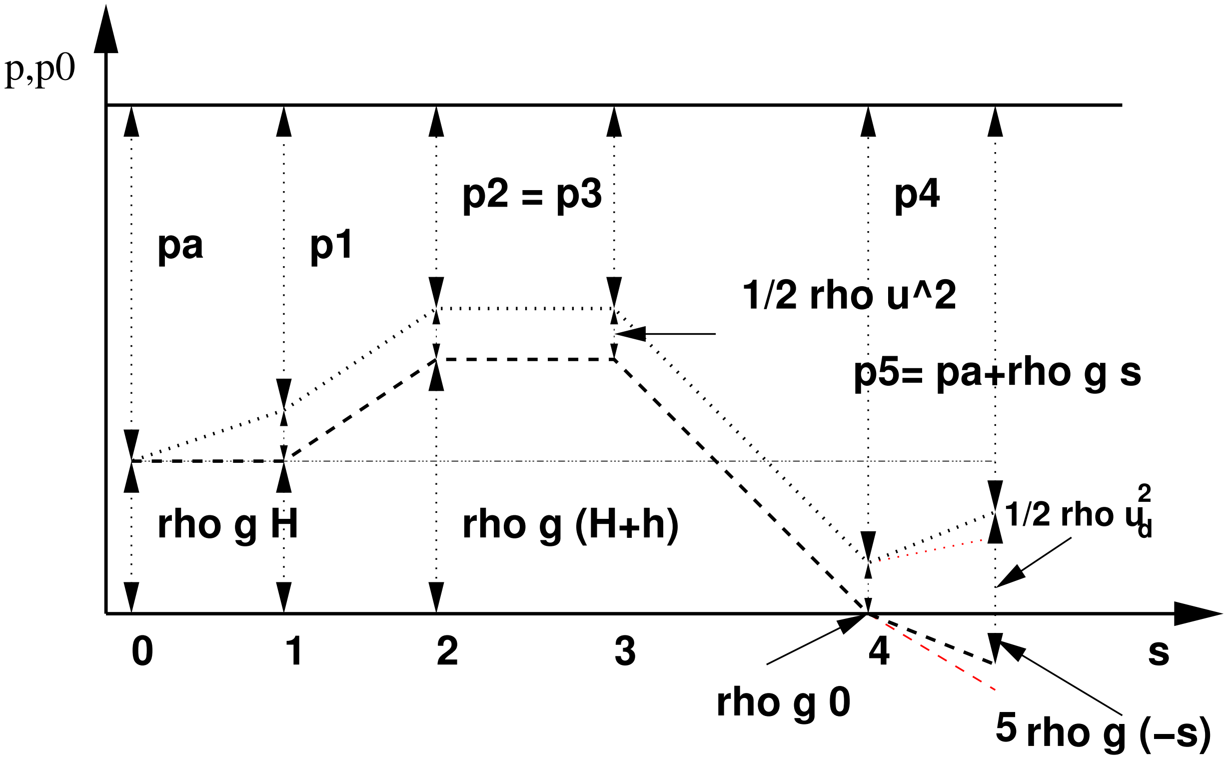
$\longrightarrow$ pressure is constant in the exit cross-section

$$\longrightarrow p_5 = p_a + \rho g s$$

$$p_a + \rho g H = p_a + \rho g s - \rho g s + \frac{1}{2}\rho v_5^2 \longrightarrow v_5 = \sqrt{2gH} \neq f(A_d, s)$$

$$\longrightarrow \dot{Q} = A_d v_5 = 4\frac{m^3}{s}$$

b)



