Biological & Medical Fluid Mechanics
01: Basic Equations

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Fluid Mechanics

- Fluid Mechanics is concerned with the behavior of fluids at rest and in motion

- A Fluid can be defined as a substance which can deform continuously when being subjected to shear stress at any magnitude.

- In other words, it can flow continuously as a result of shearing action. This includes any liquid or gas.

- A gas is a fluid that is easily compressed. It fills any vessel in which it is contained.

- A liquid is a fluid which is hard to compress. A given mass of liquid will occupy a fixed volume, irrespective of the size of the container.

- If a fluid is at rest, we know that the forces on it are in balance.

Fluid Mechanics

Hydrostatics, aerostatics | Aerodynamics, hydrodynamics

Compressible fluids

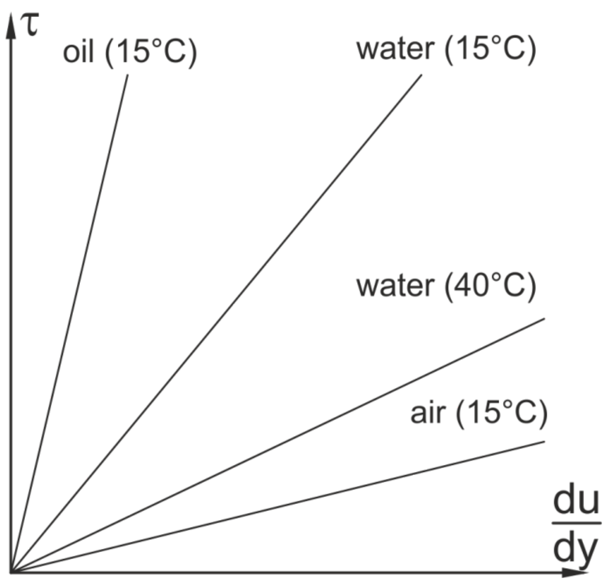
Gases, density ρ depends on p,T: $\rho = \rho(p,T)$

Incompressible fluids

Liquids, density ρ is constant

Newtonian fluids

Viscosity η is constant



Newtonian fluids:
Water, air, oil

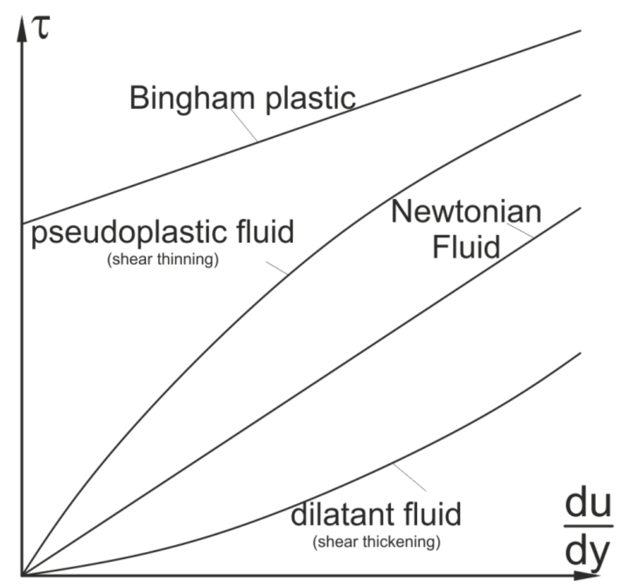
Bingham plastic:
tooth paste, mayonnaise

Dilatant fluids:
corn starch

Pseudoplastic fluids:
lava, ketchup,
whipped cream

Non-Newtonian fluids

Viscosity η depends on du/dy



Fluid Mechanics

- Quantities concerning the fluid
  - Density of the fluid ρ
  - Dynamic Viscosity η
  - Kinematic Viscosity $\nu = \eta / \rho$
  - Specific heat capacity c_p
- Quantities concerning the flow
  - Velocity field $\vec{v}(x, y, z, t)$
  - Static pressure p
  - Temperature T
  - Shear stress tensor $\bar{\bar{\tau}}$
- Tensors

Rank 0: scalar

$$p, \quad \rho, \quad T, \ldots$$

Rank 1: vektors

$$\vec{F}, \quad \vec{v}, \quad \vec{I}, \ldots$$

Rank 2: dyadic

$$\underbrace{\bar{\bar{\sigma}}}_{\text{shear stress sensor}} , \bar{\bar{\tau}}, \ldots$$

Vectors, tensors, calculation rules I

- Tensors
  - Rank 0: scalar   Rank 1: vector   Rank 2: dyadic

$$p, \quad \rho, \quad T, \ldots \qquad \vec{F}, \quad \vec{v}, \quad \vec{I}, \ldots \qquad \underbrace{\bar{\bar{\sigma}}}_{\text{stress tensor}}, \bar{\bar{\tau}}, \ldots$$

- Scalar – vector → vector

$$a\,\vec{b} = a \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a\,b_x \\ a\,b_y \\ a\,b_z \end{pmatrix} = \vec{c}$$

- Vector – vector → scalar (scalar product, dot product)

$$\vec{a} \cdot \vec{b} = (a_x, a_y, a_z) \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z = c$$

Vectors, tensors, calculation rules II

- Vector – vector → vector (curl, cross product)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{pmatrix} a_y b_z & - & a_z b_y \\ a_z b_x & - & a_x b_z \\ a_x b_y & - & a_y b_x \end{pmatrix} = \vec{c}$$

- Vector – vector → dyadic/second rank tensor

$$\vec{a}\vec{b} = (a_x, a_y, a_z) \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix} = \bar{\bar{c}}$$

- Vector – dyadic → vector

$$\vec{a} \cdot \bar{\bar{b}} = (a_x, a_y, a_z) \cdot \begin{pmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{pmatrix} = \begin{pmatrix} a_x b_{xx} + a_y b_{yx} + a_z b_{zx} \\ a_x b_{xy} + a_y b_{yy} + a_z b_{zy} \\ a_x b_{xz} + a_y b_{yz} + a_z b_{zz} \end{pmatrix}$$

Operators I

- Differential operators (in cartesian coordinates)
  - Nabla operator

$$\nabla = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

  - Laplacian operator

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Differential operations using the Nabla operator
  - Nabla operator – scalar → gradient

$$\operatorname{grad} p = \nabla p = \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix} = (\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z})^T$$

  - Nabla operator – vector → divergence

$$\operatorname{div} \vec{v} = \nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

- Nabla operator – vector → curl (cross product)

$$\mathrm{rot}\,\vec{v} = \nabla \times \vec{v} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}$$

- Differential operations using the Laplacian operator

$$\Delta p = \nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$

$$\Delta \vec{v} = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix}$$

- Second derivatives
  - Divergence of gradient

$$\mathrm{rot}\,(\mathrm{grad}\,a) = \nabla \times (\nabla a) = 0$$

  - Divergence of curl

$$\mathrm{div}\,(\mathrm{rot}\,\vec{v}) = \nabla \cdot (\nabla \times \vec{a}) = 0$$

  - Curl of curl

$$\vec{v} \times (\mathrm{rot}\,\vec{v}) = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \times \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix} = \frac{1}{2}\nabla \vec{v}^2 - (\vec{v} \cdot \nabla)\vec{v}$$

Derivatives

- Total derivative of a function $f(x, y, z)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

- The total derivative describes the increase of a function

  - Total derivative

$$\vec{v} = \vec{v}(t, x, y, z) \quad \rightarrow \quad d\vec{v} = \frac{\partial \vec{v}}{\partial t}dt + \frac{\partial \vec{v}}{\partial x}dx + \frac{\partial \vec{v}}{\partial y}dy + \frac{\partial \vec{v}}{\partial z}dz \Big| : dt$$

  - Substantial derivative

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}\underbrace{\frac{dx}{dt}}_{u} + \frac{\partial \vec{v}}{\partial y}\underbrace{\frac{dy}{dt}}_{v} + \frac{\partial \vec{v}}{\partial z}\underbrace{\frac{dz}{dt}}_{w}$$

$$\underbrace{\frac{d\vec{v}}{dt}}_{\textbf{substantial}} = \overbrace{\frac{\partial \vec{v}}{\partial t}}^{\textbf{local}} + \underbrace{u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}}_{\textbf{convective}}$$

Basic equations

- Continuity equation

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla(\rho \vec{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

- Navier-Stokes equations

$$\frac{\partial}{\partial t}(\rho \vec{v}) + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla p - \nabla \overset{=}{\tau} + \rho \vec{g}$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} \right) = -\nabla p - \nabla \overset{=}{\tau} + \rho \vec{g}$$

$$\rho \frac{d\vec{v}}{dt} = -\nabla p - \nabla \overset{=}{\tau} + \rho \vec{g}$$

- Energy equation

$$\rho c_p \frac{dT}{dt} = -\nabla \vec{q} + \frac{dp}{dt} + \overset{=}{\tau} \cdot \nabla \vec{v}$$

- fluid-mechanical properties

  - Density of a fluid ρ
  - Dynamic viscosity η
  - Kinematic viscosity $\nu = \eta / \rho$
  - Specific heat capacity c_p
  - Velocity field $\vec{v}(x, y, z, t)$
  - Static pressure p
  - Temperature T
  - Stress tensor $\overset{=}{\tau}$

Simplifications for certain flow types

- Steady flow

$$\boxed{\frac{\partial}{\partial t} = 0 \quad \underline{\text{not}} \quad \frac{d}{dt} = 0}$$

- Incompressible flow

$$\boxed{\rho = \text{const.}}$$

- Symmetrical flow

$$\boxed{\frac{\partial}{\partial \theta} = 0}$$

- frictionless flow

$$\boxed{\eta = 0 \quad \nu = 0 \quad \lambda = 0}$$ (λ: heat conductivity)

- 2-Dimensional flow

$$\boxed{\frac{\partial}{\partial z} = 0 \quad w = 0}$$

(reduced number of equations)
(reduced number of derivatives)

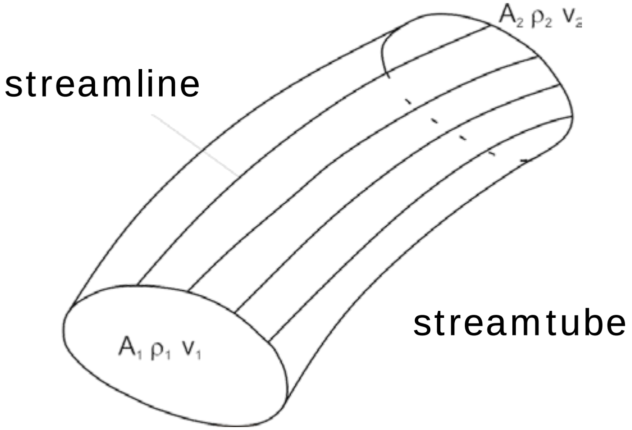
- fully developed flow

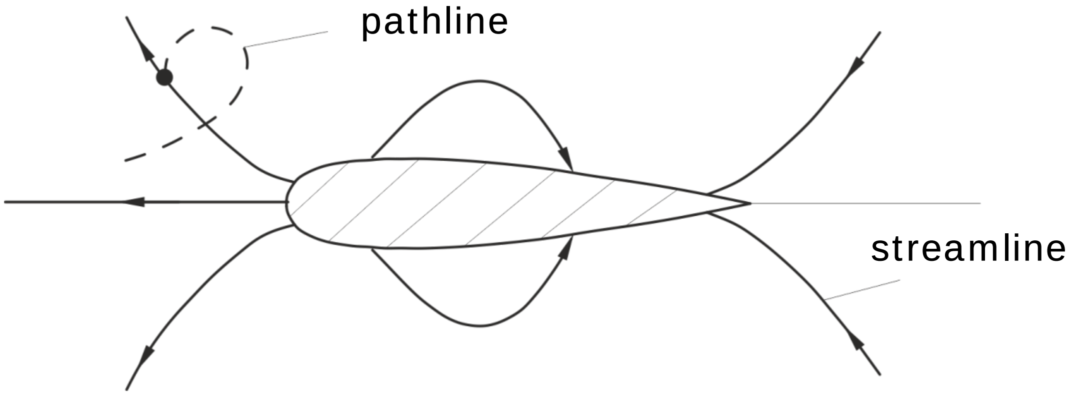
$$\boxed{\frac{\partial}{\partial x} = 0}$$

Basic quantities

- Basic quantities
  - Volume flux $$\frac{\partial V}{\partial t} = \dot{V} = Q = A \cdot v \qquad \left[\frac{m^3}{s}\right]$$
  - Mass flux $$Q \cdot \rho = \dot{m} \qquad \left[\frac{kg}{s}\right]$$
  - Momentum $$\dot{m} \cdot v = \dot{I} \qquad [N]$$
  - Kinetic energy $$\frac{1}{2}\dot{m} \cdot v^2 = \dot{E} \qquad [W]$$
- Simplified equations
  - Continuity $$\dot{m} = \rho \cdot v \cdot A \qquad \dot{m}_{in} = \dot{m}_{out}$$
  - Momentum $$\dot{I} = \dot{m} \cdot v \qquad \dot{I}_{in} - \dot{I}_{out} + \sum \vec{F} = 0$$
  - Energy $$\frac{1}{2}\rho \cdot v^2 + p + \rho \cdot g \cdot z = const.$$

- Streamlines and pathlines





- Steady and unsteady flow
  - Steady state flow: A flow is said to be in steady state if the flow field is only a function of position (x, y, z) but not of time t:

$$\frac{\partial}{\partial t} = 0 \qquad \vec{v} = \vec{v}\,(x, y, z)\,, \quad \rho = \rho\,(x, y, z)\,, \quad p = p\,(x, y, z)$$

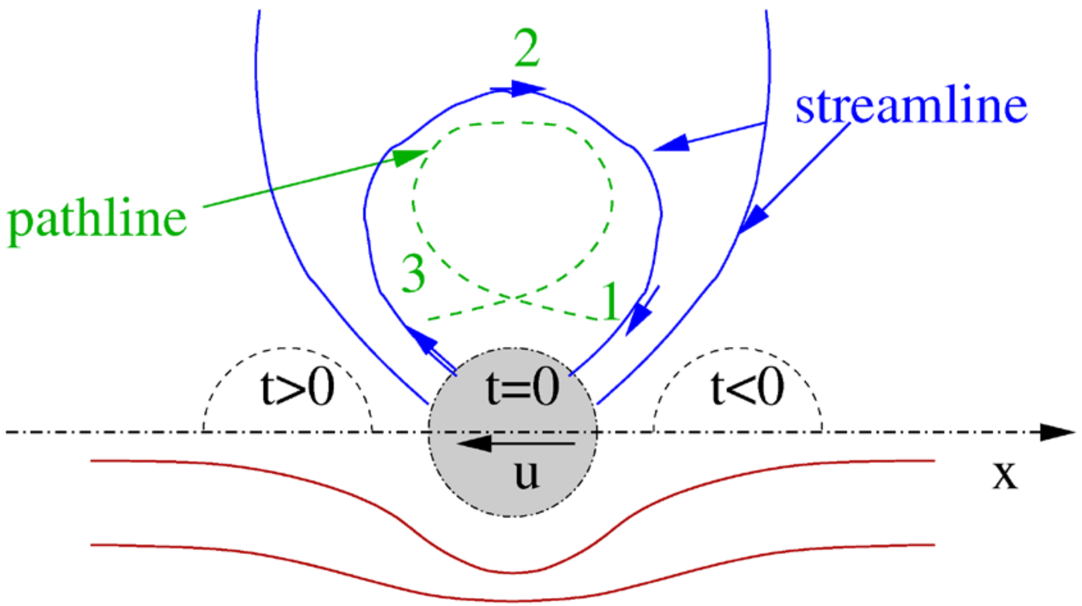
Examples: airplane at constant speed, pipe flow, most technical applications if the boundary conditions are independent of time or the changes in time are very slow (quasi-steady)

  - Unsteady flow: if the flow field is both a function of position (x, y, z) and time t, the flow is said to be unsteady:

$$\frac{\partial}{\partial t} \neq 0 \qquad \vec{v} = \vec{v}\,(t, x, y, z)\,, \quad \rho = \rho\,(t, x, y, z)\,, \quad p = p\,(t, x, y, z)$$

Examples: start-up procedures, flow in internal combustion engines, bird flight, beating heart,...

Streamlies and pathlines



- Streamlines: curves that are instantaneously tangent to the velocity
- Pathlines: trajectories that individual fluid particles follow → In steady flow, the streamlines and pathlines coincide
- Unsteady flow:
  pathline ≠ streamline
- Steady flow:
  pathline = streamline

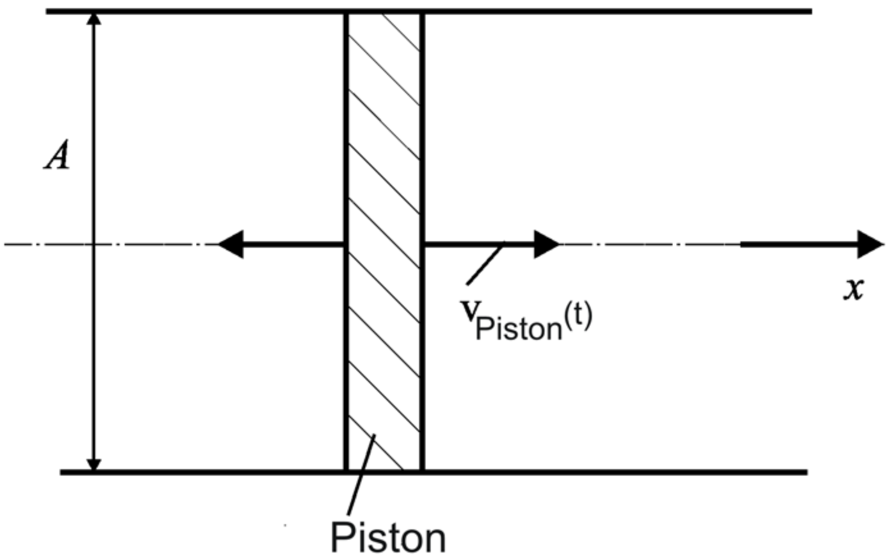
- Motionless environment, constant velocity u of the object
  - Unsteady flow for observer looking at the moving object
  - Steady flow for observer moving with the object

- Eulerian approach: analysis is performed by defining a control volume to represent a fluid domain which allows the fluid to flow across the volume. This approach is more suitable to be used in fluid mechanics.

- Lagrangian approach: analysis is performed by tracking down all motion parameters and deformation of a domain or particle as it moves. This approach is widely used for particle and solid mechanics.

Example I: task

- A piston is moving in a tube of infinite length and with constant cross section A with the velocity $v_{piston}(t)$. The density of the fluid is constant.



- Determine the substantial acceleration in the tube.

Example I: solution

- Substantial derivative

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$$

- Convective acceleration: continuity

$$A(x) \cdot v(x) = const. \quad \text{and} \quad A(x) = const. \quad \rightarrow v(x) = const. \rightarrow \frac{\partial v}{\partial x} = 0$$

- local acceleration

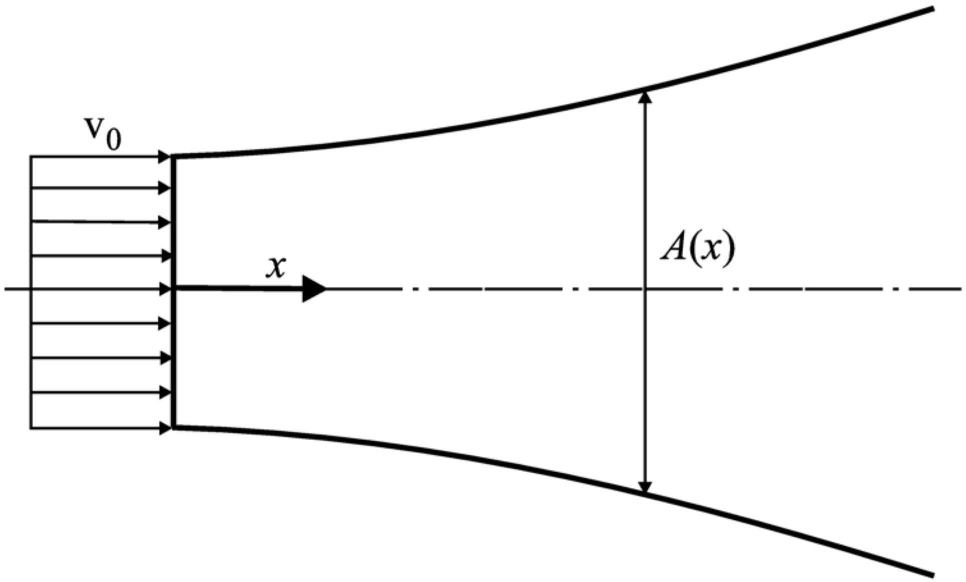
$$\frac{\partial v}{\partial t} = \frac{\partial v_{piston}}{\partial t}$$

- Hence: $\boxed{\frac{dv}{dt} = \frac{\partial v_{piston}}{\partial t}}$

→ only local acceleration

Example II: task

- A fluid of constant density flows into a diffuser with the constant velocity $v = v_0$. The cross section of the diffuser is $A(x)$.



- Determine the substantial acceleration of the fluid along the axis x.

Example II: solution

- Substantial derivate:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$$

- Constant inflow velocity:

$$v = v_0 \longrightarrow \frac{\partial v}{\partial t} = 0$$

- Continuity and 1st derivative:

$$A(x) \cdot v(x) = A_0 \cdot v_0 = const.$$

$$\frac{\partial A(x)}{\partial x} \cdot v(x) + A(x) \cdot \frac{\partial v(x)}{\partial x} = 0 \qquad \longrightarrow \qquad \frac{\partial v(x)}{\partial x} = -\frac{v(x)}{A(x)} \cdot \frac{\partial A(x)}{\partial x}$$

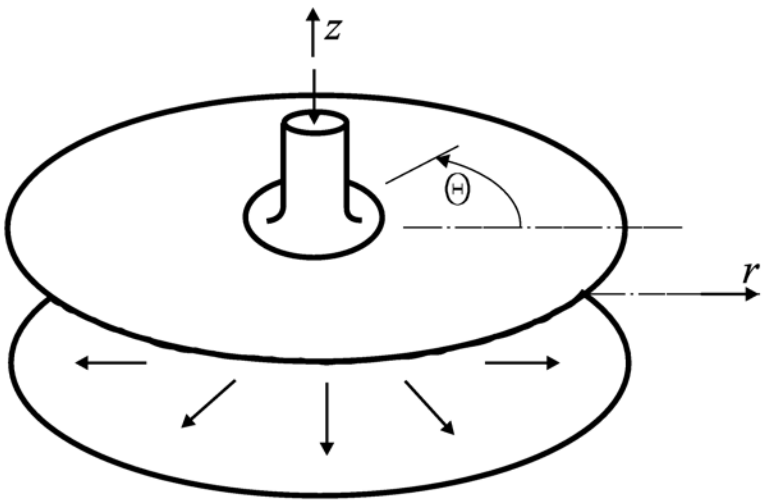
- Hence:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 + v(x) \cdot \left(-\frac{v(x)}{A(x)} \cdot \frac{\partial A(x)}{\partial x}\right)$$

$$v(x) = \frac{A_0 \cdot v_0}{A(x)} \qquad \boxed{\frac{dv}{dt} = -\frac{v_0^2 \cdot A_0^2}{A^3(x)}\frac{\partial A(x)}{\partial x}}$$

Example III: task

- An incompressible fluid with the viscosity η is flowing laminar and steady between two parallel plates. The flow is radial from inside to outside.