# 6.5

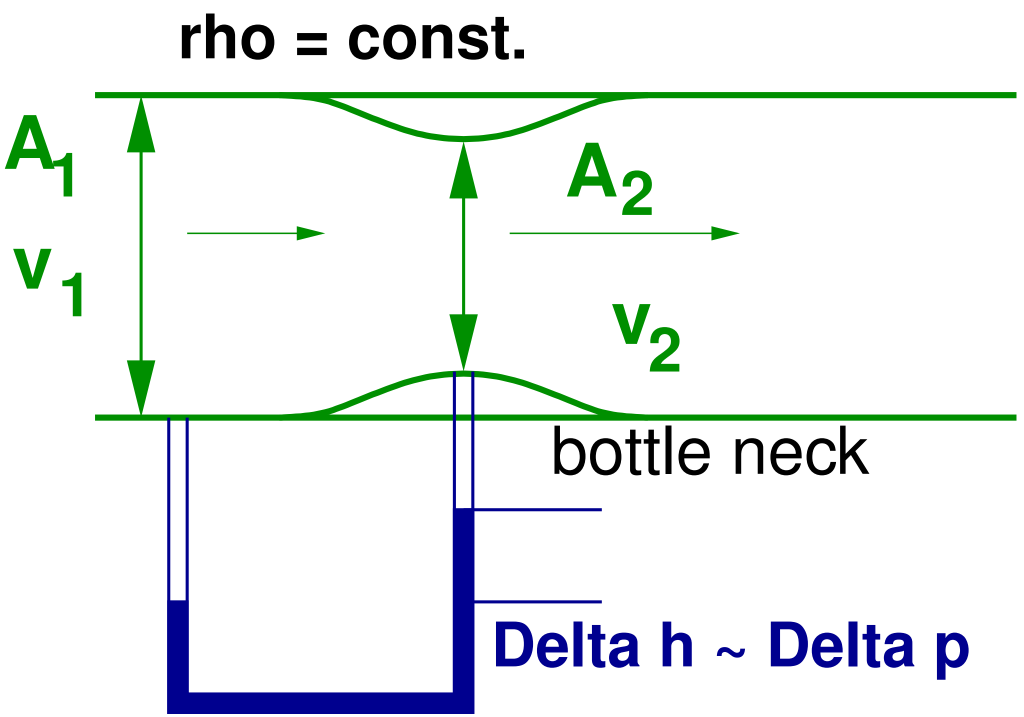
c) minimum pressure between 2 and 3: $p_2 = p_3 = p_D$

continuity: $v_5^* A_D^* = v^* A$

Bernoulli: $p_a = p_D + \rho g h + \frac{1}{2} \rho {v^*}^2$

$$\longrightarrow A_d = A \sqrt{\frac{p_a - p_D}{\rho g H} - \frac{h}{H}} = \ 0.244 \ m^2$$

# extended Bernoulli




Theoretical volume flux: $\dot{Q}_{th}$ for frictionless flow

1. Bernoulli: $p_1 + \frac{\rho}{2} v_1^2 = p_2 + \frac{\rho}{2} v_2^2$
2. continuity: $v_1 A_1 = v_2 A_2$

# extended Bernoulli

ratio of areas: $m = \frac{A_2}{A_1}$: $\longrightarrow$ conti $v_1 = v_2 m$

$$\longrightarrow \text{Bernoulli: } \frac{p_1}{\rho} + \frac{1}{2} v_2^2 m^2 = \frac{p_2}{\rho} + \frac{1}{2} v_2^2$$

$$\longrightarrow \quad v_2^2 \left(1 - m^2\right) = 2\frac{p_1 - p_2}{\rho} = 2\frac{\Delta p}{\rho}$$

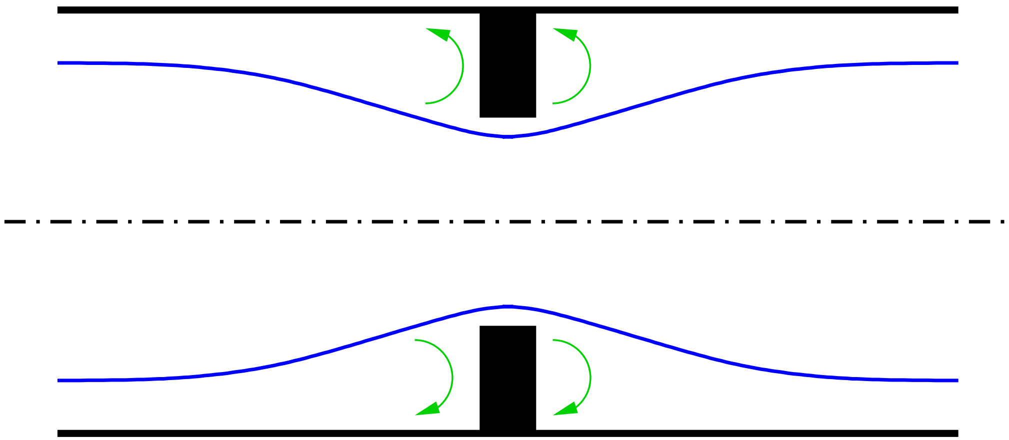
$$\longrightarrow \quad v_2 = \sqrt{\frac{2\Delta p}{\rho(1 - m^2)}}$$

$$\longrightarrow \boxed{\dot{Q}_{th} = A_2 \sqrt{\frac{2\Delta p}{\rho(1 - m^2)}}}$$

# extended Bernoulli (Cont'd)

In reality losses from friction, vortices, bottle necks . . . occur.
$\longrightarrow$ the flow is no longer frictionless

The losses and the ration of aread are put together in the
discharge coefficient $\alpha$



vortex, dissipation

$$\dot{Q}_{real} = \alpha A_2 \sqrt{\frac{2\Delta p}{\rho(1-m^2)}}$$

$$\alpha^\star = \alpha \sqrt{\frac{1}{1-m^2}}$$

$\alpha^\star$ from experiments

losses in pipe flows can be predicted similar.

# extended Bernoulli (Cont'd)

pressure loss across a constructive element (ellbow, valve, . . .)

$$\Delta p_v = \zeta \cdot \frac{1}{2}\rho v^2$$

coeffizient: $\zeta = \frac{\Delta p_v}{\frac{1}{2}\rho v^2} = \frac{\text{pressure loss}}{\text{dynamic pressure}}$

$$\longrightarrow v = \frac{1}{\sqrt{\zeta}}\sqrt{\frac{2\Delta p}{\rho(1-m^2)}} \Longrightarrow \dot{Q} = v \cdot A = \frac{1}{\sqrt{\zeta}}\,A\,\sqrt{\frac{2\Delta p}{\rho(1-m^2)}}$$

(Experiments, standards $\longrightarrow$ catalogue)

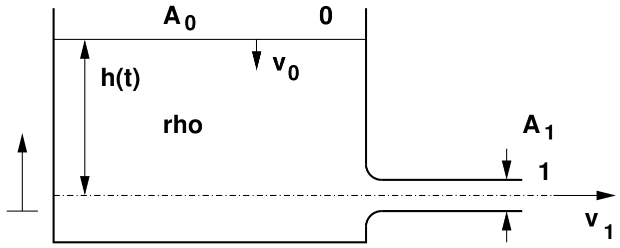
# Hydrodynamics: unsteady Bernoulli

assumption

$$\frac{A_1}{A_0} \ll 1 \quad \longrightarrow v_0 \ll v_1$$

$v_0$ is neglectable

but $h = h(t)$ and $v_1 = v_1(t)$




unsteady Bernoulli from "0" to "1"

$$p_a + \frac{\rho}{2} v_0^2(t) + \rho g h(t) = p_a + \frac{\rho}{2} v_1^2(t) + \int_0^1 \rho \frac{\partial v}{\partial t} ds$$

Assumption $v_1(t) = \sqrt{2gh(t)}$

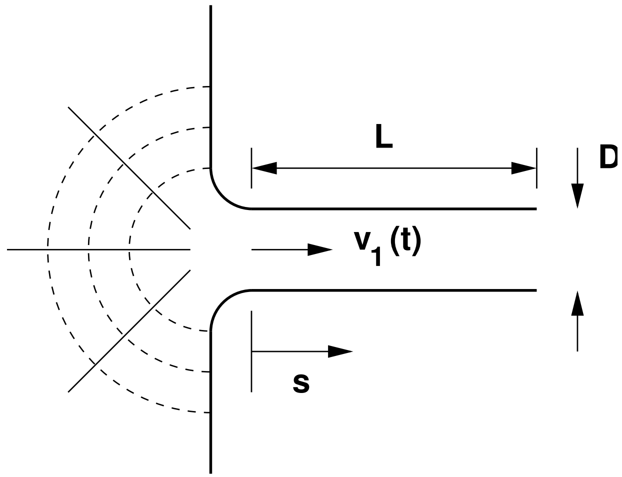
continuity equation: $v_1(t) A_1 = -\frac{dh}{dt} A_0$