- The differential equations in cylindrical coordinates are:

$$\frac{1}{r}\,\frac{\partial(\rho\ r\ v_r)}{\partial r} + \frac{1}{r}\,\frac{\partial(\rho\ v_\Theta)}{\partial \Theta} + \frac{\partial(\rho\ v_z)}{\partial z} = 0$$

$$\rho\left(v_r\,\frac{\partial v_r}{\partial r} + \frac{v_\Theta}{r}\,\frac{\partial v_r}{\partial \Theta} - \frac{v_\Theta^2}{r} + v_z\,\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \eta\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial(r\ v_r)}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2}\,\frac{\partial^2\ v_r}{\partial \Theta^2} - \frac{2}{r^2}\,\frac{\partial v_\Theta}{\partial \Theta} + \frac{\partial^2\ v_r}{\partial z^2}\right)$$

- Simplify the equations for the flow problem described above.

Beispielaufgabe III: solution

- Continuity

$$\underbrace{\frac{1}{r}\frac{\partial \rho r v_r}{\partial r}}_{\rho=const.} + \underbrace{\frac{1}{r}\frac{\partial \rho v_\theta}{\partial \theta}}_{\text{radial flow}} + \underbrace{\frac{\partial \rho v_z}{\partial z}}_{\text{parallel plates}} = 0$$

$$\boxed{\frac{\partial r v_r}{\partial r} = 0}$$

- Radial momentum equation, left side

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \cancel{\frac{v_\Theta}{r}\frac{\partial v_r}{\partial \Theta}} - \cancel{\frac{v_\Theta^2}{r}} + \cancel{v_z\frac{\partial v_r}{\partial z}}\right)$$

$$v_\theta = 0 \quad v_\theta = 0 \quad v_z = 0$$

$$\frac{\partial}{\partial \theta} = 0$$

$$\boxed{\rho v_r\frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} + \eta\frac{\partial^2 v_r}{\partial z^2}}$$

- Radial momentum equation, right side

$$-\frac{\partial p}{\partial r} + \eta\left(\frac{\partial}{\partial r}\left(\cancel{\frac{1}{r}\frac{\partial (r v_r)}{\partial r}}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \Theta^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\Theta}{\partial \Theta}} + \frac{\partial^2 v_r}{\partial z^2}\right)$$

$$\frac{\partial r v_r}{\partial r} = 0 \quad \frac{\partial}{\partial \theta} = 0 \quad v_\theta = 0$$

$$\frac{\partial}{\partial \theta} = 0$$

Example IV: task

- The Navier-Stokes equations for unsteady, incompressible flows in a graviational field read:

$$\nabla \cdot \vec{v} \;=\; 0$$

$$\rho \frac{d\vec{v}}{dt} \;=\; -\nabla p \;+\; \eta \nabla^2 \vec{v} \;+\; \rho \vec{g}$$

- Formulate the equations for a steady, frictionless, two-dimensional flow in a cartesian coordinate system (x,y).

Example IV: solution

- Continuity:

$$\nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

- Momentum equation, $\eta=0$ (frictionless)

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \cancel{\eta \nabla^2 \vec{v}} + \rho \vec{g} \quad \rightarrow \quad \rho \frac{d\vec{v}}{dt} = -\nabla p + \rho \vec{g}$$

- Momentum equation, x-direction

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \rho g_x$$

$$\frac{du}{dt} = \rho \left(\cancel{\frac{\partial u}{\partial t}} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) \qquad \boxed{\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho g_x}$$

- Momentum equation, y-direction

$$\rho \frac{dv}{dt} = -\frac{\partial p}{\partial y} + \rho g_y$$

$$\frac{dv}{dt} = \rho \left(\cancel{\frac{\partial v}{\partial t}} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) \qquad \boxed{\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \rho g_y}$$

Example V: task

- The continuity equation and the Navier-Stokes equations for two-dimensional flows read:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0$$

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla p - \nabla \overset{=}{\tau} + \rho \vec{g}$$

$$\overset{=}{\tau} = \begin{pmatrix} -2\eta\frac{\partial u}{\partial x} + \frac{3}{2}\eta(\nabla \cdot \vec{v}) & -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & -2\eta\frac{\partial v}{\partial y} + \frac{3}{2}\eta(\nabla \cdot \vec{v}) \end{pmatrix}$$

- The equations are to be simplified for:
  - Steady flows,
  - Steady and incompressible flows,
  - Steady and incompressible flows with constant viscosity
  - Steady, incompressible, and frictionless flows.

Example V: solution

- Continuity:

$$\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0 \qquad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} = 0$$

  - Steady flow:

$$\frac{\partial}{\partial t} = 0 \rightarrow \quad u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \boxed{\nabla \cdot (\rho\vec{v}) = 0}$$

  - Steady and incompressible flow:

$$\frac{\partial}{\partial t} = 0 \qquad \rho = const.$$

$$\nabla \cdot (\rho\vec{v}) = 0 \qquad \rho \cdot \nabla \cdot \vec{v} = 0 \qquad \nabla \cdot \vec{v} = 0 \qquad \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

  - Steady and incompressible flow with constant viscosity (also $\eta = 0$ / frictionless):

$$\frac{\partial}{\partial t} = 0 \qquad \rho = const. \qquad \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

Example V: solution

- Momentum equation:

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla p - \nabla \overline{\overline{\tau}} + \rho \vec{g}$$

$$\overline{\overline{\tau}} = \begin{pmatrix} -2\eta\frac{\partial u}{\partial x} + \frac{3}{2}\eta(\nabla \cdot \vec{v}) & -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & -2\eta\frac{\partial v}{\partial y} + \frac{3}{2}\eta(\nabla \cdot \vec{v}) \end{pmatrix}$$

  - Steady flow:

$$\frac{\partial}{\partial t} = 0 \qquad \boxed{\nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla p - \nabla \overline{\overline{\tau}} + \rho \vec{g}}$$

  - Steady and incompressible flow:

$$\frac{\partial}{\partial t} = 0 \qquad \rho = const.$$

Left side:

$$\nabla \cdot (\rho \vec{v}\vec{v}) = \rho \nabla \cdot (\vec{v}\vec{v}) = \rho \nabla \cdot \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix} = \rho \begin{pmatrix} \frac{\partial u^2}{\partial x} & + & \frac{\partial uv}{\partial y} \\ \frac{\partial uv}{\partial x} & + & \frac{\partial v^2}{\partial y} \end{pmatrix}$$

Example V: solution

- Steady and incompressible flow:

  Left side (cont'd)

$$\nabla \cdot (\rho \vec{v}\vec{v}) = \rho \nabla \cdot (\vec{v}\vec{v}) = \rho \nabla \cdot \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix} = \rho \begin{pmatrix} \frac{\partial u^2}{\partial x} & + & \frac{\partial uv}{\partial y} \\ \frac{\partial uv}{\partial x} & + & \frac{\partial v^2}{\partial y} \end{pmatrix}$$

$$\rho \begin{pmatrix} \frac{\partial u^2}{\partial x} & + & \frac{\partial uv}{\partial y} \\ \frac{\partial uv}{\partial x} & + & \frac{\partial v^2}{\partial y} \end{pmatrix} = \rho \begin{pmatrix} u(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ v(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{pmatrix} = \rho(\vec{v} \cdot \nabla) \cdot \vec{v}$$

$$\nabla \cdot (\rho \vec{v}\vec{v}) = \rho(\vec{v} \cdot \nabla) \cdot \vec{v}$$

Right side:

$$-\nabla p - \nabla \overline{\overline{\tau}} + \rho \vec{g} \qquad \overline{\overline{\tau}} = \begin{pmatrix} -2\eta\frac{\partial u}{\partial x} + \frac{3}{2}\eta(\nabla \cdot \vec{v}) & -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & -2\eta\frac{\partial v}{\partial y} + \frac{3}{2}\eta(\nabla \cdot \vec{v}) \end{pmatrix}$$

$$\nabla \cdot \vec{v} = 0 \qquad \overline{\overline{\tau}} = \begin{pmatrix} -2\eta\frac{\partial u}{\partial x} + & -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & -2\eta\frac{\partial v}{\partial y} \end{pmatrix}$$

Example V: solution

- Steady and incompressible flow:

$$\boxed{\rho(\vec{v}\cdot\nabla)\cdot\vec{v} = -\nabla p - \nabla\,\overline{\overline{\tau}} + \rho\vec{g}}$$

$$\overline{\overline{\tau}} = \begin{pmatrix} -2\eta\frac{\partial u}{\partial x} + & -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & -2\eta\frac{\partial v}{\partial y} \end{pmatrix}$$

- Steady and incompressible flow with constant viscosity (η = const.):

$$\frac{\partial}{\partial t} = 0 \qquad \rho = const. \qquad \eta = const.$$

Left side: $\rho(\vec{v}\cdot\nabla)\cdot\vec{v}$ (no changes)

Right side: $-\nabla p - \nabla\,\overline{\overline{\tau}} + \rho\vec{g}$

$$\nabla\,\overline{\overline{\tau}} = \eta\nabla \begin{pmatrix} -2\frac{\partial u}{\partial x} + & -(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ -(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & -2\frac{\partial v}{\partial y} \end{pmatrix} = -\eta \begin{pmatrix} 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ \frac{\partial}{\partial x}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + 2\frac{\partial^2 v}{\partial y^2} \end{pmatrix}$$

$$= -\eta \begin{pmatrix} 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} \end{pmatrix} = -\eta \begin{pmatrix} \frac{\partial}{\partial x}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial}{\partial y}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \end{pmatrix}$$

Example V: solution

- Steady and incompressible flow with constant viscosity (η = const.):

$$-\eta \begin{pmatrix} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \end{pmatrix} = -\eta \nabla^2 \cdot \vec{v} = -\eta \Delta \cdot \vec{v}$$

$$\boxed{\rho(\vec{v} \cdot \nabla) \cdot \vec{v} = -\nabla p + \eta \nabla^2 \cdot \vec{v} + \rho \vec{g}}$$

- Steady, incompressible and frictionless flow ($\eta=0$):

$$\frac{\partial}{\partial t} = 0 \qquad \rho = const. \qquad \eta = 0$$

$$\boxed{\rho(\vec{v} \cdot \nabla) \cdot \vec{v} = -\nabla p + + \rho \vec{g}}$$

Biological & Medical Fluid Mechanics
02: Hydrostatics

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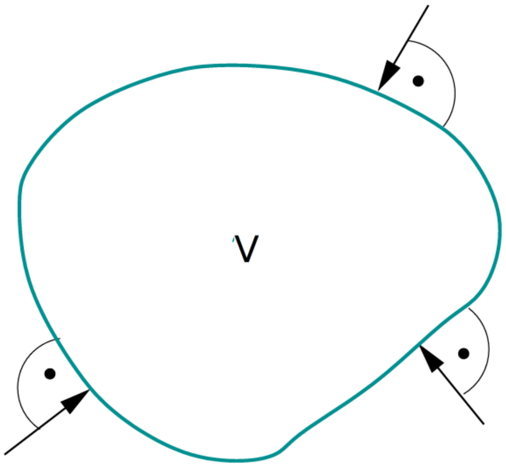
Definitions

- Hydrostatics: mechanics of fluids in static equilibrium / fluids at rest

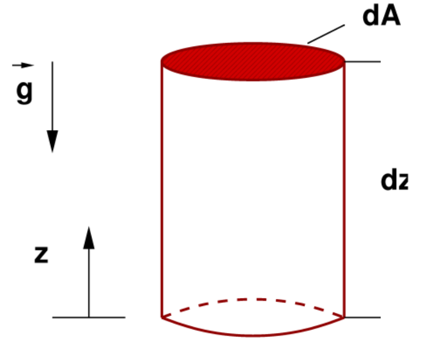
- Fluids: materials that are deformed due to shear stress
  → A fluid starts to move if a shear stress is applied
  → No shear stress in a fluid at rest

- Hydrostatics:
  - Fluids at rest are in stable equilibrium, the sum of all external forces equals zero
  - Fluid elements are not moving or are moving with constant velocity
  - Only normal stresses, no shear stresses
  - Normal stresses are pressures (no internal molecular forces), the pressure on a fluid at rest is isotropic



Basic hydrostatic equation

- Derivation of the basic hydrostatic equation:
  - All quantities (pressure p, density ρ,...) are functions of the coordinate z: p(z), ρ(z),...

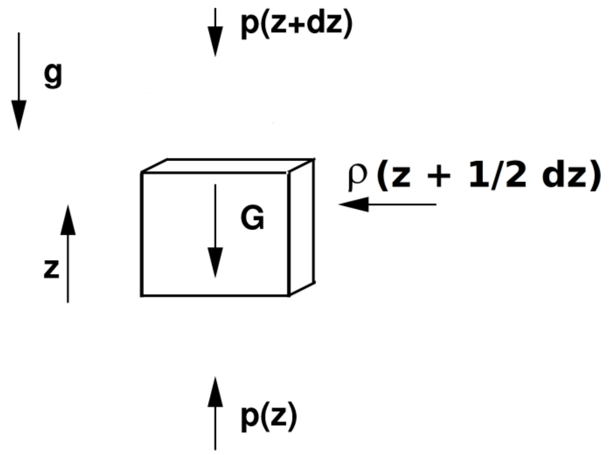


  - Force balance equation for a differential cube (Eulerian cube)

$$\sum F_z = 0$$

$$p(z)dA - p(z+dz)dA - G = 0$$

$$G = \rho(z + \frac{dz}{2})\, g\, dz\, dA$$



Basic hydrostatic equation

- Derivation of the basic hydrostatic equation:
  - Taylor series of p and ρ:

$$p(z+dz) = p(z) + \frac{dp}{dz}dz + \underbrace{\cancel{\frac{d^2p}{dz^2}\frac{dz^2}{2}}}_{\approx 0} + \cdots$$

$$\rho(z+\frac{dz}{2}) = \rho(z) + \frac{d\rho}{dz}\frac{dz}{2} + \underbrace{\cancel{\frac{d^2\rho}{dz^2}\frac{dz^2}{4}}}_{\approx 0} + \cdots$$

  - Hence:

$$p\,dA - \left(p + \frac{dp}{dz}dz - (\rho + \frac{d\rho}{dz}\frac{dz}{2})g\,dz\right)dA = 0$$

$$-\frac{dp}{dz}dz dA - \rho\,g\;dz\,dA - \underbrace{\frac{d\rho}{dz}\frac{dz^2}{2}\,g\;dA}_{\approx 0} = 0 \longrightarrow \boxed{\frac{dp}{dz} = -\rho\,g}$$

  - Integration for incompressible fluids in a constant gravitational field:

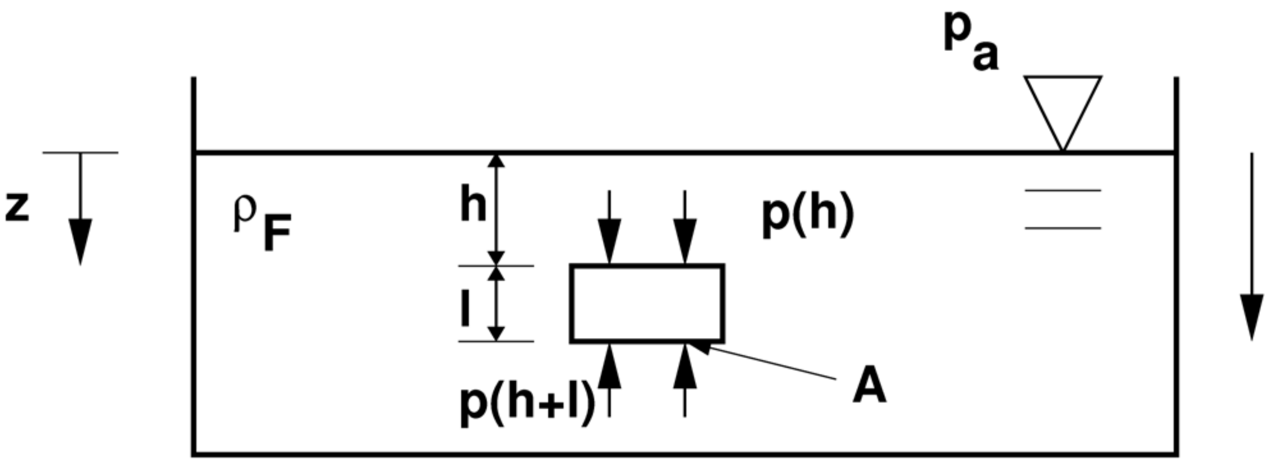
$$\rho = konst. \quad \vec{g} = konst.$$

$$\frac{dp}{dz} = -\rho\,g \quad \longrightarrow \quad dp = -\rho\,g dz \quad \longrightarrow \quad \boxed{p + \rho\,g z = konst.}$$

Basic hydrostatic equation

- Submerged objects that are either partly or completely below a free surface (liquid-gas interface) or within a completely full vessel experience a force that is equal to the weight of the fluid displaced by the object → buoyancy.

- Parallelepiped in a fluid with the density ρ_F



- Force F_p in z-direction: $$F_p = (p(h) - p(h+l))\,A$$

- Hydrostatic pressure: $$p(z) = p_a + \rho_F\, g z$$

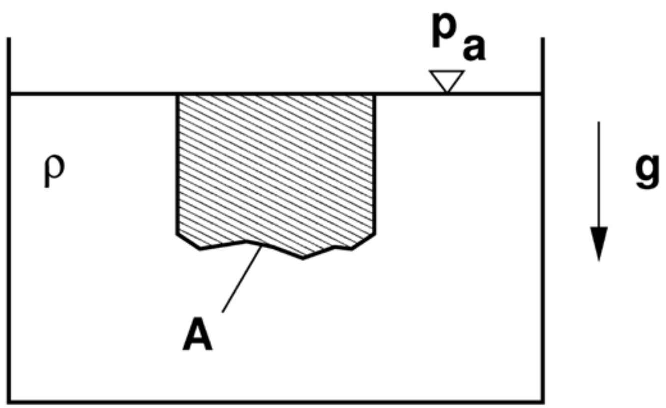
$$\longrightarrow F_p = (p_a + \rho_F g h - p_a - \rho_F g(h+l))\,A$$

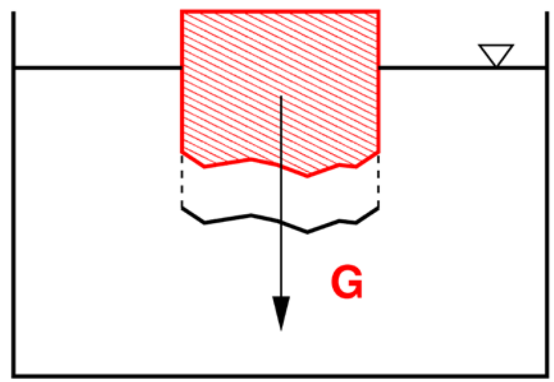
$$F_p = -\rho_F\, g \underbrace{lA}_{volume} = \boxed{-\rho_F\, g\, V = F_L} \text{ (ARCHIMEDES' PRINCIPLE)}$$

Stevin's principle

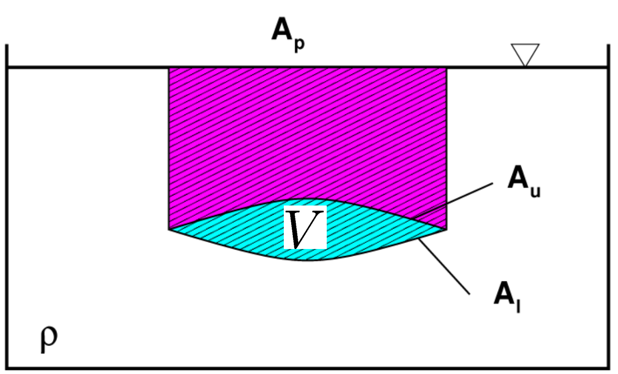
- The force on an arbitrary area A in the fluid corresponds to weight of the fluid column above the area and the outer pressure multiplied with the projected area.

$$F = G + p_a A$$





- Force on an object with the volume V



V_o

$V_u = V_o + V$

V

$$F_L = p_a A_p + \rho g V_o - p_a A_p - \rho g V_u$$

$$F_L = \rho g V_o - \rho g V_u = -\rho g (V_u - V_o)$$

$$\boxed{F_L = -\rho g V}$$

Basic hydrostatic equation

- Integration for compressible fluids
  - Assumption: perfect gas:

$$\rho = \frac{p}{RT}$$

  - Isothermal atmosphere:

$$T = T_0 = konst.$$

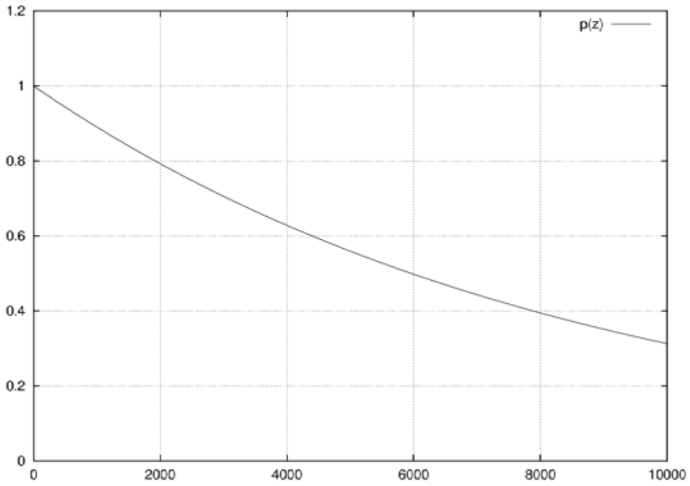
  - Hence:

$$\frac{dp}{dz} = -\rho\, g \rightarrow dp = -\rho(z)\, g\, dz = -\frac{p(z)}{RT} g\, dz$$

  - Integration:

$$\int_{p_0}^{p_1} \frac{dp}{p} = -\int_{z_0}^{z_1} \frac{g}{RT} dz$$

$$\ln p_1 - \ln p_0 = \ln \frac{p_1}{p_0} = -\frac{g(z_1 - z_0)}{RT_0}$$



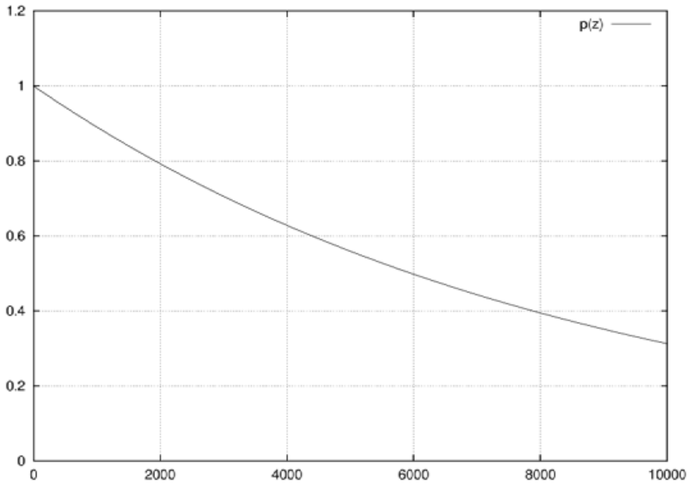
$$\boxed{p_1 = p_0\, e^{-\frac{g\Delta z}{RT_0}}}$$

Barometric formula

Balloon in atmosphere

- Atmosphere: perfect gas, density ρ depens on height z
  - Perfect gas:

$$\rho = \frac{p}{RT} \quad \text{and} \quad \rho = \rho(z)$$



  - Barometric formula:

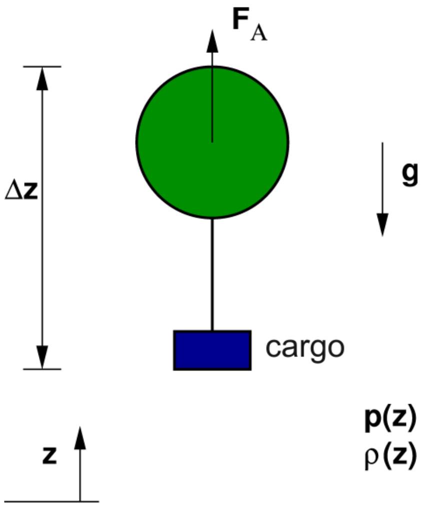
$$\frac{p_1}{p_0} = \frac{\rho_1}{\rho_0} = e^{-\frac{g\Delta z}{RT_0}}$$

  - Typical values:

$$\Delta z = 10m$$

$$T_0 = 290K$$

$$R_L = 288\frac{Nm}{kgK}$$



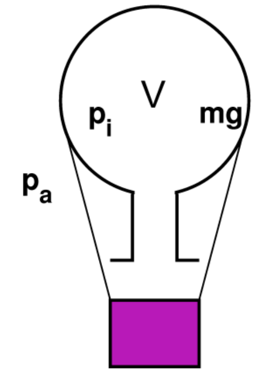
  - Change of the density across the height of the balloon:

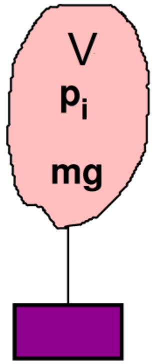
$$\frac{\rho(z+dz) - \rho(z)}{\rho(z)} = e^{-\frac{g\Delta z}{R_L T_0}} - 1 \approx 1.2 \cdot 10^{-3}$$

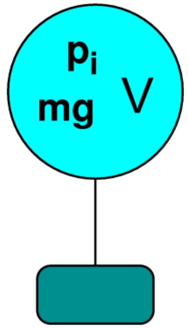
  - The change of the density across the height of the balloon is negligible

Balloon in atmosphere

- Different types of balloons
  - Rigid & open (hot-air balloon)
    - Open → pressure balance inside/outside $p_i = p_a$
    - Rigid → constant volume $V = konst.$
    - Open → loss of mass $m \neq konst.$
  - Perfectely loose & closed (weather balloon)
    - Perfectely loose → no forces across envelope $p_i = p_a$
    - Closed → no loss of mass $m = konst.$
    - Perfectely loose → volume change $V \neq konst.$
  - Rigid & closed (Zeppelin)
    - Closed → no pressure balance inside/outside $p_i \neq p_a$
    - Closed → no loss of mass $m = konst.$
    - Rigid → constant volume $V = konst.$

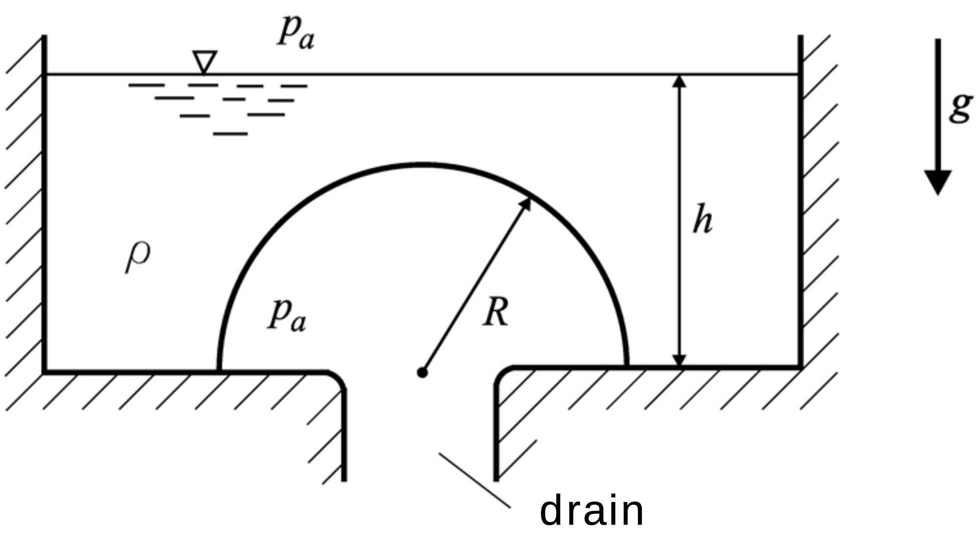






Example I: task

- A container is filled with a fluid of the density ρ. The drain of the container, filled up to a height h, is closed with a hollow hemisphere (radius R, weight G).



- Given: h, ρ, R, G, g
- Determine the necessary force F to open the drain.
- Hint: volume of a sphere: $V_s = \frac{4}{3}\pi R^3$

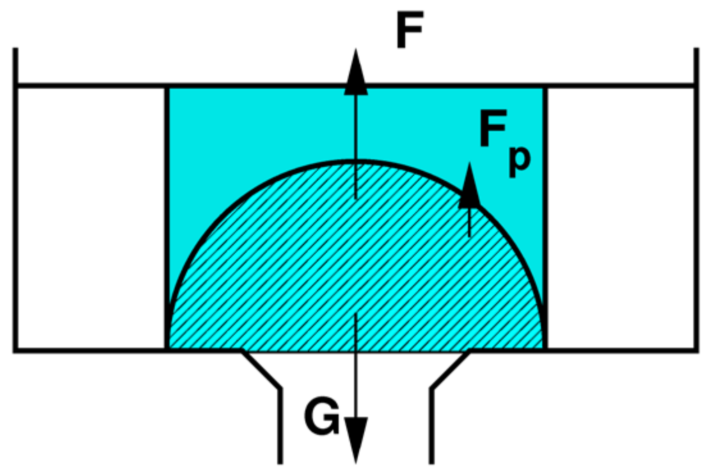
Example I: solution

- Force balance equation:

$$\sum F = 0$$

$$F - G + F_p = 0$$

$$F = G - F_p$$



- The hemisphere is not fully covered with fluid:

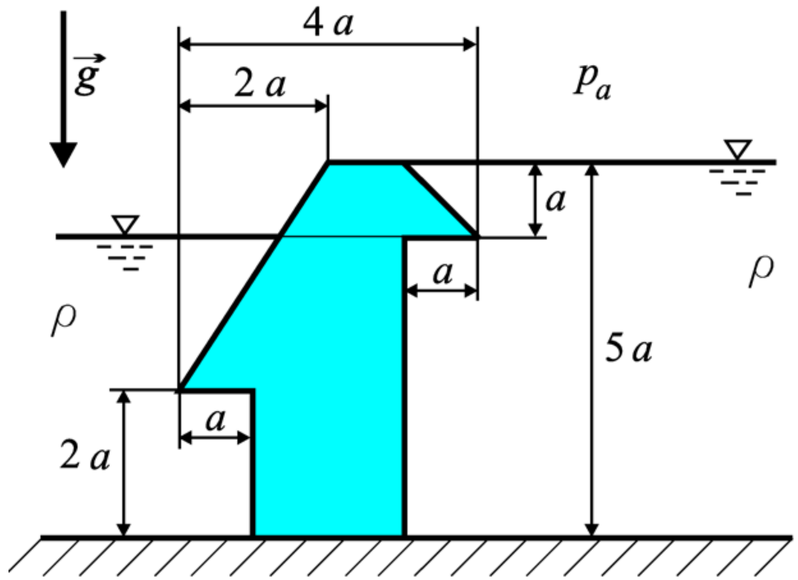
$$F_p \;=\; V_{hs}\,\rho_w\,g \;-\; \rho_w\,g\,h A_{hs}$$

$$F_p \;=\; \frac{1}{2}\,\frac{4}{3}\,\pi\,R^3\,\rho_w\,g \;-\; \rho_w\,g\,h\,\pi\,R^2$$

$$\boxed{F \;=\; G \;-\; \rho_w\,g\,\pi\,R^2\,(\frac{2}{3}\,R \;-\; h)}$$

Example II: task

- The sketched weir of length L seperates two basins of different depth.

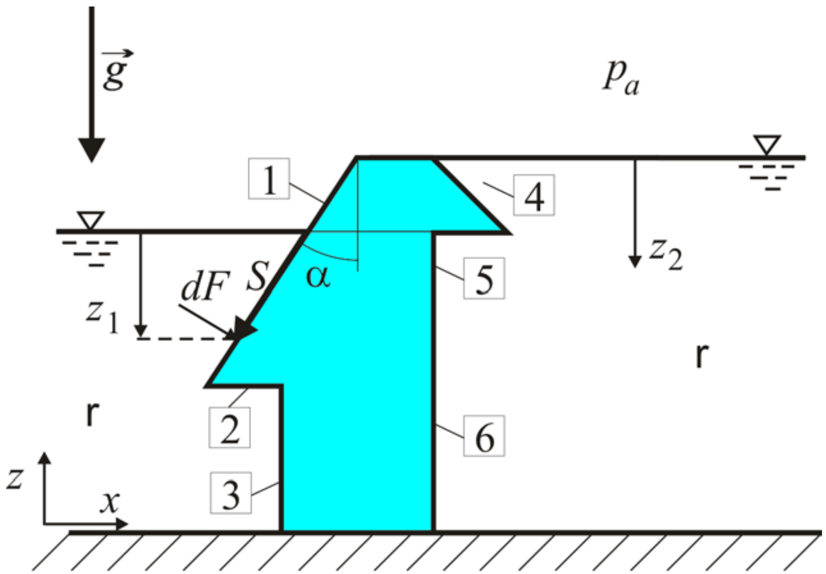


- Determine the force of the water on the weir.

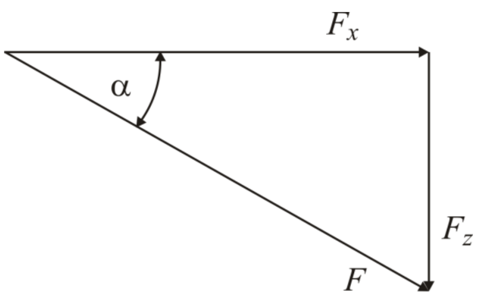
- Given: ρ, g, L, a, p_a

Example II: solution

- Surfaces:



- Surface 1:
  - Force on surface 1: $$F_1 = \int d\,F_1 = \int p\,(z_1) \cdot L \cdot ds$$
  - Coordinate transformation: $$s = \frac{z_1}{\cos\alpha} \quad ; \quad ds = \frac{d\,z_1}{\cos\alpha}$$
  - Forces in x- and z-direction:



$$F_{1x} = F_1 \cdot \cos\alpha \qquad F_{1z} = -F_1 \cdot \sin\alpha$$

Example II: solution

- Surface 1:
  - Force on surface 1 with $\tan\alpha = \dfrac{2}{3}$:

$$F_{1x} = \int_0^{2a} \cos\alpha p(z_1)\cdot L\frac{dz_1}{\cos\alpha} = \int_0^{2a} \rho g z_1 L dz_1 = 2\rho g a^2 L$$

$$F_{1z} = -\int_0^{2a} \sin\alpha p(z_1)\cdot L\frac{dz_1}{\cos\alpha} = -\int_0^{2a} \tan\alpha\cdot\rho g z_1 L dz_1 = -\frac{4}{3}\rho g a^2 L$$

- Surface 2:

$$F_{2x} = 0 \quad ; \quad F_{2z} = 2\rho g a^2 L$$

- Surface 3:

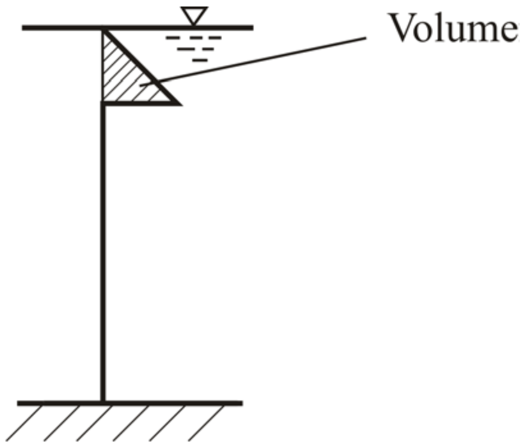
$$F_{3x} = +\int_{2a}^{4a} p(z_1)\cdot L dz_1 = +\int_{2a}^{4a} \rho g z_1 L dz_1 = 6\rho g a^2 L \qquad F_{3z} = 0$$

Example II: solution

- Surfaces 4-5-6:

$$F_{45z} = \frac{1}{2}\rho \cdot g \cdot a^2 \cdot L \qquad F_{6z} = 0$$

$$F_{456x} = -\frac{\rho g 5a}{2} \cdot 5a \cdot L = -\frac{25}{2}\rho g a^2 \cdot L$$



- Sum of all forces:

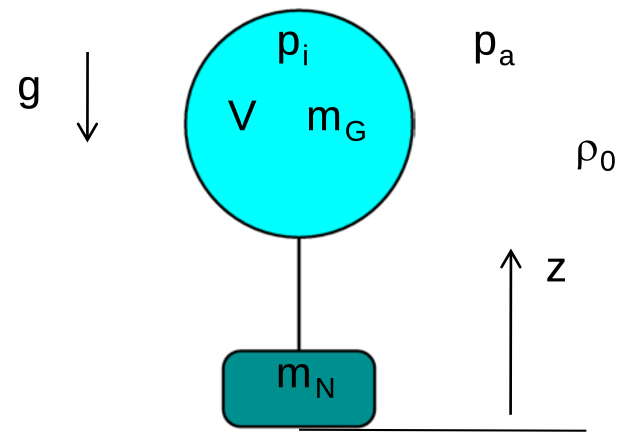
$$F_x = \sum_i F_{ix} = -\frac{9}{2}\rho g a^2 L$$

$$F_z = \sum_i F_{iz} = +\frac{7}{6}\rho g a^2 L$$

$$\boxed{F_{tot} = \sqrt{F_x^2 + F_z^2} = 4.65\rho g a^2 L}$$

Example III: task

- A rigid, closed balloon has a mass of m_N (including payload) and is filled with gas (mass m_G, Volume V, and pressure p_i). The volume V_N of the payload is negligible. The temperature of the gas (gas constant R_G) equals the temperature of the isothermal atmosphere (gas constant R_L, temperature T_0).



- Given: g, V, $V_N << V$, m_G, m_N, ρ_0, $T_i = T = T_0 = const.$, R_L, R_G
- Determine the ceiling $z_{max,1}$ of the balloon if the ballon must be tied to the ground at sea level (z=0).
- When the balloon has reached the ceiling, a hole is punched in the bottom of the envelope. Will the balloon rise or sink?
- Determine the new ceiling $z_{max,2}$ for $p_i > p_a(z_{max,1})$

Example III: solution

- Ceiling
  - Balance of forces, maximum height: $\sum F = 0 \qquad F_A - F_G - F_N = 0$
  - Lift: $F_A = \rho_L(z_{max,1}) g V$
  - Total weight: $F_G + F_N = (m_G + m_N) g$
  - Hence: $F_A = \rho_L(z_{max,1}) g V = (m_G + m_N) g = F_G + F_N$

$$\longrightarrow \rho_L(z_{max,1}) = \frac{(m_G + m_N)}{V}$$

  - Barometric formula for a compressible fluid, isothermal atmosphere:

$$\frac{p_L(z)}{p_0} = \frac{\rho_L(z)}{\rho_0} = e^{-\frac{gz}{R_L T_0}} \qquad \longrightarrow \qquad \rho_L(z) = \rho_0 \cdot e^{-\frac{gz}{R_L T_0}}$$

  - Thus:

$$\frac{(m_G + m_N)}{V} = \rho_0 \cdot e^{-\frac{g z_{max,1}}{R_L T_0}}$$

  - Finally:

$$\ln\left(\frac{m_G + m_N}{V \rho_0}\right) = -\frac{g z_{max,1}}{R_L T_0} \qquad \longrightarrow \qquad \boxed{z_{max,1} = \frac{R_L T_0}{g} \ln\left(\frac{V \rho_0}{m_G + m_N}\right)}$$

Example III: solution

- Will the balloon rise or sink?
  - Case 1: $p_i > p_a \rightarrow m_G$ decreases $\rightarrow z_{max}$ increases
  - Case 2: $p_i < p_a \rightarrow z_{max}$ decreases
- Ceiling $z_{max,2}$ for $p_i > p_a(z_{max,1})$, i.e., case 1:
  - The balloon rises and gas escapes from the balloon until a new equillibrium (pressure balance) is reached at $z_{max,2}$.
  - Balance of forces: $\sum F = 0 \quad \longrightarrow \quad F_A - F_{G,2} - F_N = 0$
  - Lift: $F_A = \rho_L(z_{max,2})gV$
  - Weight of the remaining gas: $F_{G,2} = \rho_{G,2}gV$

$$\rho_{G,2} = \frac{p_a}{R_G T_0} = \frac{R_L T_0 \rho_L(z_{max,2})}{R_G T_0} = \frac{R_L}{R_G}\rho_L(z_{max,2})$$

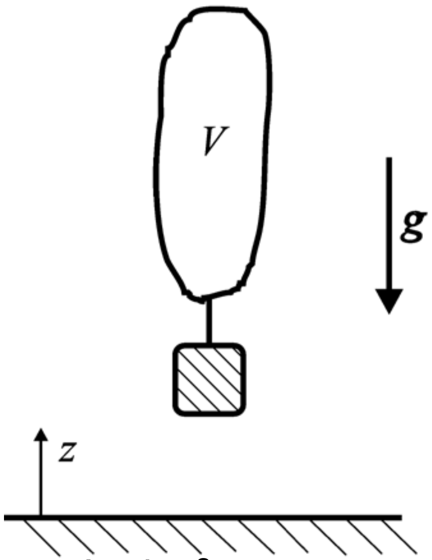
  - Hence:

$$\rho_L(z_{max,2})gV - \frac{R_L}{R_G}\rho_L(z_{max,2})gV - F_N = 0$$

$$\rho_L(z_{max,2}) = \rho_0 \cdot e^{-\frac{g z_{max,2}}{R_L T_0}} \quad \longrightarrow \quad \boxed{z_{max,2} = \frac{R_L T_0}{g} \ln\left(\frac{V\rho_0}{m_N} \cdot \frac{R_G - R_L}{R_G}\right)}$$

Example IV: task

- A weather balloon with the mass m and the initial volume V_0 ascends in an isothermal atmosphere. Its envelope is loose until the balloon reaches the maximal volume V_1.



- Given: $p_0 = 10^5\ N/m^2$, $\rho_0 = 1{,}27\ kg/m^3$, $m = 2{,}5\ kg$, $V_0 = 2{,}8\ m^3$, $V_1 = 10\ m^3$, $R = 287\ Nm/kgK$, $g = 10\ m/s^2$

- What is the necessary force to hold down the balloon before launch?
- At which altitude will the balloon reach its maximum volume V_1?
- Determine the ceiling of the balloon.

Example IV: solution

- Before start:

$$\sum F_z = 0$$

$$\sum F_z = F_A - F_G - F_N - F_H = 0$$

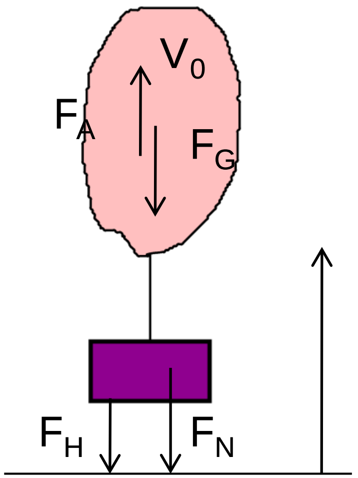
$$F_A = \rho_L(z=0)V_0 g$$

$$F_N = m_N g$$

$$F_G = m_G g$$

$$F_H = F_A - F_N - F_G = \rho_L(z=0)gV_0 - m_N g - m_G g$$

$$F_H = \left(\rho_L(z=0)V_0 - \underbrace{(m_N + m_G)}_{m}\right) g = 10.6N$$



- Altitude z_1 for maximum Volume V_1:
  - Envelope is perfectely loose and closed for $V < V_1$
    → no loss of mass, volume change

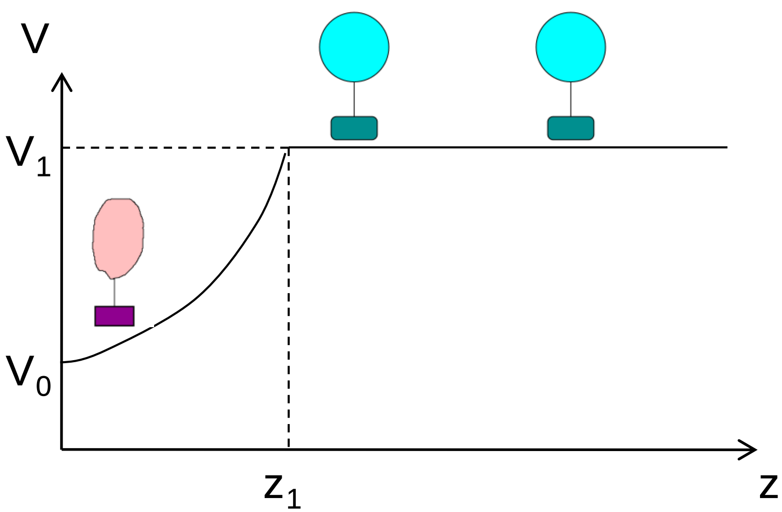
$$m_G = const. = \rho_G V = \frac{p_G}{R_G T_G} V \quad \text{with} \quad p_G = p_i = p_a$$

Example IV: solution

- The ballon rises very slowly: $T_i = T_a$
- Isothermal atmosphere → barometric formula

$$V = \frac{m_G R_G T_G}{p_G} \sim \frac{1}{p_G} = \frac{1}{p_L}$$

- Volume as function of heigt:



$$V = V_0 \cdot e^{\frac{g z}{R T_0}}$$

$$z = z_0 = 0 \quad \longrightarrow \quad V = V_0 \cdot e^{\frac{g z_0}{R T_0}} = V_0$$

$$z = z_1 \quad \longrightarrow \quad V_1 = V_0 \cdot e^{\frac{g z_1}{R T_0}}$$

$$\longrightarrow \quad z_1 = \frac{R_L T_0}{g} \ln \frac{V_1}{V_0} = \frac{p_0}{\rho_0 g} \ln \frac{V_1}{V_0}$$

$$\longrightarrow \quad \boxed{z_1 = 10.0 km}$$

Example IV: solution

- Ceiling:
  - $z < z_1$: with $p_L = p_G$ and $T_L = T_G$

$$z \leq z_1 \quad \longrightarrow \quad F_A = \rho_L V g = \frac{p_L}{R_L T_0} \frac{m_G R_G T_G g}{p_G} = const.$$

→ The lift force on a perfectly loose, closed balloon is constant.

$$F_A(z \leq z_1) = \rho_0 V_0 g = \rho_L(z_1) V_1 g = const.$$

  - $z > z_1$: $V = V_1 = const.$

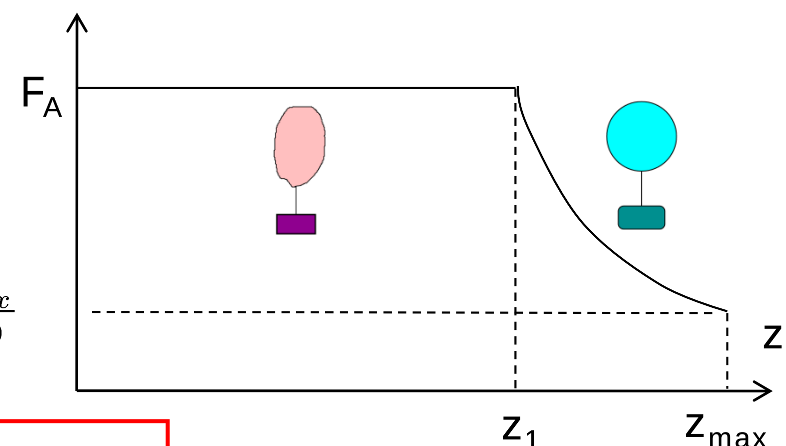
$$z > z_1 \quad \longrightarrow \quad F_A(z > z_1) = \rho_L(z) V_1 g = F_A(z \leq z_1) \cdot \frac{\rho_L(z)}{\rho_L(z_1)}$$

$$F_A(z > z_1) = F_A(z \leq z_1) \cdot e^{-\frac{g(z - z_1)}{R_L T_0}}$$

  - Ceiling:

$$\sum F_z = 0$$

$$mg = \rho(z_{max}) V_1 g \qquad \frac{m}{V_1} = \rho_0 \cdot e^{-\frac{g z_{max}}{R_L T_0}}$$



$$\boxed{z_{max} = \frac{R_L T_0}{g} \ln \frac{V_1 \rho_0}{m} = \frac{p_0}{\rho_0 g} \ln \frac{V_1 \rho_0}{m} = 12.8 km}$$