$\longrightarrow$ differential equation for für $h(t)$

# Hydrodynamics: unsteady Bernoulli



"well rounded" inlet




assumption:

$$s < -\frac{D}{\sqrt{8}} : \text{ radial flow with } \dot{Q} = v \cdot \frac{\pi s^2}{2}$$

$$s \geq -\frac{D}{\sqrt{8}} : v = v_1$$

# Hydrodynamics: unsteady Bernoulli



Potential flow without any losses $\longrightarrow$ Bernoulligleichung

$$\int_{-\infty}^{L} \frac{\partial v}{\partial t} ds = \int_{-\infty}^{-D/\sqrt{8}} \frac{\partial v(s)}{\partial t} ds + \int_{-D/\sqrt{8}}^{L} \frac{\partial v_1}{\partial t} ds = \int_{-\infty}^{-D/\sqrt{8}} \frac{\partial}{\partial t} \left( \frac{v_1 \pi \frac{D^2}{4}}{2\pi s^2} \right) ds + \int_{-D/\sqrt{8}}^{L} \frac{\partial v_1}{\partial t} ds$$

$$= \frac{dv_1(t)}{dt} \int_{-\infty}^{-D/\sqrt{8}} \frac{D^2}{8s^2} ds + \frac{dv_1(t)}{dt} \int_{-D/\sqrt{8}}^{L} ds$$

$$= \left( \frac{D}{\sqrt{8}} + L + \frac{D}{\sqrt{8}} \right) \cdot \frac{dv_1(t)}{dt} = \underbrace{\left( \frac{D}{\sqrt{2}} + L \right) \frac{dv_1(t)}{dt}}_{\text{well rounde inlet}}$$

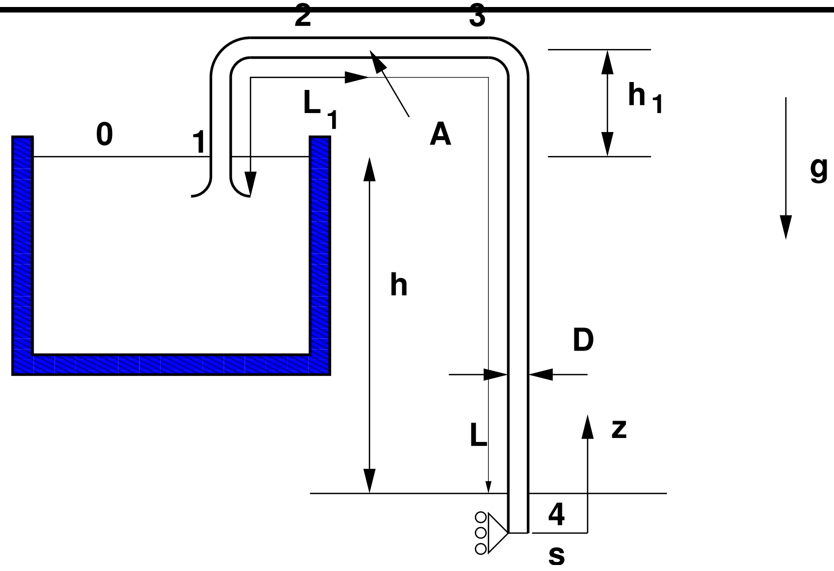
if $L \gg D \longrightarrow \int_{-\infty}^{L} \frac{\partial v}{\partial t} ds = L \cdot \frac{dv_1(t)}{dt}$

# Example: duct from a big tank

$$L = 20m \gg D,\ L_1 = 5m$$

$$h = 5m$$

$$\rho = 10^3 \frac{kg}{m^3},\ g = 10\frac{m}{s^2}$$




a) At what time after the immediately opening of the flap the flow reaches 99 % of its final value?

b) At that time, what is the difference between the current pressure and the final pressre at point “A”?

# Example: duct from a big tank

a) Bernoulli: $p_a + \rho g(h+s) = p_a + \rho g s + \frac{\rho}{2} v_4^2 + \rho \int\limits_{s_0}^{s_4} \frac{\partial v}{\partial t} ds$

well rounded inlet: $\int\limits_{s_0}^{s_4} \frac{\partial v}{\partial t} ds = L \frac{dv_4}{dt}$

$$\longrightarrow \quad \rho g h = \frac{\rho}{2} v_4^2 + L\rho \frac{dv_4}{dt} \qquad \longrightarrow \quad \int_0^T dt = L \frac{dv_4}{gh - \frac{2}{v4}^{2}}$$

Integration

$$T = 2L \int\limits_0^{0.99\sqrt{2gh}} \frac{dv_4}{2gh - v_4^2}$$

$$= \frac{L}{\sqrt{2gh}} \ln \left. \frac{\sqrt{2gh} + v_4}{\sqrt{2gh} - v_4} \right|_0^{0.99\sqrt{2gh}} = 10.6\, s$$

# Example (cont'd)

the accelerated initial flow is dependent of $L$,
but $v_4(t \to \infty)$ is independent of $L$.

b)

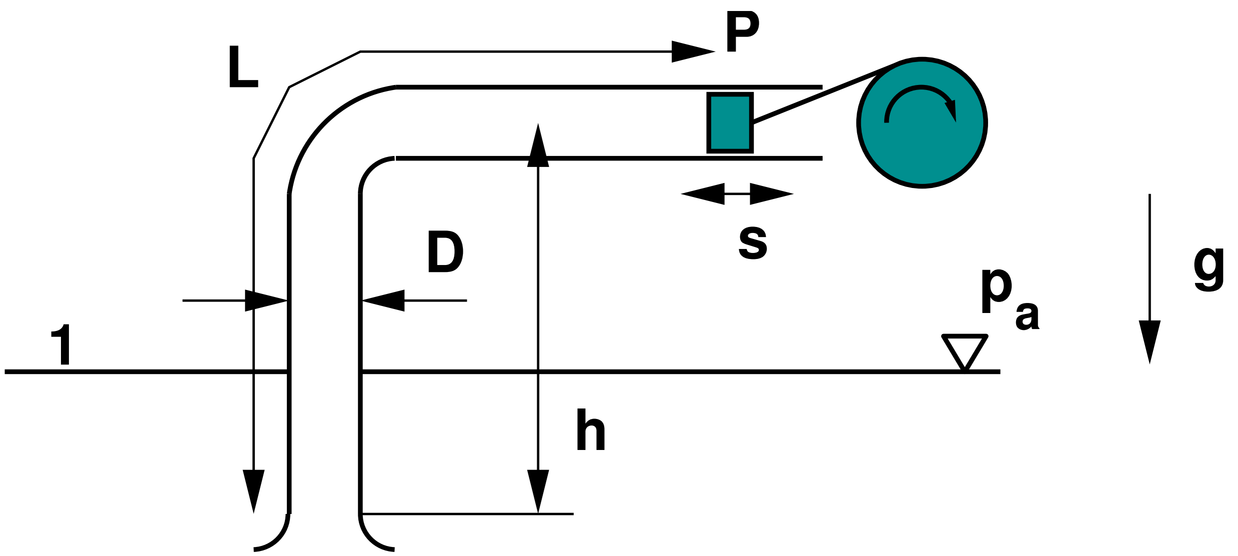
$$p_a = p_A + \rho g h_1 + \frac{\rho}{2} v_4^2 + L_1 \frac{dv_4}{dt}$$

$$t \to \infty : p_a = p_{A,\infty} + \rho g h_1 + \frac{\rho}{2} 2gh$$

from a) $\displaystyle \frac{dv_4}{dt} = \frac{1}{L}\left(gh - \frac{v_4^2}{2}\right)$

$$\Longrightarrow \qquad p_A - p_{A,\infty} = \rho g h \left(1 - 0.99^2\right)\left(1 - \frac{L_1}{L}\right) = 746 \, \frac{N}{m^2}$$

# example: moving piston




A piston is moving in a duct: $s = s_0 \cdot \sin \omega t$

$$p_a = 1 \text{ bar} \;\; L = 10m \;\; \ll D \;\; h = 2m \qquad g = 10 \frac{m}{s^2}$$

$$s_0 = 0.1m \;\; \rho = 10^3 \frac{kg}{m^3} \qquad p_D = 2500 \frac{N}{m^2}$$

At what angular speed $\omega$ the pressure at the bottom of the piston reaches the vapour pressure $p_D$?