Biological & Medical Fluid Mechanics
03: Continuity equation & Bernoulli equation

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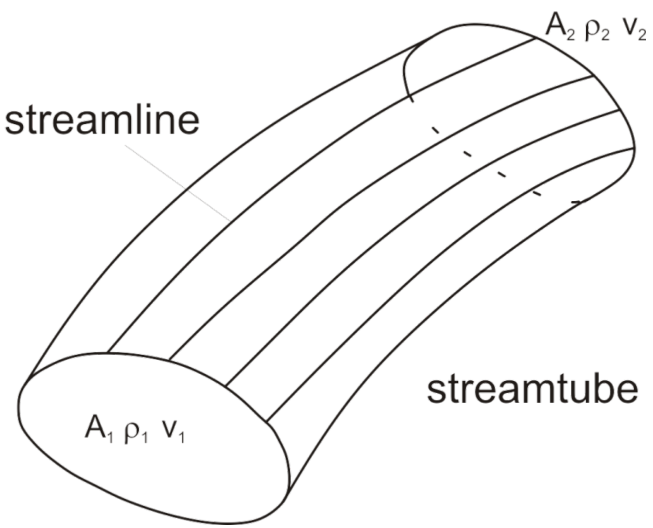
Continuity equation

- Continuity equation = conservation of mass/conservation of volume flux:



- Conservation of mass/mass flux:

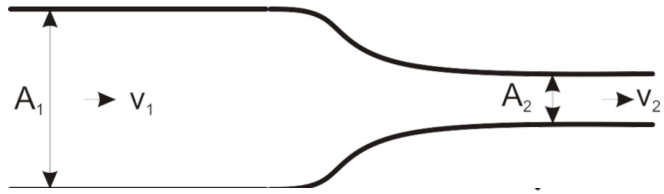
$$\underbrace{\rho_1 v_1 A_1}_{\dot{m}_1} = \underbrace{\rho_2 v_2 A_2}_{\dot{m}_2}$$

- Conservation of volume flux for an incompressible fluid:

$$\rho_1 = \rho_2 = const. \quad \longrightarrow \quad \underbrace{v_1 A_1}_{\dot{V}_1} = \underbrace{v_2 A_2}_{\dot{V}_2}$$

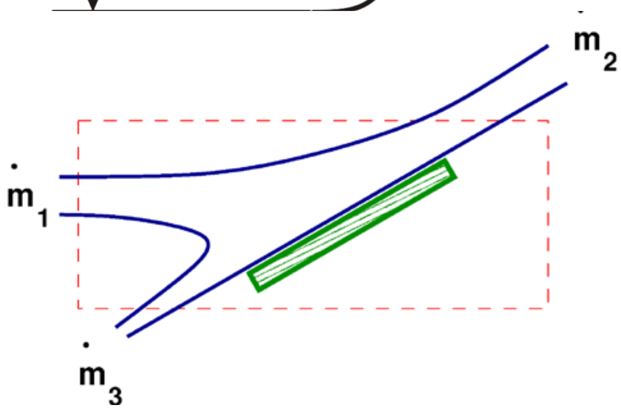
Continuity equation

- Examples:
  - Pipe flow:



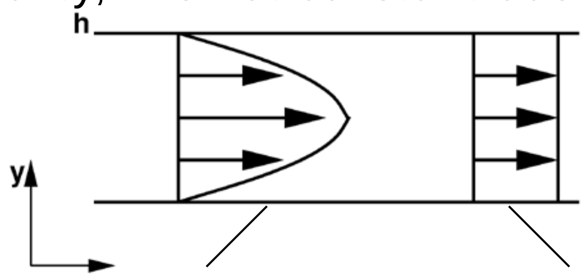
$$v_2 = v_1 \cdot \frac{A_1}{A_2}$$

  - Water Jet:



$$\dot{m}_1 = \dot{m}_2 + \dot{m}_3$$

- The one-dimensional continuity equation contains an average value of the velocity. In reality, v is not constant due to friction, vortices,...



Reality: $\vec{v} = \vec{v}(y)$ One-dimensional continuity equation: $\vec{v} = const.(\neq \vec{v}(y))$

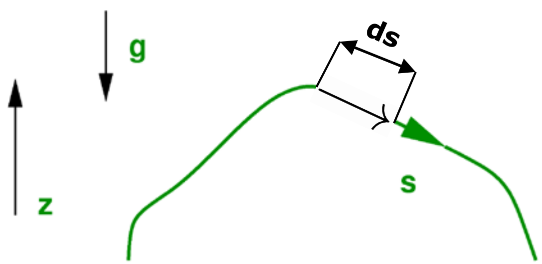
Constant mass flux: $\int \rho v(y)\, dy \;=\; \rho \bar{v} h$

Bernoulli equation: derivation

- Derivation of the Bernoulli equation:
  - 2nd Newtonian law: Mass × acceleration = sum of outer forces

$$m \cdot \frac{d\vec{v}}{dt} = \sum F_a$$

  - Equation of motion for an infinitesimal element along a streamline:



$$\underbrace{\rho \frac{d\vec{v}}{dt}}_{\text{inertia}} = -\underbrace{\frac{\partial p}{\partial s}}_{\text{pressure}} - \underbrace{\rho g \frac{dz}{ds}}_{\text{gravitation}} - \underbrace{R'}_{\text{friction}}$$

  - Velocity along a streamline: $v = v(s,t)$

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + \frac{\partial \vec{v}}{\partial s} ds \longrightarrow \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{ds}{dt}\frac{\partial \vec{v}}{\partial s} = \frac{\partial \vec{v}}{\partial t} + v \frac{\partial \vec{v}}{\partial s}$$

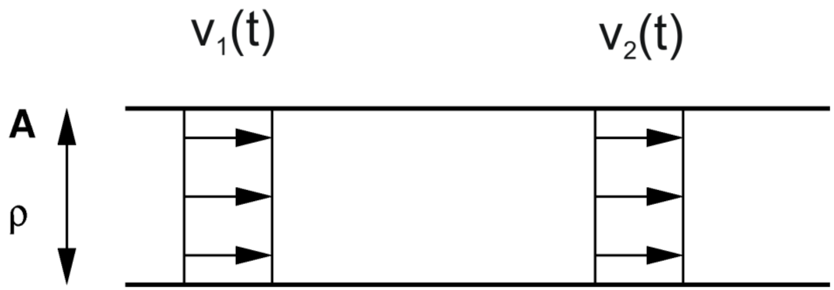
Total acceleration

Local acceleration

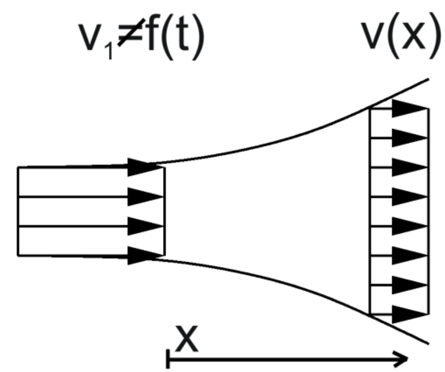
Convective acceleration

Bernoulli equation

- Pipe flow



- Diffuser



- Only local acceleration

$$A, \rho = const. \quad \longrightarrow$$

$$v_1(t) = v_2(t)$$

- Only convective acceleration

$$\rho = const. \quad \wedge \quad A \neq const. \quad \longrightarrow$$

$$v(x) = v_1 \cdot \frac{A_1}{A(x)}$$

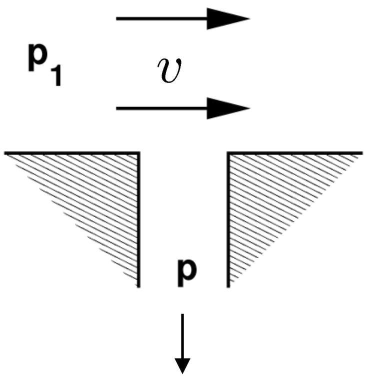
- Simplifications:
  - Incompressible fluid: $\rho = const.$
  - Frictionless flow: $R' = 0$
  - Steady flow: $\partial / \partial t = 0$
  - Constant gravity: $\vec{g} = const.$

$$\longrightarrow \quad \rho[\underbrace{\frac{\partial v}{\partial t}}_{=0} + v\frac{\partial v}{\partial s}] = -\frac{\partial p}{\partial s} - \rho g \frac{dz}{ds} - \underbrace{R'}_{=0}$$

$$\longrightarrow \quad \frac{\partial}{\partial s} \Rightarrow \frac{d}{ds} \quad \longrightarrow \quad \frac{1}{2}\rho\frac{dv^2}{ds} = -\frac{dp}{ds} - \rho g\frac{dz}{ds} \longrightarrow \boxed{\frac{\rho}{2}v^2 + p + \rho g z = const.}$$

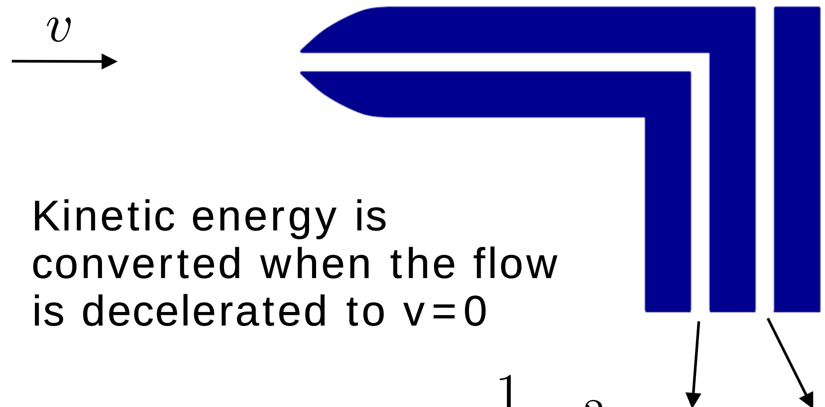
Different types of pressure

- Static pressure $p_s, p, p_a, p_1, p_\infty, \ldots$



$p, p_a, p_1, p_\infty, \ldots$

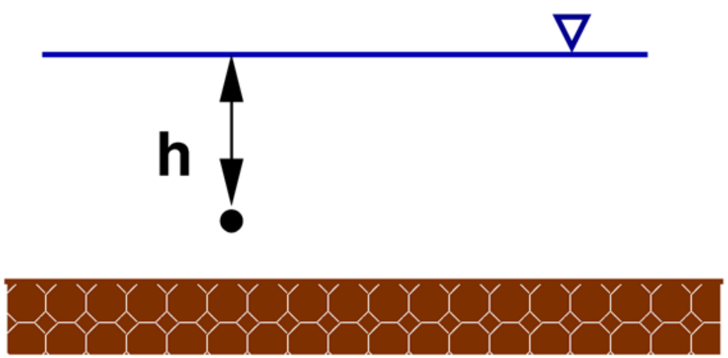
- Dynamic pressure (Prandtl tube)



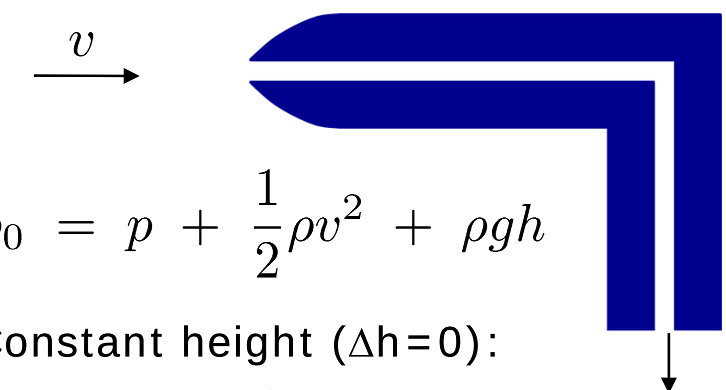
Kinetic energy is converted when the flow is decelerated to v=0

$$p_{dyn} = \frac{1}{2}\rho v^2 = p_0 - p_s$$

- Potential pressure $p_{pot} = \rho g h$



- Total pressure (Pitot tube) $p_0, p_t, p_{tot} \ldots$



$$p_0 = p + \frac{1}{2}\rho v^2 + \rho g h$$

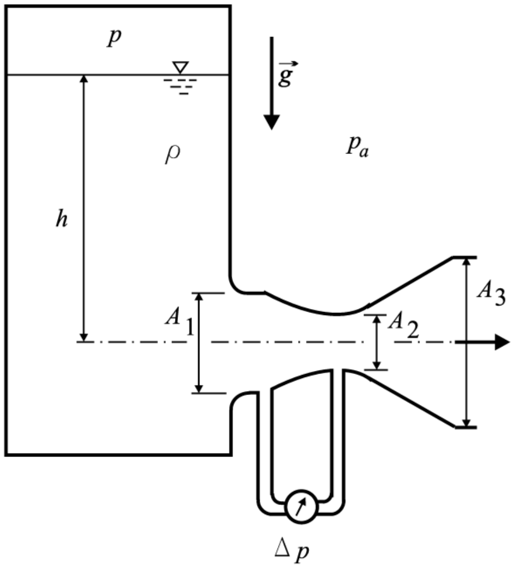
Constant height (Δh=0):

$$p_0 = p + \frac{1}{2}\rho v^2$$

p_0

Example 1: Task

- Water flows from a large pressurized tank into the open air. The pressure difference Δp is measured between A_1 and A_2



- Given: $A_1 = 0,3\ m^2, \quad A_2 = 0,1\ m^2, \quad A_3 = 0,2\ m^2, \quad h = 1\ m$

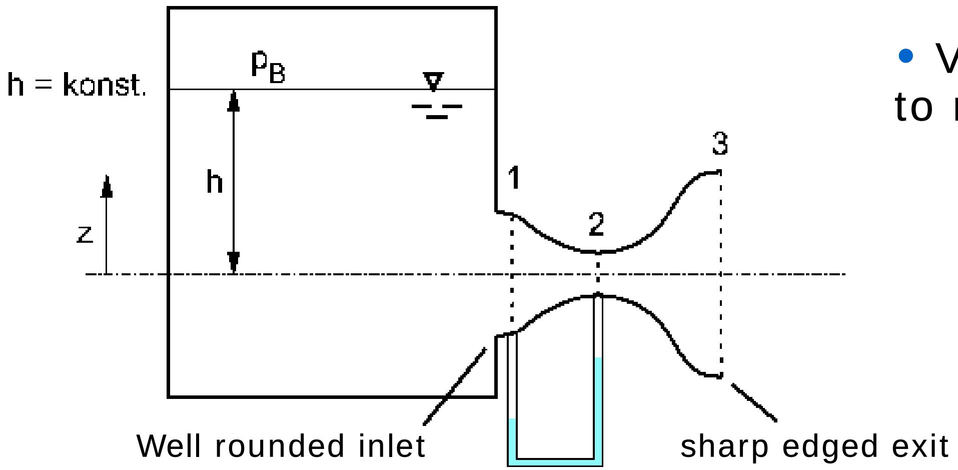
$$\rho = 10^3\ kg/m^3, \quad p_a = 10^5\ N/m^2, \quad \Delta p = 0,64 \cdot 10^5\ N/m^2$$

$$g = 10\ m/s^2$$

- Compute the velocities v_1, v_2, and v_3
- Determine the pressures p_1, p_2, and p_3 and the pressure p_B above the surface.

Example 1: solution

- Pressurized tank with well rounded inlet and sharp outlet:

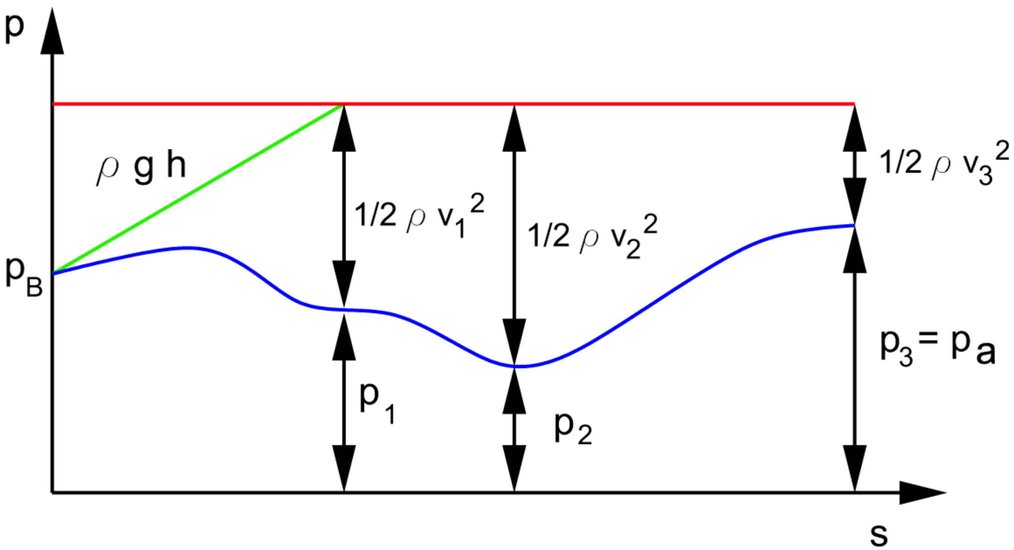


- Venturi nozzle: a Venturi nozzle is used to measure mass- and volume fluxes:

$$\dot{V} = vA = v_2 A_2$$

  - Measurement of Δp
  - Computation of v_2
  - Computation of mass- and volume flux

- Sketch of the total energy along a streamline:



→ Bernoulli

$$p_0 = p_B + \rho g h = p_i + \frac{1}{2}\rho v_i^2$$

Example 1: solution

- Continuity equation:

$$\dot{m} \;=\; \rho\dot{V} \;=\; konst.$$

$$\rho \;=\; konst \quad\Longrightarrow\quad v_1 A_1 \;=\; v_2 A_2 \;=\; v_3 A_3 \Longrightarrow A \downarrow \Longrightarrow v \uparrow \Longrightarrow p \downarrow$$

- Determination of the velocities v_1, v_2, and v_3:

  - Pressure difference:

$$\Delta p \;=\; p_1 - p_2$$

  - Bernoulli equation 1 → 2:

$$p_1 + \frac{\rho}{2} v_1^2 \;=\; p_2 + \frac{\rho}{2} v_2^2 \quad\longrightarrow\quad \Delta p \;=\; p_1 - p_2 \;=\; \frac{\rho}{2}(v_2^2 - v_1^2) \;>\; 0$$

  - Hence:

$$v_1 = v_2 \frac{A_2}{A_1} \rightarrow \Delta p \;=\; \frac{\rho}{2}\left[1 - \frac{A_2^2}{A_1^2}\right] v_2^2 \longrightarrow v_2 \;=\; \sqrt{\frac{2}{\rho}\frac{\Delta p}{\left(1 - \left(\frac{A_2}{A_1}\right)^2\right)}} = 12\frac{m}{s}$$

  - Finally:

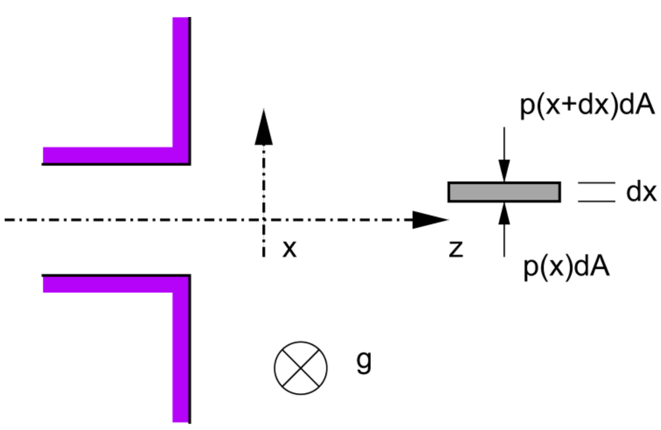
$$v_1 = v_2 \frac{A_2}{A_1} = 4\frac{m}{s} \qquad v_3 = v_2 \frac{A_2}{A_3} = 6\frac{m}{s}$$

Example 1: Task

- Determination of the pressures p_1, p_2, and p_3 and the pressure p_B above the surface.
  - The pressure p_0 represents the energy that can be converted into kinetic energy:

$$p_0 = p_B + \rho g h = p_1 + \frac{\rho}{2} v_1^2 = p_2 + \frac{\rho}{2} v_2^2 = p_3 + \frac{\rho}{2} v_3^2$$

  - If we know one pressure, we can compute the other values by using Bernoulli's equation
  - Determination of the pressure p_3 in the exit cross section
  - Equation of motion in x-direction for a moving control volume dAdx (includes always the same particles)

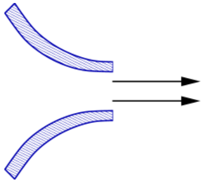


$$m \frac{du}{dt} = \rho dA dx \cdot \ddot{x} = p(x) dA - p(x+dx) dA$$

$$\longrightarrow \rho dA dx \cdot \ddot{x} = p(x) dA - \left(p + \frac{\partial p}{\partial x} dx \right) dA$$

$$\longrightarrow \rho \ddot{x} = -\frac{\partial p}{\partial x}$$

Example 1: Task



- Assumption: parallel streamlines at the sharp edged exit
- Velocity: $u = \dfrac{dx}{dt} = \dot{x}$
- Boundary condition: $\ddot{x} = 0 \longrightarrow \dfrac{\partial p}{\partial x} = 0$
- → The pressure in the exit cross-section is function of y
- Flow into air: $\dfrac{dp}{dy} = -\rho g$
- Neglect the potential energy: $p_{exit} = p_{ambience} = const.$
- Bernoulli 0 → 3: $p_B + \rho g h = p_3 + \dfrac{1}{2}\rho v_3^2 = p_a + \dfrac{1}{2}\rho v_3^2$

$$v_3 = \sqrt{\frac{2}{\rho}\left(p_B - p_a + \rho g h\right)}$$

- Open tank: $p_B = p_a$ → $v_3 = \sqrt{2gh} \neq f(A_3)$ (Theorem of Torricelli) (15.10.1608 - 25.10.1647)

Biological & Medical Fluid Mechanics
04: Momentum equation

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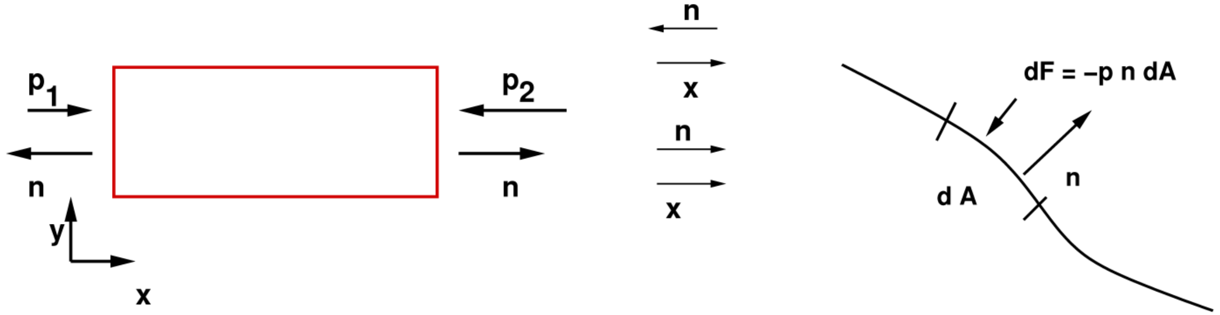
Definition