# example: moving piston

$$p_a = p_P + \rho g h + \frac{\rho}{2} v_P^2 + \rho \int_{s_1}^{s_P} \frac{\partial v}{\partial t} ds$$

$$s_0 \ll L \quad \longrightarrow \quad \int_{s_1}^{s_P} \frac{\partial v}{\partial t} ds = L \frac{d v_P}{dt}$$

$$p_P = p_a - \rho g h + \rho s_0 \omega^2 \left( L \sin \omega t - \frac{s_0}{2} \cos^2 \omega t \right)$$

$$p_{P,\text{min}} = p_D$$

$$p_D = p_{P,\text{min}} \text{ at } \cos \omega t = 0 \quad \longrightarrow \quad \frac{d p_P}{dt} = 0$$

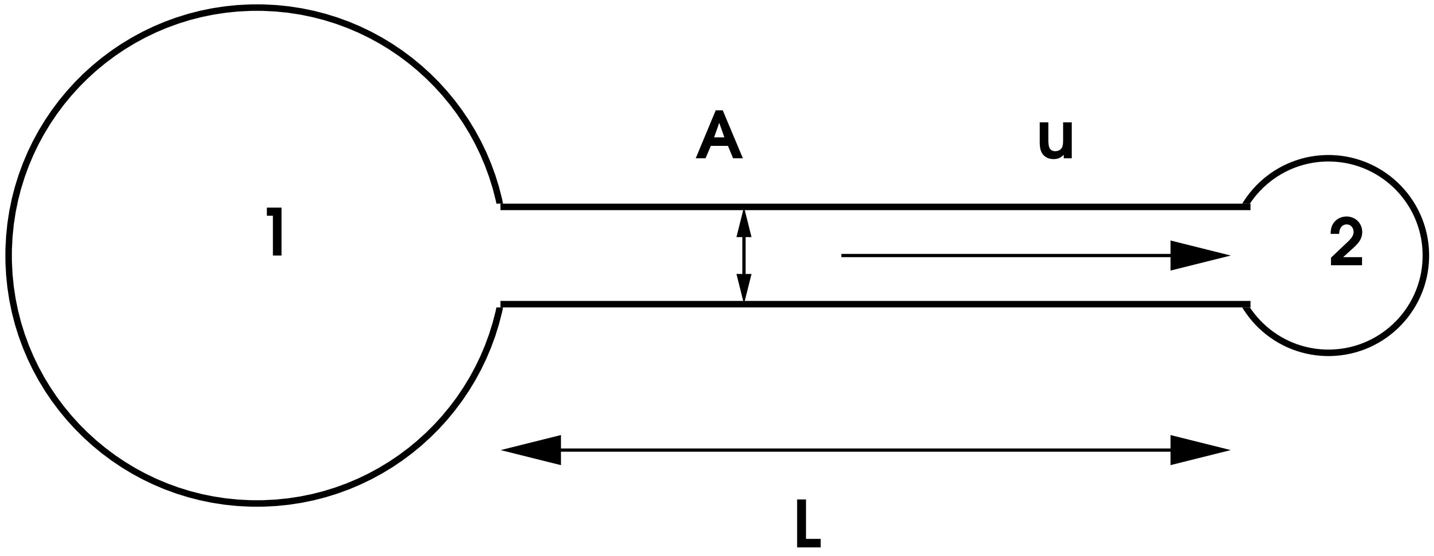
$$\Longrightarrow \quad \omega = \sqrt{\frac{p_a - p_D - \rho g h}{\rho s_0 L}} = 8.8 \ s^{-1}$$

# example: balloons

2 balloons are connected with a pipe of length $L$ and the cross-section $A$ The pressure in the balloons depends linearly on the balloon volume. $V_0$ is the volume at ambient pressure.

$$p = p_a + C(V - V_0)$$

At $t = 0$ one balloon is compressed by the volume $\Delta V$.