- Momentum equation = vector equation of motion for a continuum

- Steady flow: $\dfrac{\partial}{\partial t} = 0: \quad \dfrac{d\vec{I}}{dt} = \int_A \rho \vec{v}(\vec{v} \cdot \vec{n}) dA = \sum F_a = \vec{F}_p + \vec{F}_g(+\vec{F}_R) + \vec{F}_S$

- Pressure force:

$$\vec{F}_p = \int_A -\vec{n} p dA$$



- Volume force (incompressible flow, acceleration parallel to coordinate direction):

$$\vec{F}_g = \int_V \vec{g} dm = \int_V \vec{g} \rho dV$$

- Friction force:

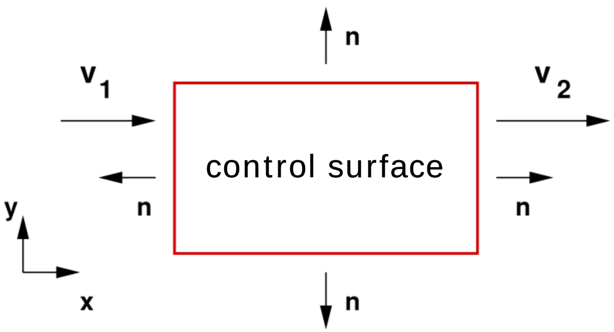
$$\vec{F}_R = -\int_A (\overline{\overline{\sigma}}' \cdot \vec{n}) dA$$

Definition

- External forces (fittings, supporting forces, casings,...)

$$\vec{F}_s$$

- Skalar product $\vec{v} \cdot \vec{n}$:
  Mass that flows normal to the surface of the control volume and that crosses the boundary of the control surface



$$\vec{v}_1 = \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} \qquad \vec{n} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$-v_{1x} = |\vec{v}_1| \, |\vec{n}| \, \cos\left(\angle(\vec{v}_1, \vec{n})\right)$$

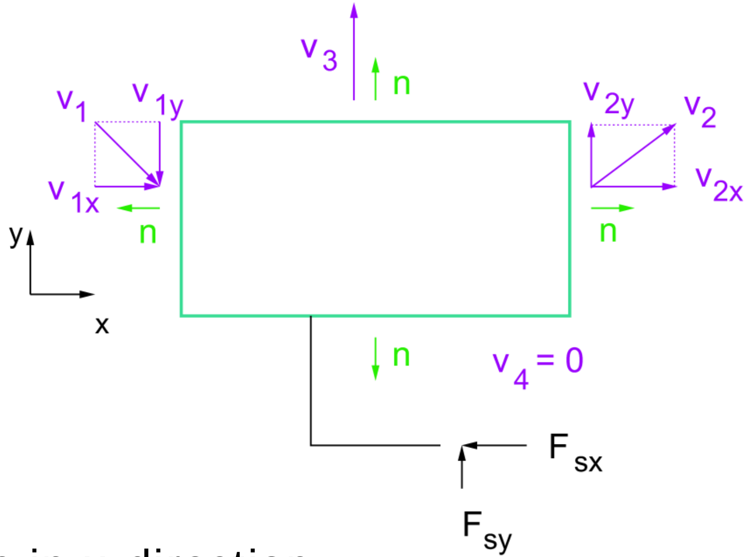
→ Incoming mass has a negative sign, outflowing mass has a positive sign

$$\begin{array}{c} \vec{n} \\ \longleftarrow \\ \longleftarrow \\ \vec{v}_1 \end{array}$$ positive sign

$$\begin{array}{c} \vec{n} \\ \longleftarrow \\ \longrightarrow \\ \vec{v}_1 \end{array}$$ negative sign

Signs

- To compute the momentum in x- and y-direction, the corresponding velocity component is used. The sign of the velocity depends on the coordinate system.
  - Velocities & forces:



  - Momentum equation in x-direction

$$\frac{dI_x}{dt} = -F_{sx} = \rho\, v_{1x} \underbrace{(-v_{1x})}_{\vec{v}_1 \cdot \vec{n}} A_1 + \rho\, v_{2x} \underbrace{(v_{2x})}_{\vec{v}_2 \cdot \vec{n}} A_2$$

  - Momentum equation in y-direction

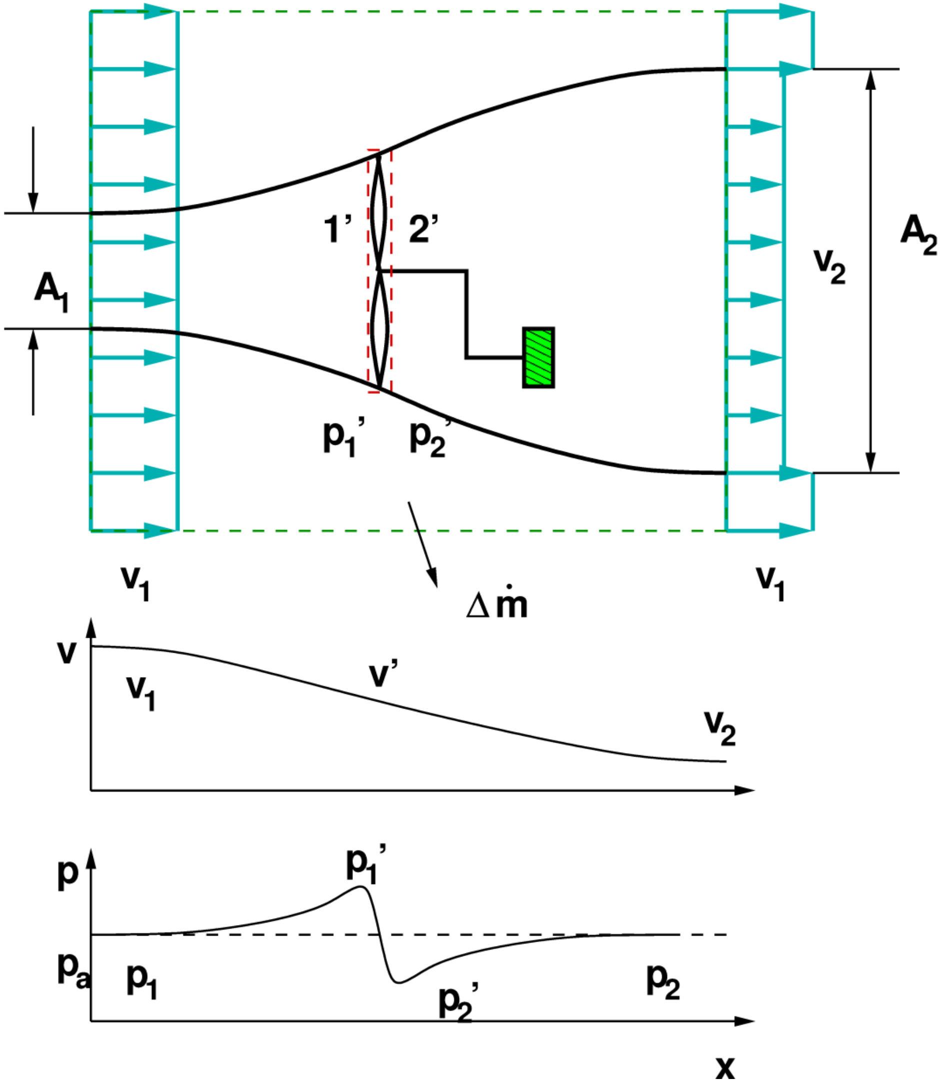
$$\frac{dI_y}{dt} = F_{sy} = \rho\,(-v_{1y})\,(-v_{1x})\,A_1 + \rho\, v_{2y}\,(v_{2x})\,A_2 + \rho v_{3y} v_{3y} A_3$$

Procedure / criterions

- Sketch the flow and define the coordinate system
- Choose the control surface such that
  - the integrands in the different directions are known or
  - the integrands are zero (symmetry plane)
  - the geometry of the control surface is simple
  - the fitting forces are included (or not)
  - if necessary use a moving control surface
  - Do not cut through walls
- Determine the integrals for the specific problem
- Important:
  - For special problems Bernoulli and Momentum equation are necessary
  - If Bernoulli is valid, the momentum equations is also valid
  - Don’t forget the continuity equation
- Rule of thumb:
  - Well rounded inlet → Bernoulli / Sharp edged inlet → Momentum
  - Sharp edged exit → Bernoulli
  - Losses (separation, mixing, ...) → Momentum
  - Power → Momentum
  - Outer forces → Momentum

Rankine‘s theory of jets

- Flow through a propeller



- Propeller, windmills, ship's screws
  - 1-dimensional flow
  - No influence of the rotation
  - Distribution of force is constant across the cross section
  - Acceleration or deceleration

Rankine's theory of jets

- Continuity equation:

$$\rho v_1 A_1 = \rho v_1^{'} A^{'} = \rho v_2^{'} A^{'} = \rho v_2 A_2 \qquad \Delta \dot{m} = \rho A_2 (v_1 - v_2)$$

- Bernoulli equation:

$$1 \rightarrow 1^{'} \quad : \quad p_a + \frac{\rho}{2} v_1^2 = p_1^{'} + \frac{\rho}{2} v_1^{'2} \qquad 2^{'} \rightarrow 2 \quad : \quad p_2^{'} + \frac{\rho}{2} v_2^{'2} = p_a + \frac{\rho}{2} v_2^2$$

- Momentum equation, red control volume:

$$-\rho v^{'2} A' + \rho v^{'2} A' = (p_1^{'} - p_2^{'}) A^{'} + F \quad \rightarrow \quad F = (p_1^{'} - p_2^{'}) A > 0$$

- Momentum equation, green control volume:

$$-\rho v_1^2 A_\infty + \rho v_2^2 A_2 + \rho v_1^2 (A_\infty - A_2) + \Delta \dot{m} v_1 = F$$

$$\rightarrow F = \rho v_2 A_2 (v_2 - v_1) = \rho v^{'} A^{'} (v_2 - v_1)$$

- Theorem of Froude:

$$v' = \frac{1}{2}(v_1 + v_2)$$

- Maximum power

$$\frac{\partial P}{\partial (\frac{v_2}{v_1})} = 0 \quad \rightarrow \quad \frac{P_{max}}{A'} = \frac{8}{27} \rho v_1^3$$

- Power:

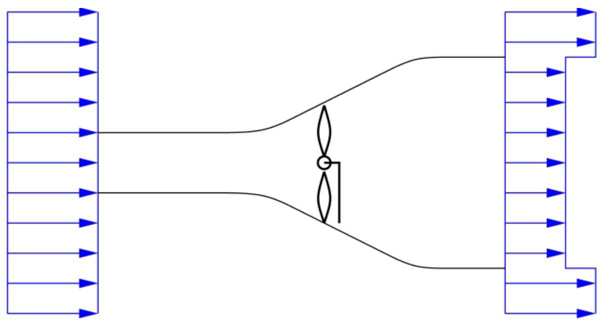
$$P = \dot{V} \Delta p_0 = \frac{\rho}{4} A' v_1^3 (1 + \frac{v_2}{v_1})(1 - \frac{v_2^2}{v_1^2}) \sim v_1^3$$

- Maximum thrust:

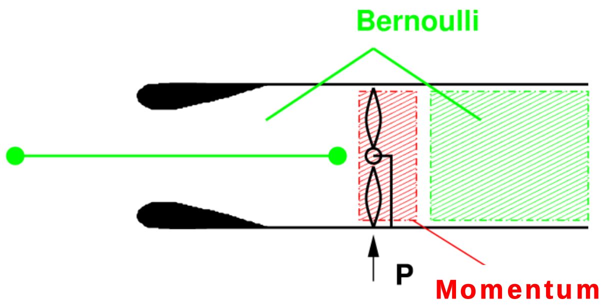
$$\frac{F}{A'} = -\frac{4}{9} \rho v_1^2 \sim v_1^2$$

Rankine‘s theory of jets

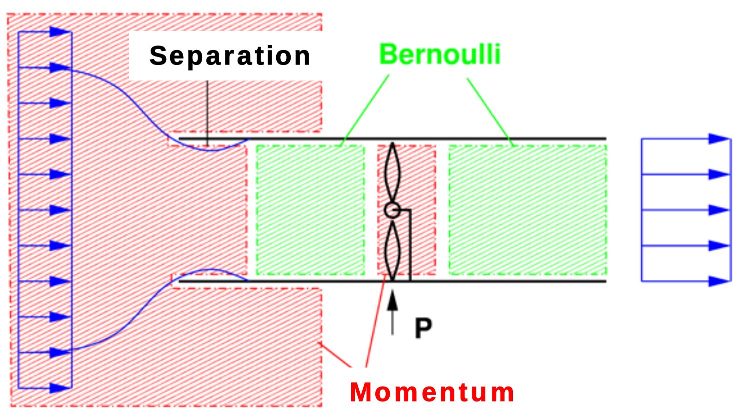
- Different forms of propellers
  - Propeller without housing



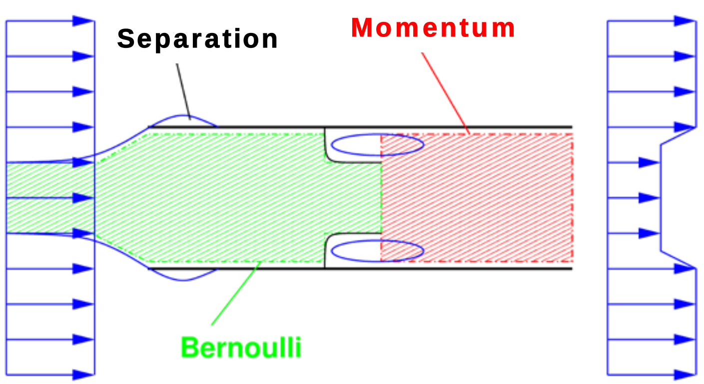
  - Propeller with housing, well rounded inlet



  - Propeller with housing, sharp edged inlet

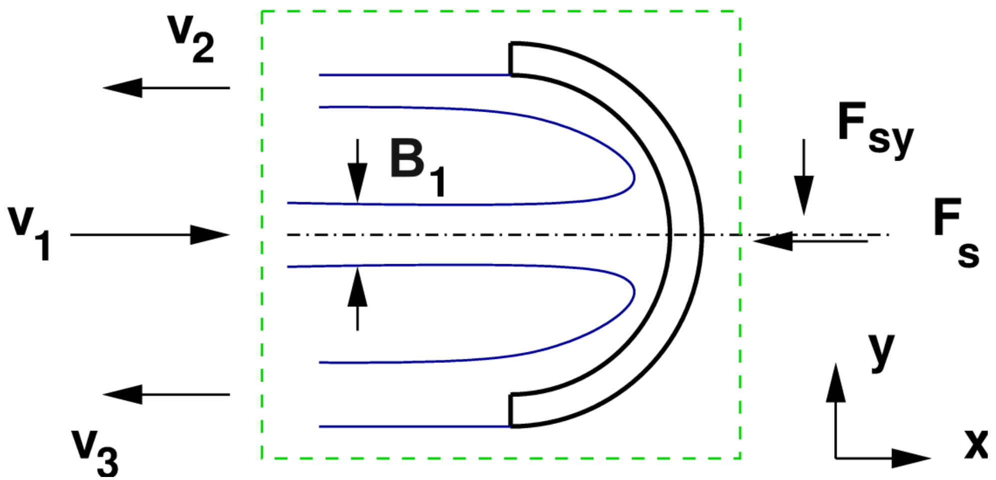


  - Pipe with nozzle



Example 1: task

- A water jet flowing into positive x-direction is reflected by a blade. The flow is 2-dimensional, frictionless and symmetrical.



- Given: v_1, ρ, B_1
- Determine the force F_s on the blade
  - for a fixed blade
  - for a blade that moves in positive x-direction with the constant velocity v_{stat}

Example 1: solution

- a) fixed blade:
  - Bernoulli equation:

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 = p_3 + \frac{1}{2}\rho v_3^2 \qquad p_1 = p_2 = p_3 \quad \rightarrow \quad v_1 = v_2 = v_3$$

  - Continuity:

$$B_1 v_1 = B_2 v_2 + B_3 v_3 \rightarrow \qquad B_2 = B_3 = \frac{1}{2} B_1$$

  - Momentum equation in x-direction:

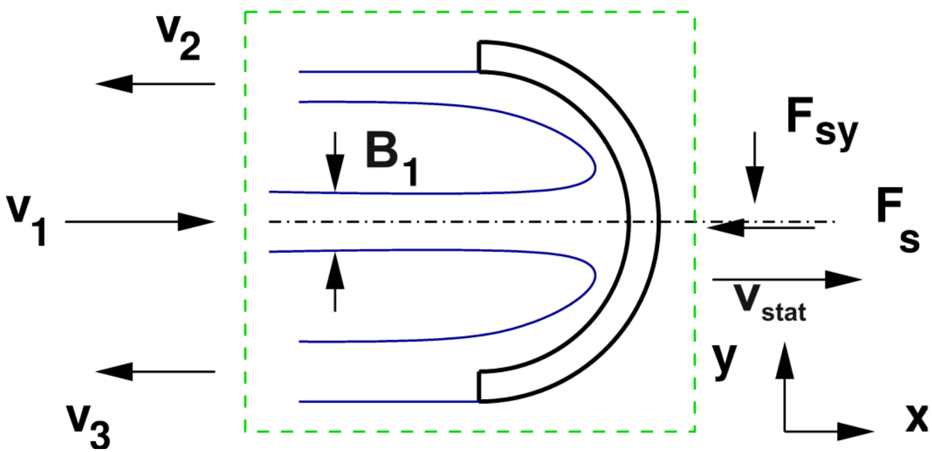
$$\frac{dI_x}{dt} = \int\limits_A \rho\vec{v}(\vec{v}\cdot\vec{n})dA = \sum F_x$$

$$\underbrace{\rho(+v_1)(-v_1)B_1}_{\text{inflow}} + \underbrace{\rho(-v_2)(+v_2)B_2}_{\text{outflow}} + \underbrace{\rho(-v_3)(+v_3)B_3}_{\text{outflow}} = -F_{sx}$$

$$\rho v_1^2 (-B_1 - \frac{1}{2}B_1 - \frac{1}{2}B_1) = -F_{sx} \qquad \rightarrow \qquad \boxed{F_{sx} = 2\rho v_1^2 B_1}$$

Example 1: solution

- b) moving blade:



$$F_{sx} = ?$$
$$v_{abs} = v_{rel} + v_{stat}$$
$$v_{rel,1} = v_{abs,1} - v_{stat}$$
$$v_{rel,2} = v_{abs,2} - v_{stat}$$
$$v_{rel,3} = v_{abs,3} - v_{stat}$$

- Bernoulli equation/continuity/symmetry:

$$v_{rel,1} = v_{rel,2} = v_{rel,3} \rightarrow B_2 = B_3 = \frac{1}{2} B_1$$

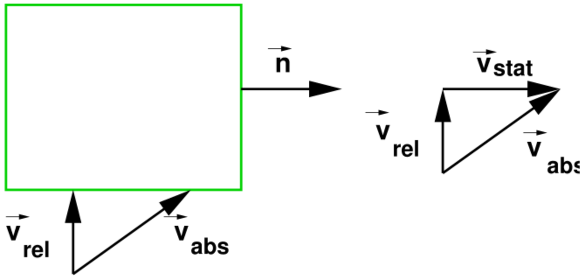
- Momentum equation in the absolute/relative system

$$\frac{dI_x}{dt} = \int\limits_A \rho \vec{v}_{abs} (\vec{v}_{rel} \cdot \vec{n}) dA = \sum F_x$$

Velocity　　　Mass flux

Example 1: solution

- b) Moving control surface



$$\vec{v}_{abs} = \vec{v}_{rel} + \vec{v}_{stat}$$

  - Momentum equation in the absolute/relative system

$$\frac{dI_x}{dt} = \int\limits_A \rho \vec{v}_{abs}(\vec{v}_{rel} \cdot \vec{n})dA = \int\limits_A \rho(\vec{v}_{rel} + \vec{v}_{stat})(\vec{v}_{rel} \cdot \vec{n})dA$$

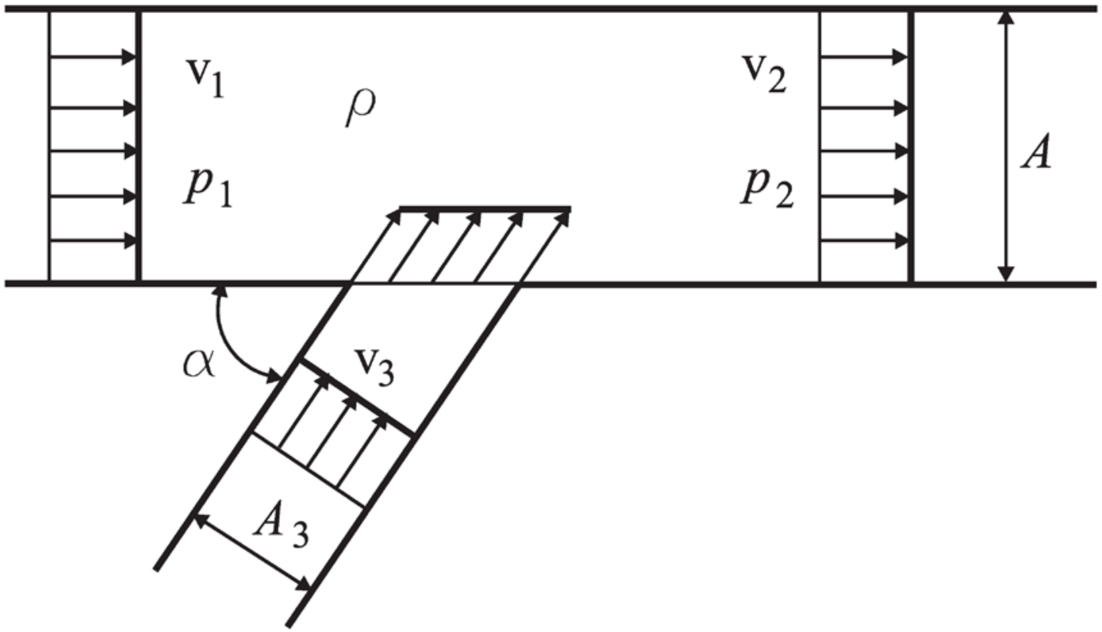
$$= \underbrace{\int\limits_A \rho \vec{v}_{stat}(\vec{v}_{rel} \cdot \vec{n})dA}_{=0} + \int\limits_A \rho \vec{v}_{rel}(\vec{v}_{rel} \cdot \vec{n})dA$$

$$\frac{dI_x}{dt} = \int\limits_A \rho \vec{v}_{abs}(\vec{v}_{rel} \cdot \vec{n})dA = \int\limits_A \rho \vec{v}_{rel}(\vec{v}_{rel} \cdot \vec{n})dA$$

$$\boxed{F_{sx} = 2\rho v_{rel,1}^2 B_1}$$

Example 2: task

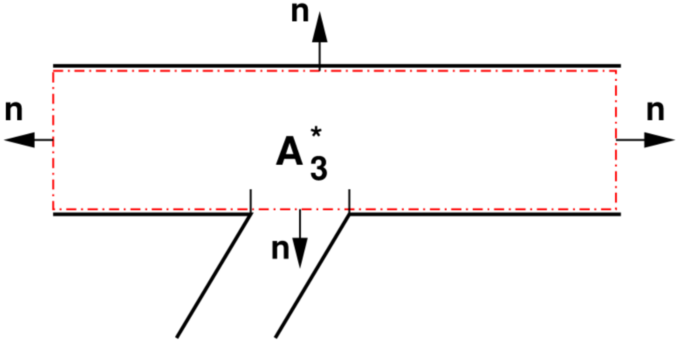
- Given: Determine the pressure difference $\Delta p = p_2 - p_1$ in the plotted bifurcation by neglecting the friction.