## example: balloons

a) Show, that the frictionless flow in the pipe is described by the equation of oscillation

$$\ddot{u} + K^2 u = 0.$$

Determine the eigenfrequency of the system.

b) Determine the maximum of the velocity in the pipe.

c) Determine the maximum of the pressure difference between the balloons.

Given: $\Delta V,\ L,\ A,\ \rho,\ C$

Hint:

- General Ansatz for the eqation of oscillations $\ddot{x} + a^2 x = 0$:

$$x = C_1 \sin(at) + C_2 \cos(at)$$

# example: balloons

a)

continuity for the balloons:

$$\dot{V}_1 = -uA \qquad \dot{V}_2 = uA$$

Bernoulli (unsteady, frictionless)

$$\rho L\dot{u} + p_2 - p_1 = 0 \qquad \Big|\Big(\frac{\partial}{\partial t}\Big)$$

$$\Longrightarrow \rho L\ddot{u} + \dot{p}_2 - \dot{p}_1 = 0 \qquad p = p_a + C(V - V_0) \Rightarrow \dot{p} = C\dot{V}$$

$$\Longrightarrow \rho L\ddot{u} + 2CAu = 0 \Rightarrow \ddot{u} + \frac{2CA}{\rho L}u = 0 \Rightarrow K = \sqrt{\frac{2CA}{\rho L}}$$

# example: balloons

b)

$$V_2(t) = V + 0 - \Delta V \cos(Kt) \Rightarrow \dot{V}_2(t) = \Delta V K \sin(Kt) = uA$$

$$\Rightarrow u(t) = \frac{\Delta V K}{A} \sin(Kt) \Longrightarrow u_{max} = \frac{\Delta V K}{A} = \Delta V \sqrt{\frac{2C}{\rho A L}}$$

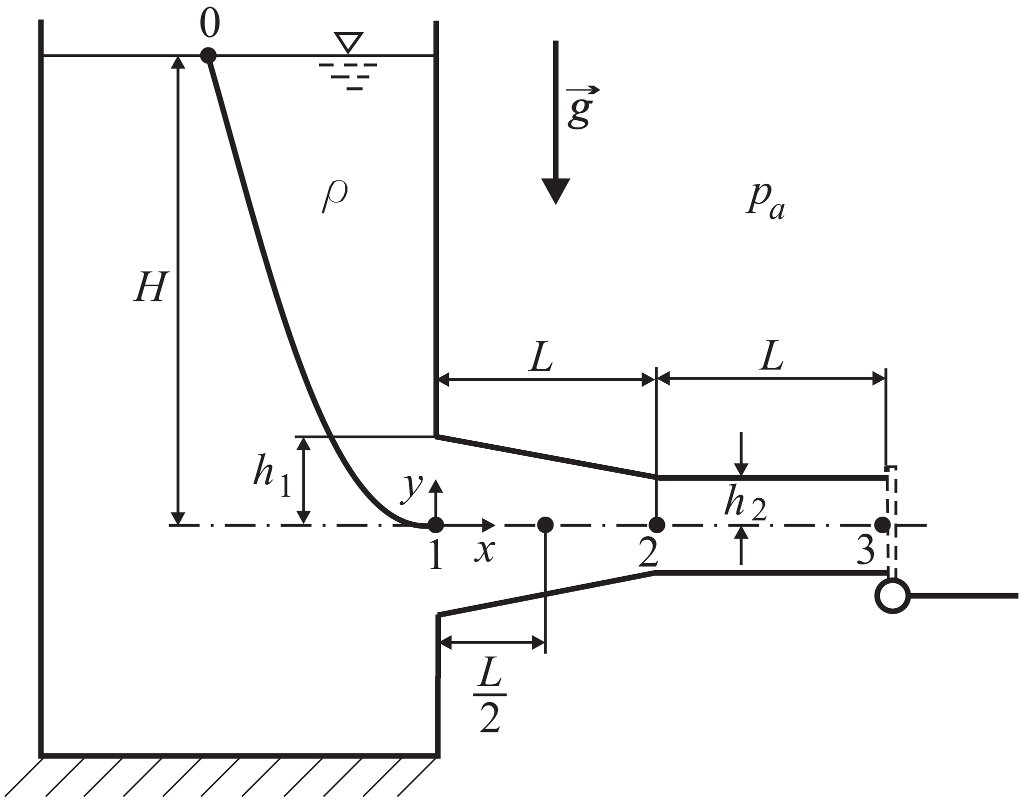
c)

$$|p_2 - p_1|_{max} = |\rho L \dot{u}|_{max}$$

$$\dot{u} = \frac{\Delta V K^2}{A} \cos(Kt) \Rightarrow |p_2 - p_1|_{max} = \frac{\rho L \Delta V}{A} K^2 = 2C\Delta V$$