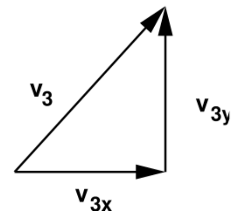


- Given: v_1, v_2, α, ρ=const., A_3 , $\frac{1}{4} A = A_3$

Example 2: solution

- Alternative 1: control surface





$$A_3^* = A_3 / \sin\alpha$$

$$\vec{v}\cdot\vec{n} = -v_3 \sin\alpha = v_3^*$$

  - Momentum equation in x-direction:

$$\frac{dI_x}{dt} = \int_A \rho \vec{v}_x (\vec{v}\cdot\vec{n}) dA = \sum F_x$$

$$\frac{dI_x}{dt} = \rho v_1(-v_1)A_1 + \rho v_2 v_2 A + \rho v_3 \cos\alpha(-v_3 A_3) = \sum F_x$$

  - Pressure force:

$$\sum F_x = -\int p\vec{n} dA = (p_1 - p_2)A$$

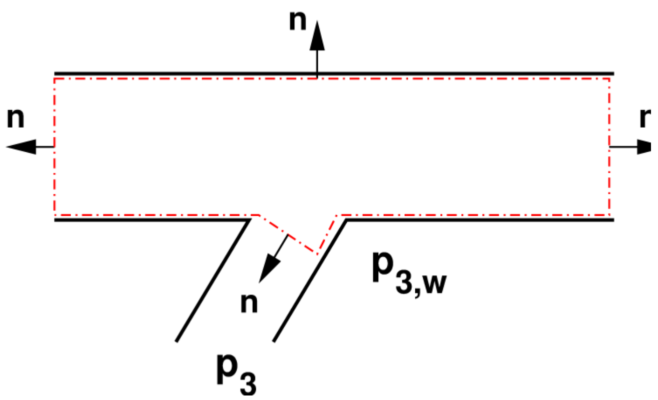
  - Continuity:

$$v_1 A_1 + v_3 A_3 = v_2 A_2 \quad \rightarrow \quad v_3 = 4(v_2 - v_1)$$

$$\rightarrow \quad \Delta p = p_2 - p_1 = \rho(v_1^2 - v_2^2 + 4(v_2 - v_1)^2 \cos\alpha)$$

Example 2: solution

- Alternative 2: control surface



  - Momentum equation in x-direction:

$$\frac{dI_x}{dt} = \int_A \rho \vec{v}_x (\vec{v} \cdot \vec{n}) dA = \sum F_x$$

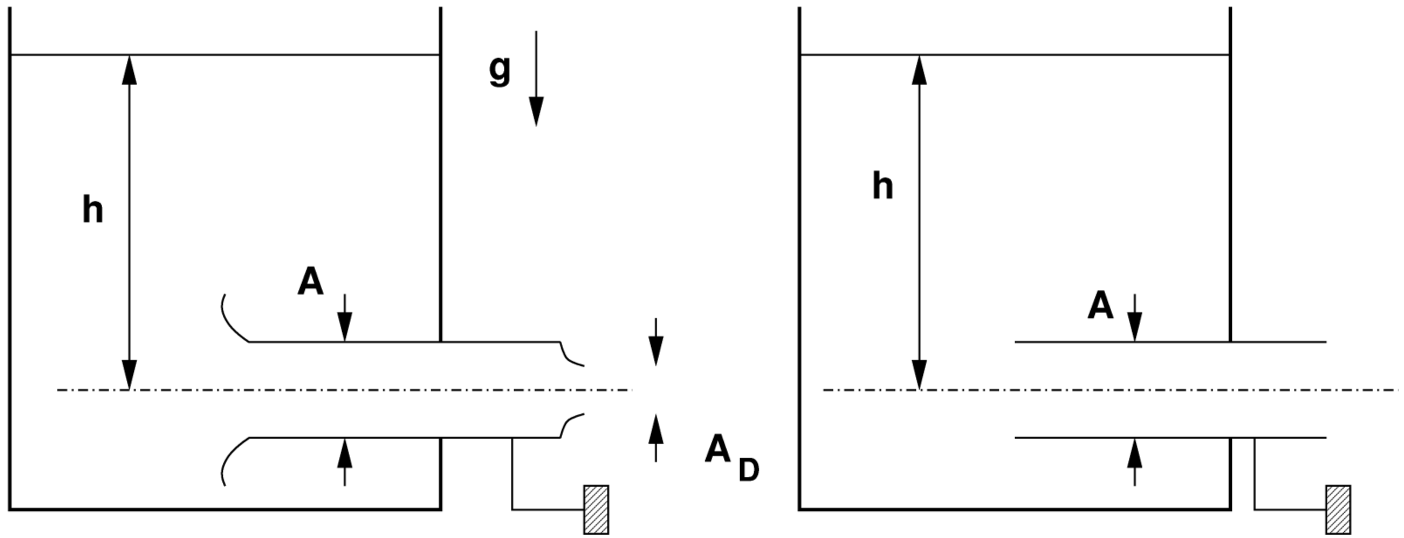
$$\int \rho \underbrace{\vec{v}}_{v_3 \cos \alpha} \underbrace{(\vec{v} \cdot \vec{n}) dA}_{-v_3 A_3}$$

  - Pressure force:

$p_3, \quad p_{3,w}$ unknown $\longrightarrow \int p\vec{n}\, dA$ cannot be computed

Example 3: task

- Water is flowing steadily from a large container into the open air. The inlet is well rounded. The exit possesses the shape of a nozzle.



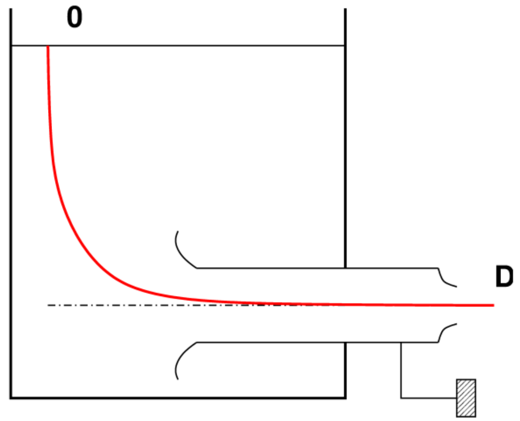
- Determine the fitting force
  a) for the standard configuration
  b) without inlet and nozzle

- Given: ρ=const., A , A_D, h, g

Example 3: solution

- Mass flux: $$\dot{V} = vA = v_D A_D$$

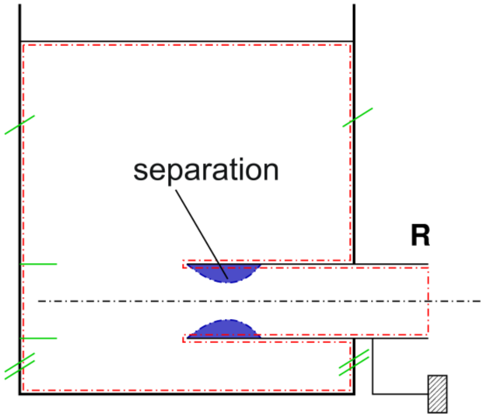
a) well rounded inlet and nozzle:
no losses → Bernoulli equation



$$p_a + \rho g h = p_a + \frac{1}{2}\rho v_{D,a}^2$$

$$\longrightarrow v_{D,a} = \sqrt{2gh}$$

b) Borda estuary
Losses → no Bernoulli equation → Momentum equation



$$\frac{dI_x}{dt} = \int_A \rho \vec{v}_x (\vec{v} \cdot \vec{n}) dA = \sum F_a$$

$$\int_A \rho \vec{v}_x (\vec{v} \cdot \vec{n}) dA = \rho v_R A_R v_R = \dot{m} v_R$$

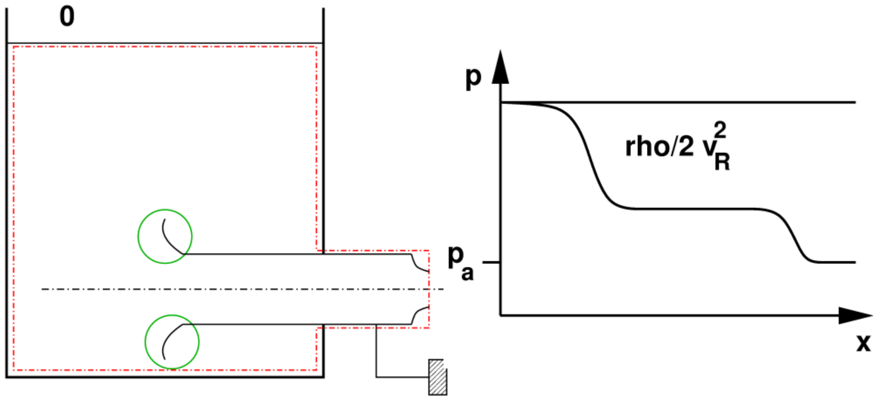
$$\sum F_a = F_{p,x} = (p_a + \rho g h) A_R - p_a A_R$$

$$\longrightarrow v_{R,b} = \sqrt{gh} < v_{D,a}$$

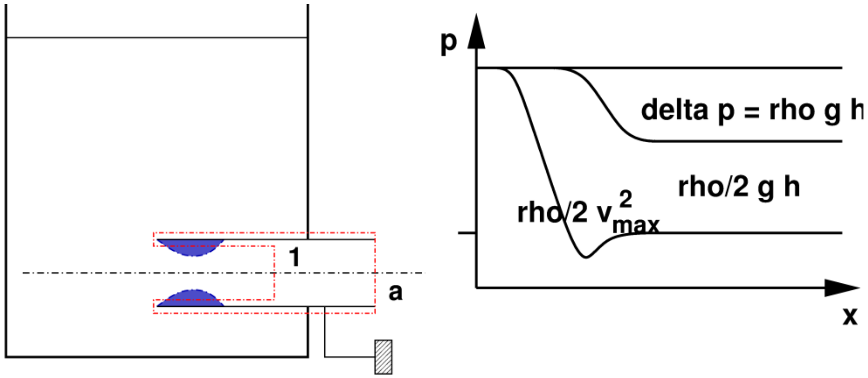
Example 3: solution

- Forces:

- Well rounded inlet and nozzle:



- Borda estuary:



$$\rho v_D^2 A_D = (p_a + \rho g h) A_R - p_a A_R + F_x$$

$$v_D = \sqrt{2gh} \rightarrow F_x = \rho g h (2A_D - A_R)$$

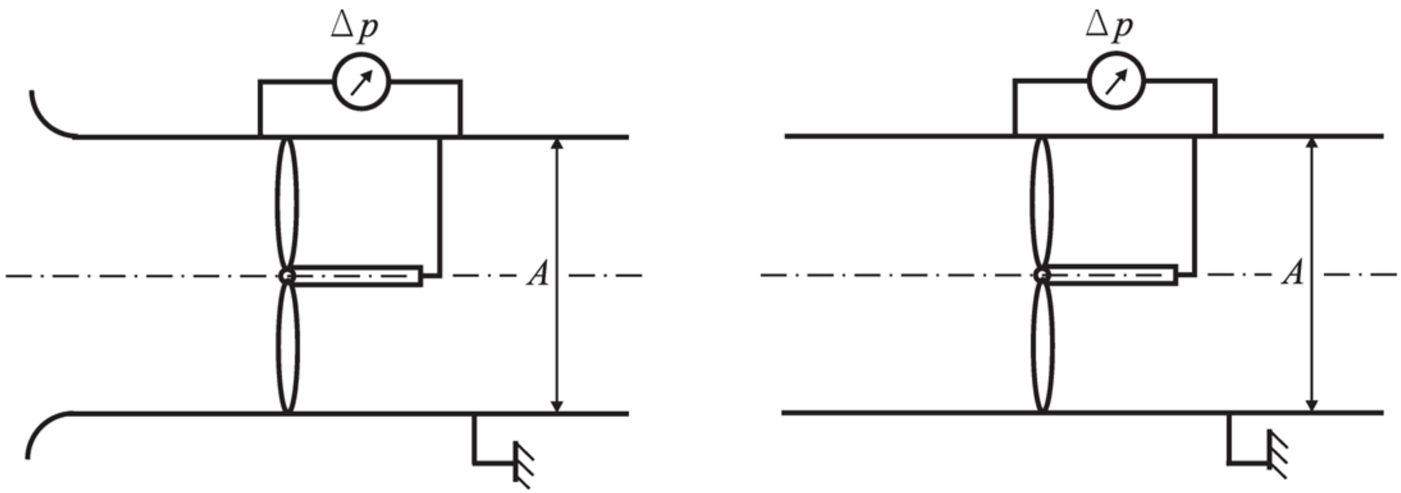
$$-\rho v_R^2 A_R + \rho v_R^2 A_R = F_x + (p_1 - p_a) A_R$$

$$p_1 + \frac{1}{2}\rho v_1^2 = p_a + \frac{1}{2}\rho v_a^2 \rightarrow v_1 = v_a = v_R$$

$$\rightarrow p_1 = p_a \rightarrow F_x = 0$$

Example 4: task

- Two fans sucking air from the surrounding differ in their inlets. The flow is incompressible.



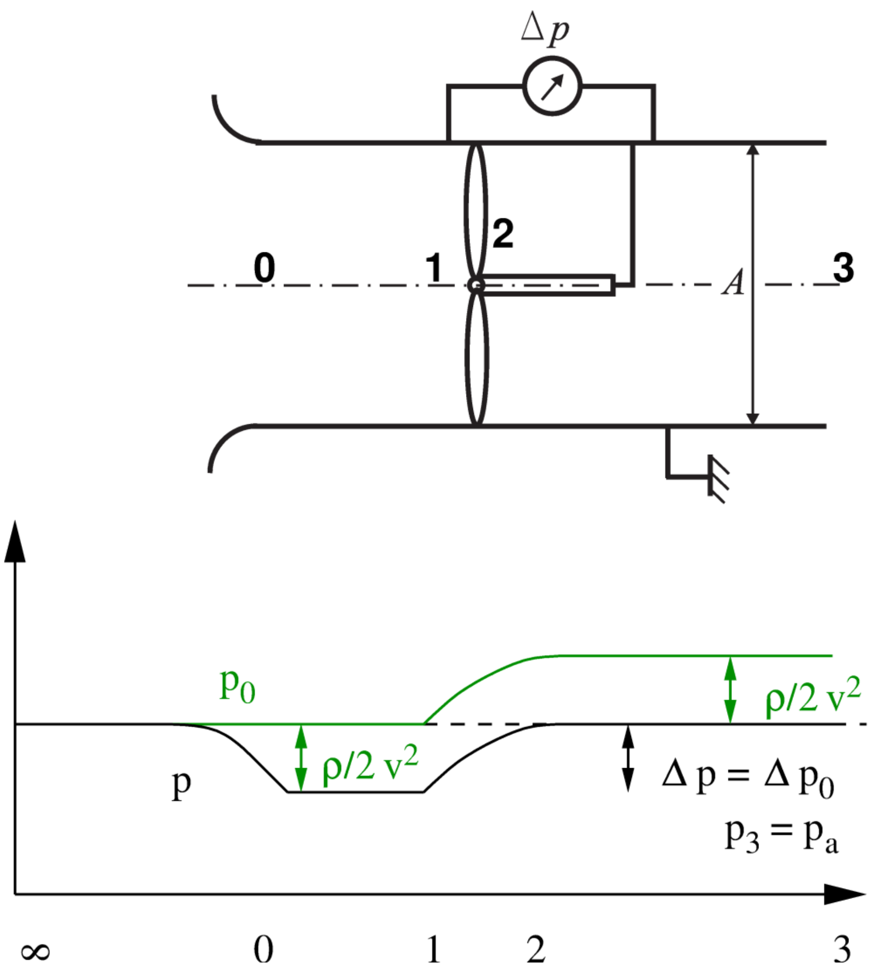
- Compute
  a) the volume flux,
  b) the power of the fans, and
  c) the force on the fitting.

- Given: ρ=const., A , Δp

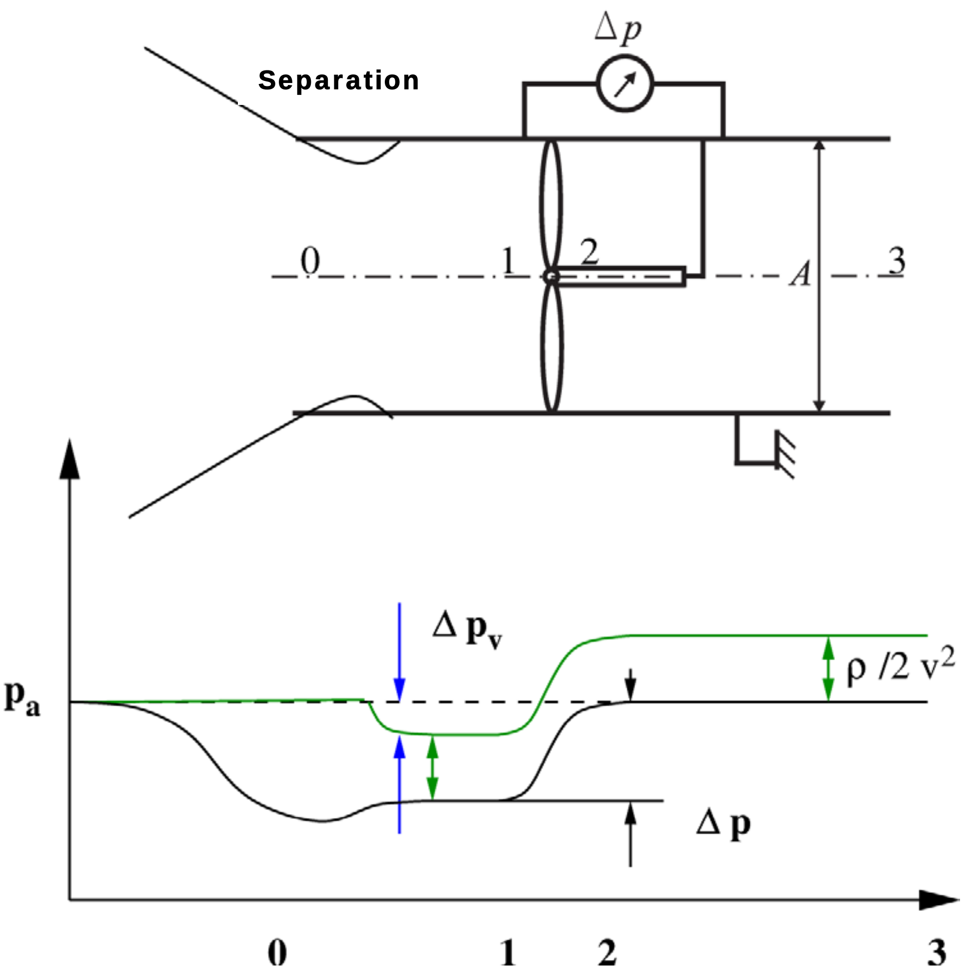
Example 4: solution

- Basic situation: Total pressure, static pressure, and dynamic pressure

- Well rounded inlet:

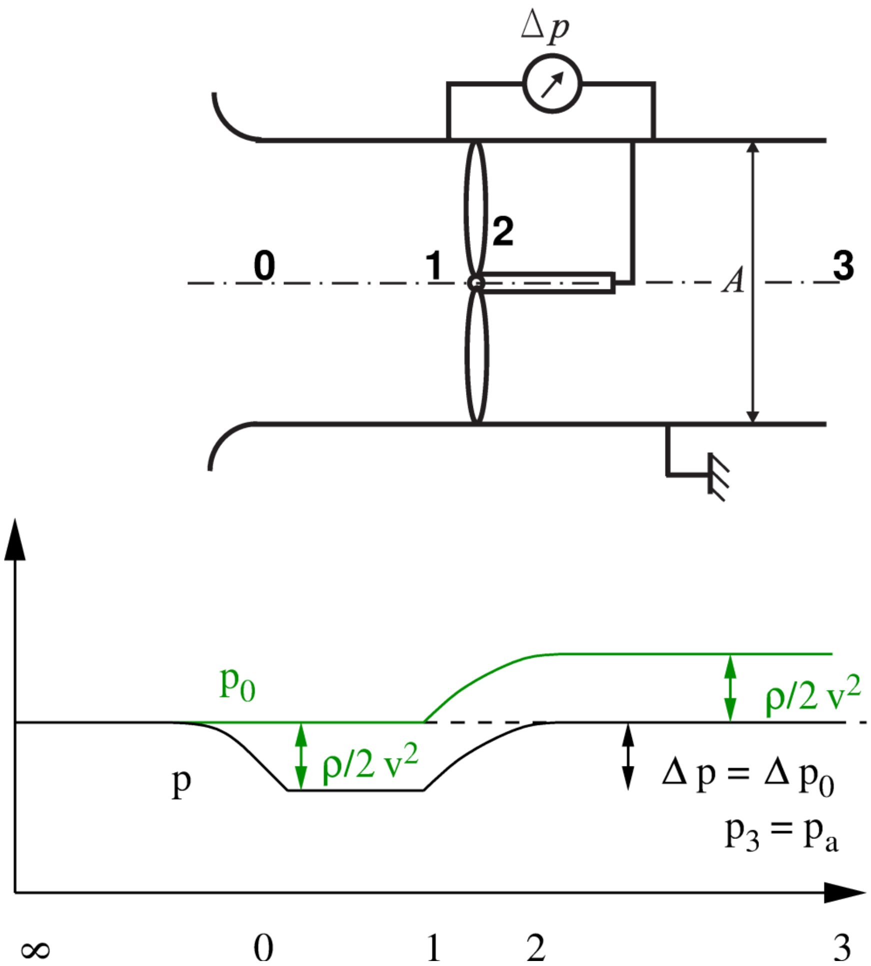


- Sharp edged inlet:



Example 4: solution

- Well rounded inlet:



- Volume flux:

$$\dot{V} = v_1 A$$

- Bernoulli equation 2 → 3:

$$p_2 + \frac{\rho}{2} v_2^2 = p_3 + \frac{\rho}{2} v_3^2$$

$$\rightarrow \quad p_2 = p_3 = p_a$$

- Bernoulli equation -∞ → 1:

$$p_a + 0 = p_1 + \frac{\rho}{2} v_1^2$$

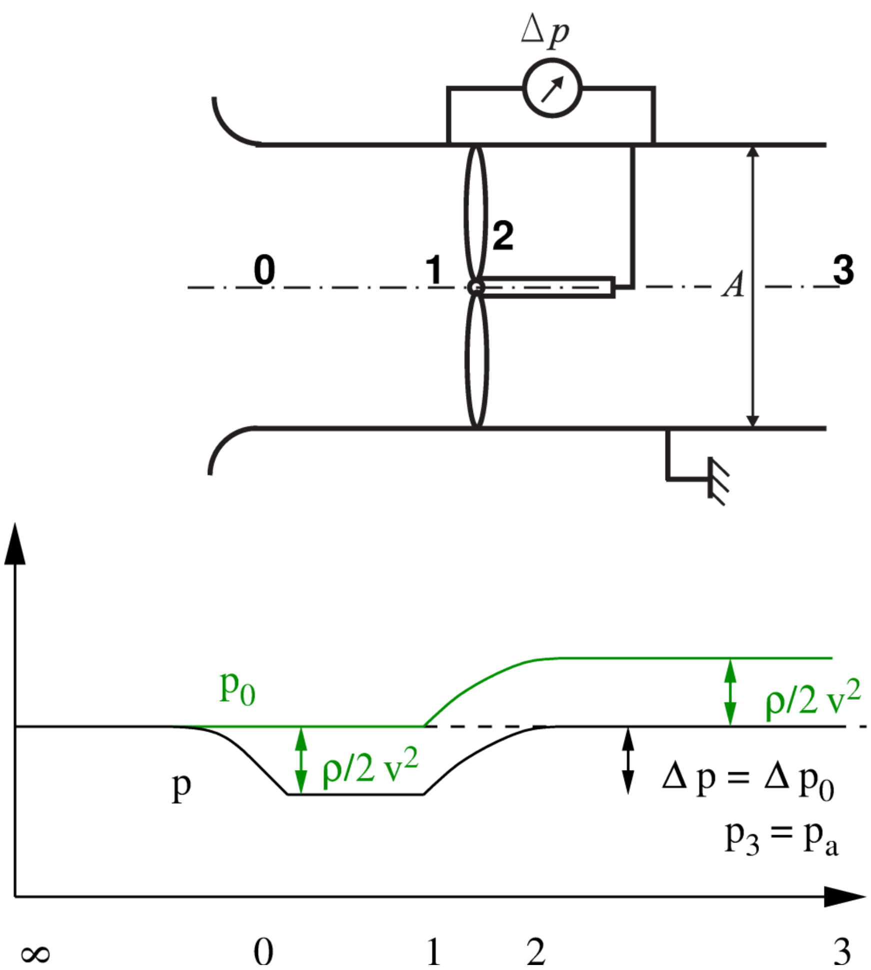
$$(\Delta p = p_2 - p_1)$$

$$\longrightarrow v_1 = \sqrt{\frac{2}{\rho} \Delta p}$$

$$\dot{V} = v_1 A = \sqrt{\frac{2}{\rho} \Delta p} A$$

Example 4: solution

- Well rounded inlet:



- Power:

$$P = \dot{V} \Delta p_0$$

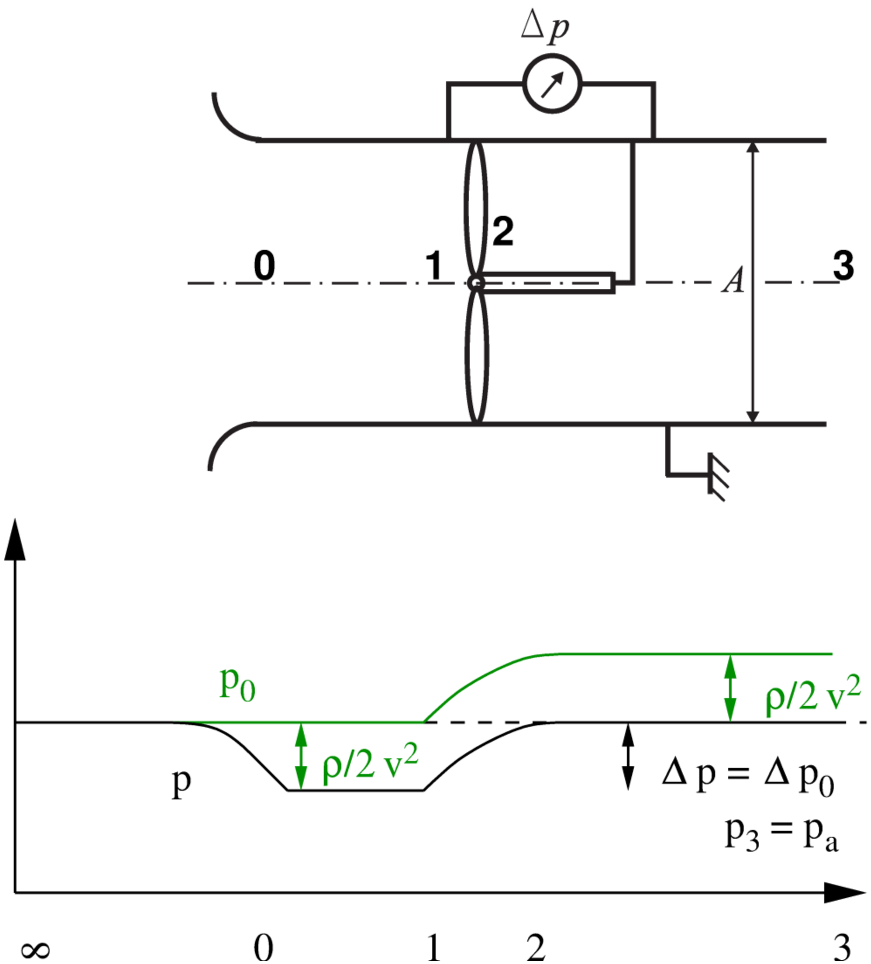
- Here:

$$\begin{aligned} \Delta p_0 &= p_{02} - p_{01} \\ &= p_2 + \frac{\rho}{2} v_2^2 - p_1 - \frac{\rho}{2} v_1^2 \\ &= p_2 - p_1 = \Delta p \end{aligned}$$

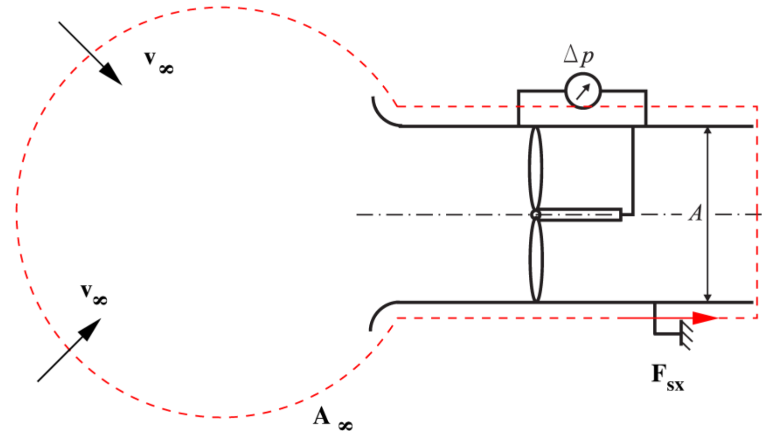
$$P = \Delta p A \sqrt{\frac{2}{\rho} \Delta p}$$

Example 4: solution

- Well rounded inlet:



- Fitting force:



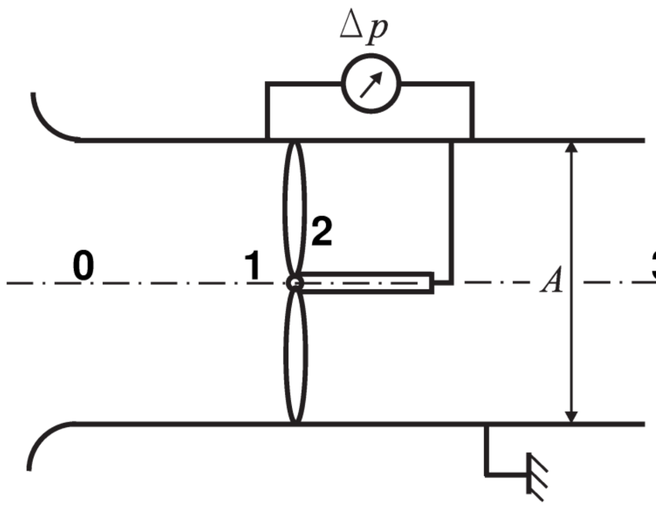
- Flow field can be described using a point sink
    - No direction at infinity
    - The velocity is constant

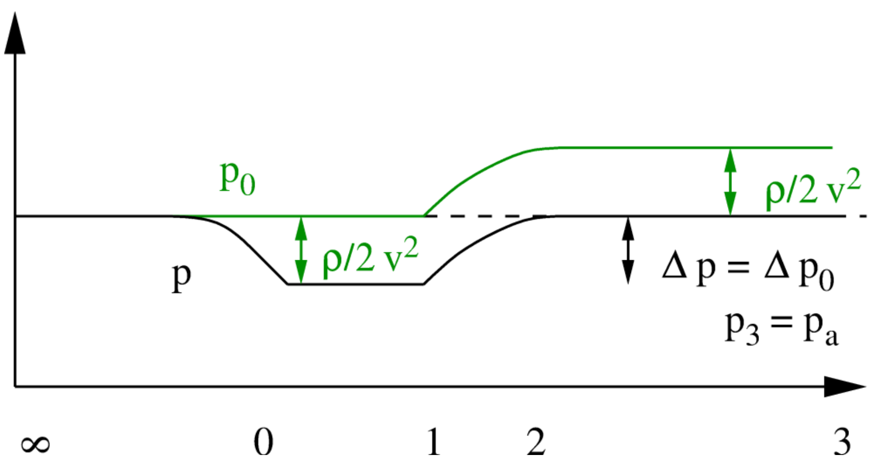
$$A_\infty v_\infty = Av$$

$$v_\infty = \frac{Av}{A_\infty}$$

Example 4: solution

- Well rounded inlet:





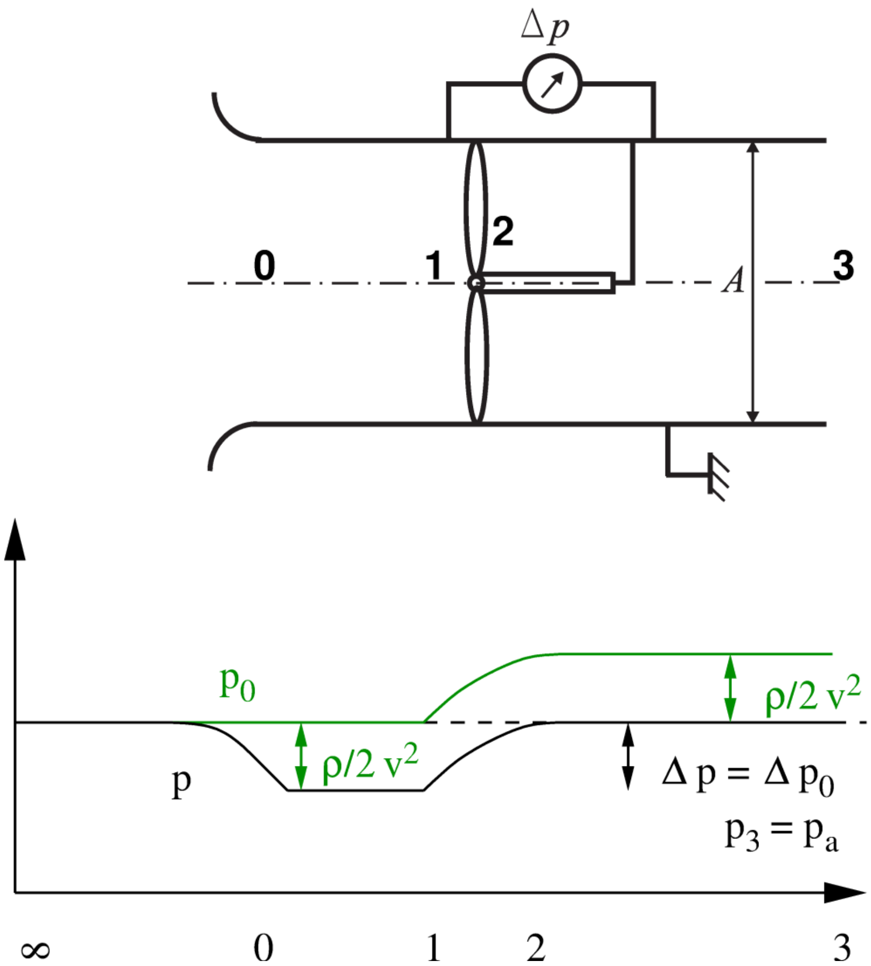
- Momentum flux for A_∞:

$$
\begin{aligned}
\left|\frac{d\vec{I}}{dt}\right| &= \sqrt{\left(\frac{dI}{dt}\right)_x^2 + \left(\frac{dI}{dt}\right)_y^2} \\
&= \left|\oint_{A_\infty} \rho\vec{v}\,(\vec{v}\cdot\vec{n})\,dA\right| \\
&= \oint_{A_\infty} |\rho\,\underbrace{\vec{v}}_{v_\infty}\,\underbrace{(\vec{v}\cdot\vec{n})}_{\le v_\infty}\,dA| \\
&\le \oint_{A_\infty} \rho v_\infty^2\,dA = \rho v_\infty^2 A_\infty
\end{aligned}
$$

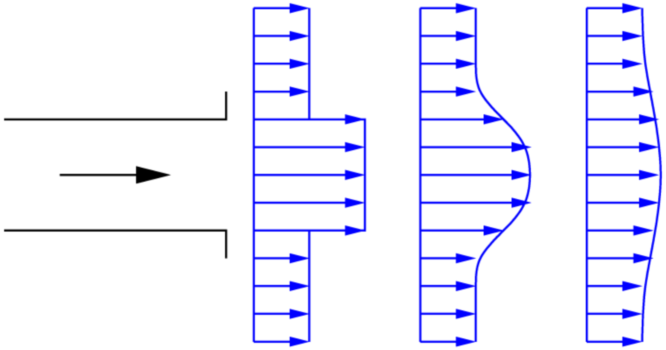
$$
\begin{aligned}
v_\infty &= \frac{Av}{A_\infty} \rightarrow \left|\frac{d\vec{I}}{dt}\right| < \frac{\rho v^2 A^2}{A_\infty} \\
&= \frac{\dot{m}^2}{\rho A_\infty} \rightarrow 0 \text{ für } A_\infty \rightarrow \infty
\end{aligned}
$$

Example 4: solution

- Well rounded inlet:



- Exit:

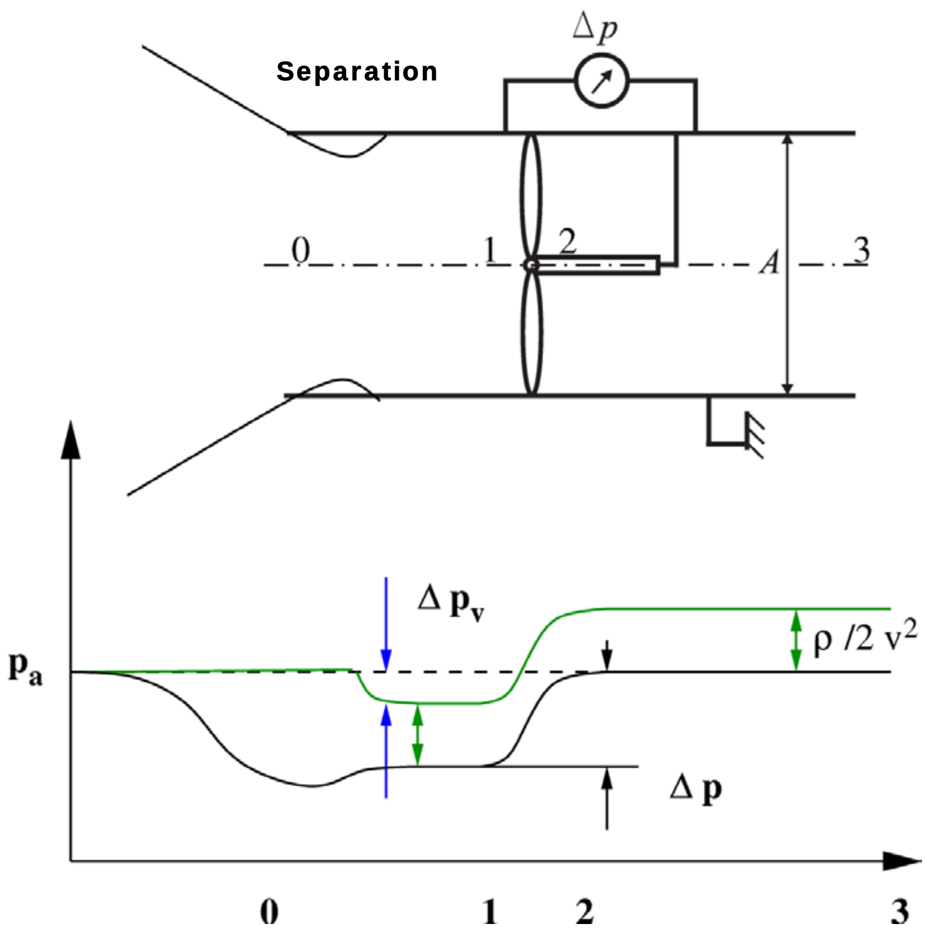


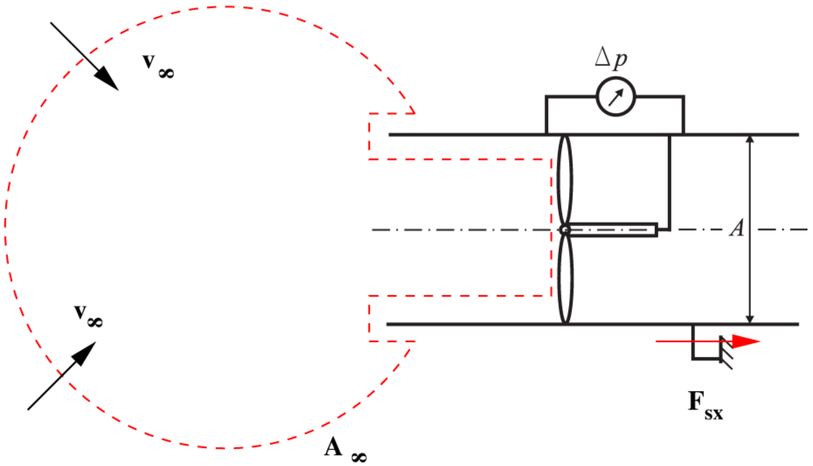
$$\left|\frac{dI}{dt}\right|_x = \rho v^2 A = -p_a \oint_{A_\infty} \vec{n} dA + F_{sx}$$

$$\longrightarrow F_{sx} = \rho v^2 A = 2\Delta p \cdot A$$

Example 4: solution

- Sharp edged inlet:





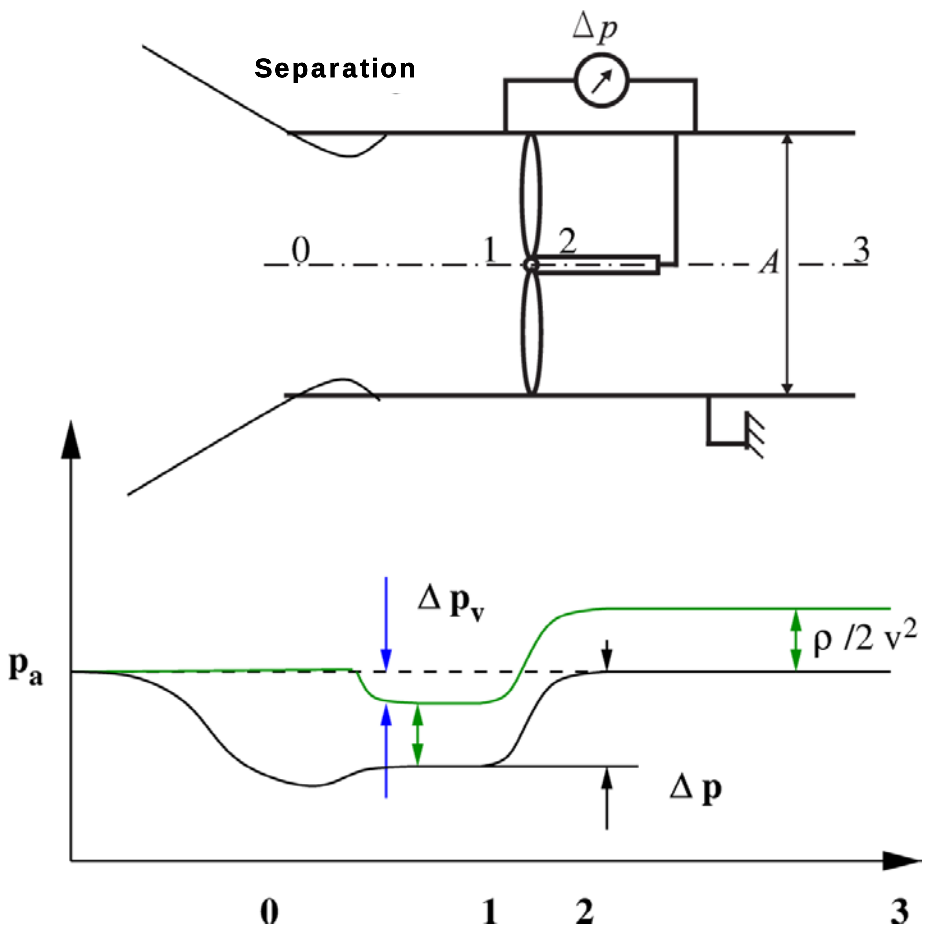
- Momentum equation:

$$\rho v^2 A = p_a A_\infty - (p_a(A_\infty - A) + p_1 A)$$
$$= (p_a - p_1)A = \Delta p A$$

$$v_1 = v = \sqrt{\frac{\Delta p}{\rho}} \rightarrow \dot{V} = \sqrt{\frac{\Delta p}{\rho}} A$$

Example 4: solution

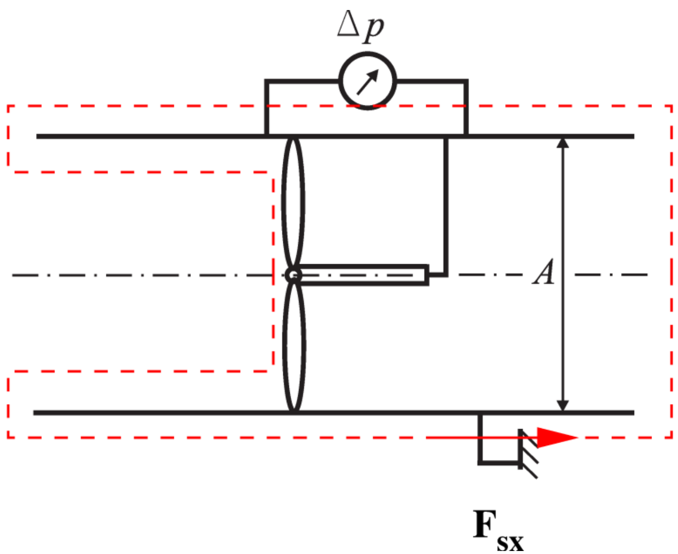
- Sharp edged inlet:



- Power:

$$P = \Delta p_0 \dot{V} = \Delta p A \sqrt{\frac{\Delta p}{\rho}}$$

- Fitting force:



$$\rho v(-v)A + \rho v v A = (p_1 - p_3)A + F_{sx}$$

$$\rightarrow \quad F_{sx} = \Delta p A$$

Biological & Medical Fluid Mechanics (BMF/BME)
05: Similarity rules

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Motivation

- Initial situation
  - Exact analytical solution of the conservation equations is usually not possible

    ⇒ Experimental and numerical investigations are necessary
  - Fundamental questions:
    - When can experimental results be transferred to the realistic conditions?
    - How can we design an experiment as general as possible?
    - How can we reduce the complexity of the problem?