# 6.6

The flap at the exit of the water pipe (constant width $B$) of a large container is opened abruptly. The appearing flow is without any losses.




Given: $H,\ h_1,\ h_2,\ g,\ L \quad ; \quad L >> h_1$

# 6.6

Determine

a) the differential equation for the exit velocity $v_3$
b) - the local acceleration
- the convective acceleration
- the substantial acceleration

at $x = \frac{L}{2}$ when the exit velocity reaches half of its asymptotic final value!

Hint: The computation of $v(t)$ is not necessary for solving this problem.

# 6.6

Bernoulli from "0" to "3"

$$p_a + \rho\, g\, H = p_3 + \frac{\rho}{2} v_3^2 + \int_0^3 \rho \frac{\partial v}{\partial t} ds \quad , \quad p_3 = p_a$$

Splitting of the integral

$$\int_0^1 \rho \frac{\partial v}{\partial t} ds \approx 0 (h_1 << L)$$

$$\int_1^2 \rho \frac{\partial v}{\partial t} ds \quad , \quad v = v_2 \frac{h_2}{h} \quad , \quad h = h_1 + \frac{h_2 - h_1}{L} x$$

$$\Rightarrow \rho \frac{dv_2}{dt} \int_1^2 \frac{h_2}{h_1 + \frac{h_2 - h_1}{L} x} dx = \rho \frac{dv_2}{dt} \frac{h_2\, L}{h_2 - h_1} \ln \frac{h_2}{h_1} = \rho \frac{dv_2}{dt} \overline{L}$$

# 6.6

$$\rho \int_2^3 \frac{\partial v}{\partial t}\, ds = \rho\, L\, \frac{dv_2}{dt}$$

introduce in Bernoulli

$$p_a + \rho\, g\, H = p_a + \frac{\rho}{2} v_3^2 + \rho\, \frac{dv_3}{dt} (\bar{L} + L)$$

$$\frac{dv_3}{dt} = \frac{1}{\bar{L} + L} \left( g\, H - \frac{v_3^2}{2} \right)$$

$$t \to \infty : \qquad g\, H - \frac{1}{2} v_{3e}^2 = 0 \Longrightarrow v_{3e} = \sqrt{2\, g\, H}$$

local acceleration:

$$b_l = \frac{\partial v}{\partial t} = \frac{dv_3}{dt} \frac{h_2}{h} \,, \quad b_{l\,(v_3=\frac{1}{2} v_{3e},\, x=\frac{L}{2})} = \frac{1}{\bar{L} + L}\, g\, H\, \frac{3}{4} \frac{2\, h_2}{h_1 + h_2}$$

# 6.6

convective acceleration:

$$v = v_2 \frac{h_2}{h} \qquad \frac{\partial v}{\partial x} = -v_2 h_2 \frac{1}{h^2}$$

$$b_k = v \frac{\partial v}{\partial x} = -v_3^2 \frac{h_2^2}{h^3} \frac{dh}{dx} \quad , \quad \frac{dh}{dx} = \frac{h_2 - h_1}{L}$$

$$b_{k(v_3 = \frac{1}{2} v_{3e},\ x = \frac{L}{2})} = 4 \frac{g\ H}{L} \frac{h_2^2\ (h_1 - h_2)}{(h_1 + h_2)^3}$$

substantial acceleration:

$$b_s = b_l + b_k = \frac{3}{2} \frac{g\ H}{L + L} \frac{h_2}{h_1 + h_2} + 4 \frac{g\ H}{L} \frac{h_2^2\ (h_1 - h_2)}{(h_1 + h_2)^3}$$