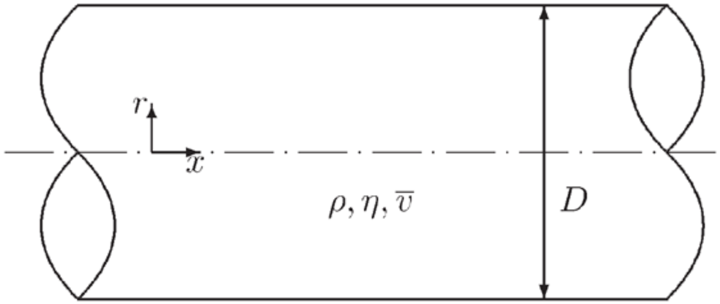
→ Similarity theory: Find a set of dimensionless similarity parameters that describe the problem

Example: pipeline problem

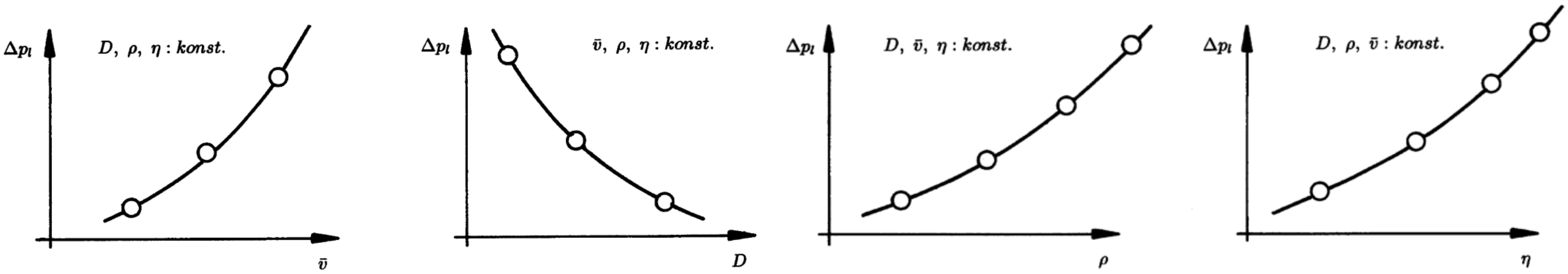
- Experimental investigation of the pressure loss for the steady, incompressible flow of a Newtonian fluid through a long horizontal tube with circular cross section

→ Find a relation for Δp_l that describes its dependence on the variables of the flow

$$\Delta p_l = f(D, \rho, \eta, \bar{v})$$



- Approach 1: Several experiments with modifications in one variable



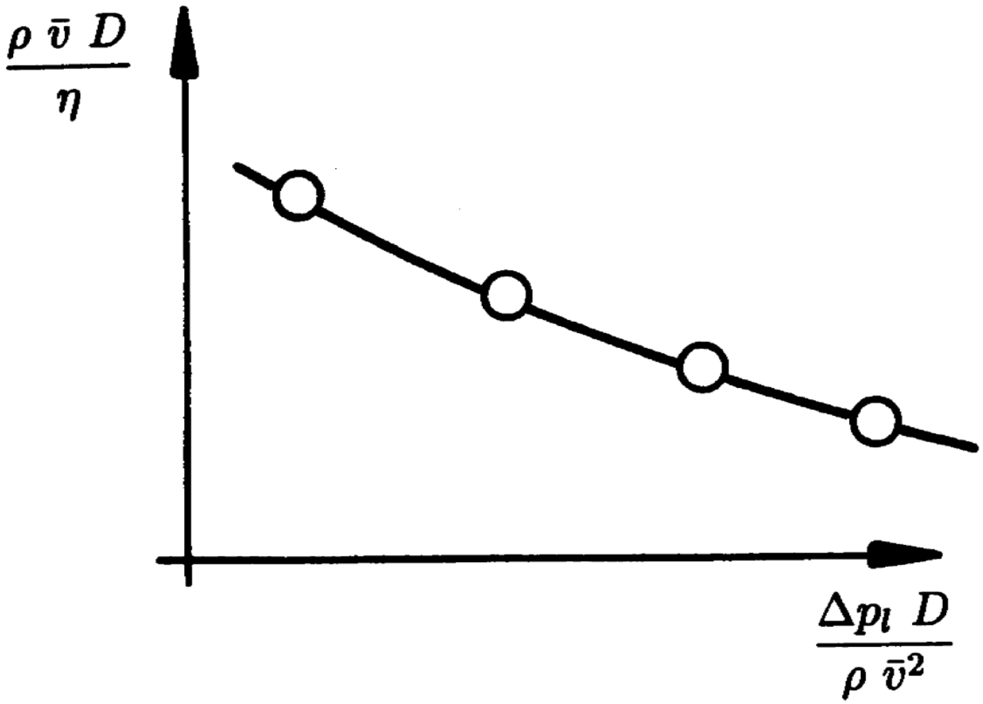
→ expensive, difficult, results not necessarily transferable to other pipelines with different flow conditions

Example: pipeline problem

- Approach 2: Combine (D, ρ, η, v) to dimensionless parameters (similarity parameters):

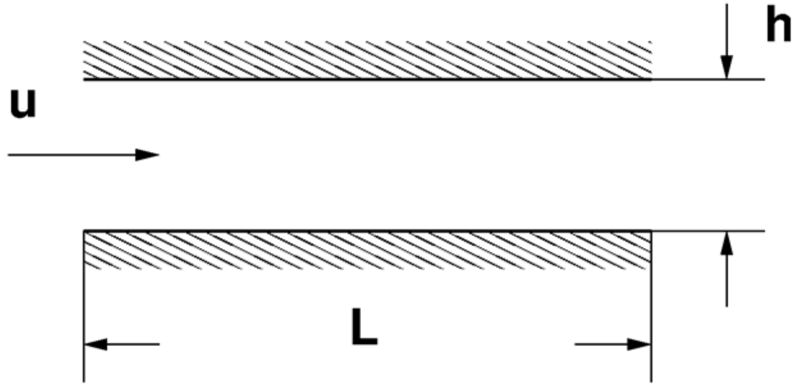
$$\frac{\Delta p_l \cdot D}{\rho \bar{v}^2} = \Phi\left(\frac{\rho \cdot \bar{v} \cdot D}{\eta}\right)$$

$$\Pi_1 = \Phi\left(\Pi_2\right)$$



Definition

- Theory of similarity:
  - Comparison of experimental results with real configurations
  - Reduction of the number of physical quantities
    $\rightarrow$ reduction of the number of experiments
  - Experimental results are independent of the scale
  - Similarity parameters are dimensionless
  - Dynamic similarity: flows are not necessarily similar, if only the flow quantities are scaled
  - Two flow fields are similar if they are geometrically and dynamically similar
  - Flow in a gap:



$\frac{h}{L}$ = parameter of the geometry

Similarity numbers

- Geometrical similarity

$$L_1 = \Omega \cdot L_2$$ → scale

→ Transfer from reality to model

- Euler number: Similarity concerning pressure

$$Eu = \frac{\Delta p}{\rho \cdot u^2}$$ → pressure force / inertia

- Reynolds number: Similarity concerning viscous stresses

$$Re = \frac{\rho \cdot u \cdot L}{\eta} = \frac{u \cdot L}{\nu}$$ → inertia /viscous forces

- $Re \rightarrow 0$ → creeping flow
- $Re \cdot h^2/L^2 \ll 1$ → gap flow
- $Re \rightarrow \infty$ → frictionless

Due to the kinematic viscosity, the Reynolds number depends on the temperature and (for gas flow) on the pressure.

Similarity numbers

- Froude number: shallow water waves / free surfaces / ship hydrodynamics

$$Fr = \frac{u}{\sqrt{g \cdot L}}$$

→ inertia / gravitational force

The Froude number is used to determine the resistance of a partially submerged object moving through water

- Strouhal number: ratio between characteristic times

$$Sr = \frac{L}{u \cdot t}$$

- Mach number: flow velocity / speed of sound

$$Ma = \frac{u}{c}$$

$Ma < 0.3$ → incompressible flow

$Ma < 1$ → subsonic

$Ma > 1$ → supersonic

$Ma \gg 1$ → hypersonic

(subsonic, supersonic, hypersonic:) compressible flow

Similarity numbers

- Prandtl number: viscous diffusion rate / thermal diffusion rate

$$Pr = \frac{\eta \cdot c_p}{\lambda} = \frac{\nu}{a}$$

→ kin. viscosity / thermal diffusivity
(c_p = specific heat)
(λ = thermal conductivity)
(a = thermal diffusivity)

- Weber number: multiphase flows

$$We = \frac{\rho \cdot u^2 \cdot L}{\sigma}$$

→ inertia / surface tension/energy

- Nusselt number: heat transfer at a boundary (surface) within a fluid

$$Nu = \frac{\alpha \cdot L}{\lambda_f}$$

→ convective/conductive heat transfer
(λ_f = thermal conductivity)
(α = convective heat transfer coefficient)

- Archimedes number: motion of fluids due to density differences

$$Ar = \frac{\Delta\rho}{\rho_f} \cdot \frac{g \cdot c_p^3}{\nu^2}$$

Methods to determine dimensionless parameters: Buckingham's Π-Theorem

- Method of dimensional analysis (Buckingham's Π–Theorem)
  - The P-Theorem determines the maximum number of parameters to be considered
  - Number of physical quantities: k
  - Number of basic dimensions: r [m], [s], [kg], [K]
  - Number of dimensionless parameters: m= k – r

- Procedure
  - Determine the number of physical quantitites k $G_1 = f(G_2, G_3, \ldots, G_k)$
  - Decompose and determine the number of basic dimensions r
  - Determine m
  - Choose r recurring variables
    - Include all basic dimensions
    - Linearly independent
    - Don't choose the variables that are hard to measure
  - Determine the dimensionless parameters $\Pi_i = G_i \cdot (G_1^{\alpha_1} \cdot G_2^{\alpha_2} \cdot \ldots \cdot G_r^{\alpha_r})$
  - Check the dimensions
  - Formulate $\Pi_1 = g(\Pi_2, \Pi_3, \ldots, \Pi_m)$

Methods to determine dimensionless parameters: differential equations

- Starting point: differential equation that describes a physical (fluid mechanical) problem

- Determination of the similarity numbers:

  - Differential equation that describes the
  - Problem:

$$\frac{\partial p}{\partial x} = \eta \frac{\partial^2 u}{\partial y^2}$$

  - Introduce of dimensionless quantities and reference quantities:

$$u_{ref}, \quad p_{ref}, \quad \eta_{ref}, \quad L_{ref}, \quad h_{ref}, \quad \dots$$

$$u_{\infty}, \quad \Delta p \quad \eta_{ref}, \quad L, \quad h, \quad \dots$$

$$\overline{u} = \frac{u}{u_{\infty}}, \quad \overline{p} = \frac{p}{\Delta p}, \quad \overline{\eta} = \frac{\eta}{\eta_{ref}}, \quad \overline{x} = \frac{x}{L}, \quad \overline{y} = \frac{y}{h}, \quad \dots$$

  - Replace the variables in the differential equation:

$$\frac{\Delta p}{L} \frac{\partial \overline{p}}{\partial \overline{x}} = \frac{\eta_{ref} u_{\infty}}{h^2} \overline{\eta} \frac{\partial^2 \overline{u}}{\partial \overline{y}^2}$$

  - Divide the complete equation by one of the coefficients of the
  - terms:

$$\frac{\partial \overline{p}}{\partial \overline{x}} = \underbrace{\frac{L}{\Delta p} \frac{\eta_{ref} u_{\infty}}{h^2}}_{\Pi} \overline{\eta} \frac{\partial^2 \overline{u}}{\partial \overline{y}^2}$$

  - m terms → m-1 similarity numbers

Methods to determine dimensionless parameters

- Buckingham's Theorem yields the maximum number of similarity numbers for a given set of influence parameters.

- Differential equations contain additional information about the relationship between the influence parameters and the similarity numbers →The number of similarity numbers derived from Buckingham's Π-Theorem can be larger than the number derived from the differential equation.

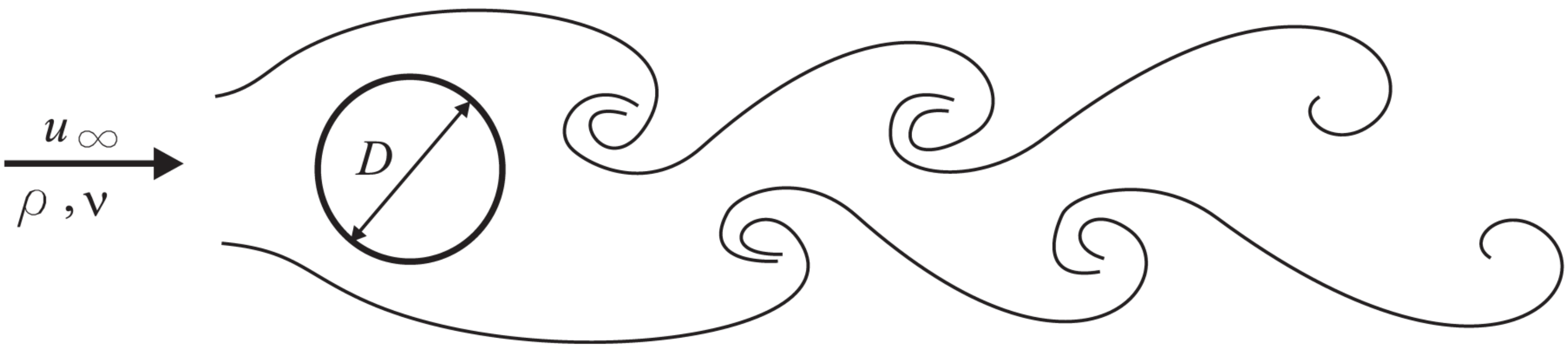
- Usually, similarity numbers determined using one of these methods can be written as a combination of known similarity numbers

- Example:

$$\frac{\partial \overline{p}}{\partial \overline{x}} = \underbrace{\frac{L}{\Delta p}\frac{\eta_{ref} u_\infty}{h^2}}_{\Pi} \overline{\eta} \frac{\partial^2 \overline{u}}{\partial \overline{y}^2} \quad \rightarrow \quad \frac{L}{\Delta p}\frac{\eta_{ref} u_\infty}{h^2} = \underbrace{\frac{\eta_{ref}}{\rho u_\infty h}}_{\frac{1}{Re}} \underbrace{\frac{\rho u_\infty^2}{\Delta p}}_{\frac{1}{Eu}} \underbrace{\frac{L}{h}}_{\text{geometry}}$$

Example 1

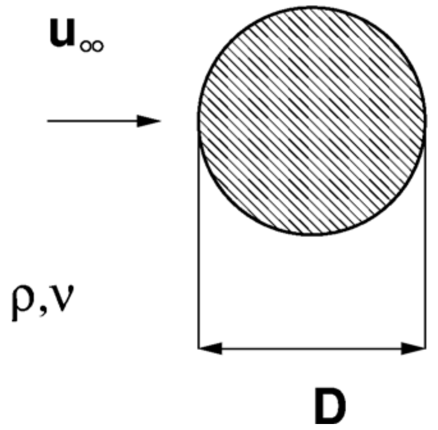
- The wake of a long cylinder with the diameter D is analyzed experimentally in a wind tunnel. Under certain conditions, a periodic vortex configuration is generated, the Kármán vortex street.



- Determine the dimensionless parameters of the problem
- How many variations of parameters are necessary in this investigation to measure the frequency of the vortex street?

Example 1

- Physical quantities
  - Freestream velocity $u_\infty \quad \left[\frac{m}{s}\right]$
  - Kinematic viscosity $\nu = \frac{\eta}{\rho} \quad \left[\frac{m^2}{s}\right]$
  - Density $\rho \quad \left[\frac{kg}{m^3}\right]$
  - Diameter of the cylinder $D \quad [m]$
  - Frequency $f \quad \left[\frac{1}{s}\right]$



- Number of similarity numbers:
  - Number of physical quantities: k = 5
  - Number of basic dimensions (m, s, kg): r = 3
  - Number of dimensionless parameters: m = k – r = 2

Example 1

- Recurring variables:
  - Freestream velocity $u_\infty \quad \left[\frac{m}{s}\right]$
  - Density $\rho \quad \left[\frac{kg}{m^3}\right]$
  - Diameter of the cylinder $D \quad [m]$
  - All dimensions are included, all variables are linearly independent

- Determination of the similarity numbers:
  - 1st number

$$\Pi_1 = \underbrace{f}_{\text{nonrecurring}} \quad \underbrace{\cdot\, u_\infty^{a_1} \cdot \rho^{b_1} \cdot D^{c_1}}_{\text{recurring}}$$

  - 2nd number:

$$\Pi_2 = \underbrace{\nu}_{\text{nonrecurring}} \quad \underbrace{\cdot\, u_\infty^{a_2} \cdot \rho^{b_2} \cdot D^{c_2}}_{\text{recurring}}$$

Example 1

- 1st similarity number

$$\Pi_1 = \underbrace{f}_{\text{nonrecurring}} \quad \underbrace{\cdot\, u_\infty^{a_1} \cdot \rho^{b_1} \cdot D^{c_1}}_{\text{recurring}}$$

  - Dimensional analysis:

$$[-] = [\frac{1}{s}] \quad [\frac{m}{s}]^{a_1} [\frac{kg}{m^3}]^{b_1} [m]^{c_1}$$

  - Comparison of the exponents:

$$\begin{aligned} [kg] &: 0 &&= b_1 && \\ [s] &: 0 &&= -1 - a_1 && \rightarrow a_1 = -1 \\ [m] &: 0 &&= a_1 - 3b_1 + c_1 && \rightarrow c_1 = 1 \end{aligned}$$

  - Hence:

$$\Pi_1 = \frac{f\ D}{u_\infty} = Sr$$

  - The first similarity number of this problem is the Strouhal number

Example 1

- 2nd similarity number

$$\Pi_1 = \underbrace{\nu}_{\text{nonrecurring}} \underbrace{\cdot u_\infty^{a_2} \cdot \rho^{b_2} \cdot D^{c_2}}_{\text{recurring}}$$

  - Dimensional analysis:

$$[-] = [\frac{m^2}{s}] \quad [\frac{m}{s}]^{a_2} [\frac{kg}{m^3}]^{b_2} [m]^{c_2}$$

  - Comparison of the exponents:

$$\begin{aligned} [kg] : 0 &= b_2 \\ [s] : 0 &= -1 - a_2 & \rightarrow a_2 = -1 \\ [m] : 0 &= 2 + a_2 - 3b_2 + c_2 & \rightarrow c_2 = -1 \end{aligned}$$

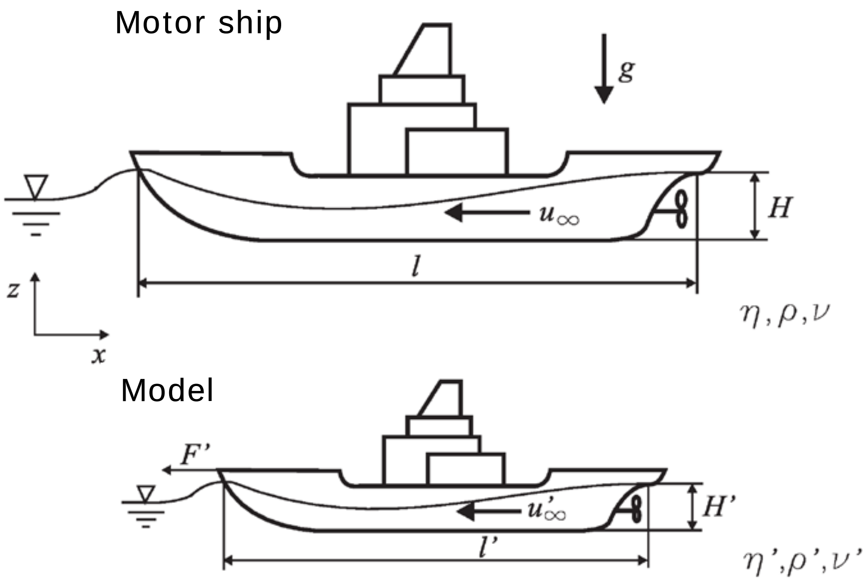
  - Hence:

$$\Pi_2 = \frac{\nu}{u_\infty \cdot D} = \frac{1}{Re}$$

  - The second similarity number of this problem is the reciprocal value of the Reynolds number
  - Sr=f(Re) → variation of 1 parameter in experiment

Example 2

- The hydrodynamic attributes of a motor ship shall be analyzed with a model in a water channel.



- Determine the dimensionless parameters of the problem with the method of differential equations using the momentum equation in z-direction, which describes the wave motion.

- Given: l, u_∞, η, ρ, g.

$$\rho \frac{dw}{dt} = -\frac{\partial p}{\partial z} - \rho g + \eta \nabla^2 w$$

Example 2

- Compute the velocity u'_∞ and the kinematic viscosity ν' of the model fluid such that the flows are similar.
- Given: u_∞, ν, $l/l' = 10$

- Compute the power of the motor ship at the velocity u_∞.

- Given: u_∞, u'_∞ , ρ', ρ, $l/l' = H/H' = 10$, drag force in the experiment F'.

Example 2

- Momentum equation in z-direction:

$$\rho \frac{dw}{dt} = -\frac{\partial p}{\partial z} - \rho g + \eta \nabla^2 w$$

- Dimensionless Terms for the derivatives:
  - 1st derivative: $$\frac{d\overline{u}}{d\overline{x}} = \frac{l}{u_\infty} \frac{du}{dx}$$
  - 2nd derivative: $$\frac{d^2\overline{u}}{d\overline{x}^2} = \frac{d}{d\overline{x}} \left(\frac{d\overline{u}}{d\overline{x}} \right) = \frac{l^2}{u_\infty} \frac{d^2u}{dx^2}$$
  - Differential operator: $$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

$$\overline{\nabla}^2 = l^2 \nabla^2$$

- The values *ρ, η, g* are given and constant

Example 2

- Dimensionless parameters:
  - Velocity: $$\overline{w} = \frac{w}{u_\infty}$$
  - Pressure: $$\frac{p}{\rho u_\infty^2} = \frac{\text{static pressure}}{\text{dynamic pressure}}$$
    - also possible: Δp as reference pressure
    - reference pressure determines similarity numbers
    - Pipe flow: Δp
    - compressible flow around wings etc.: ρu^2_∞
  - Coordinates: $$\overline{z} = \frac{z}{l} \qquad \frac{\partial}{\partial \overline{z}} = l \frac{\partial}{\partial z}$$ $$\overline{\nabla}^2 = l^2 \nabla^2$$
  - Time: $$\overline{t} = \frac{t}{l/u_\infty}$$
    - u_∞/l describes the time that a particle needs to pass a ship that has the length l and that moves with the velocity u_∞.

Example 2

- Hence: $\rightarrow \quad \rho \frac{u_\infty}{l} u_\infty \frac{d\overline{w}}{d\overline{t}} = -\frac{\rho u_\infty^2}{l} \frac{\partial \overline{p}}{\partial \overline{z}} - \rho g + \eta \frac{u_\infty}{l^2} \overline{\nabla}^2 \overline{w} \quad : \left(\frac{\rho u_\infty^2}{l} \right)$

$$\rightarrow \quad \frac{d\overline{w}}{d\overline{t}} = -\frac{\partial \overline{p}}{\partial \overline{z}} - \frac{l}{\rho u_\infty^2} \rho g + \eta \frac{u_\infty l}{l^2 u_\infty^2 \rho} \overline{\nabla}^2 \overline{w}$$

$$\rightarrow \quad \frac{d\overline{w}}{d\overline{t}} = -\frac{\partial \overline{p}}{\partial \overline{z}} - \frac{gl}{u_\infty^2} + \frac{\eta}{\rho u_\infty l} \overline{\nabla}^2 \overline{w}$$

$$\frac{d\overline{w}}{d\overline{t}} = -\frac{\partial \overline{p}}{\partial \overline{z}} - \frac{1}{Fr^2} + \frac{1}{Re} \overline{\nabla}^2 \overline{w}$$

- Velocity and viscosity in the experiment provided that the flows are similar:

$$Fr = Fr' \quad \rightarrow \quad \frac{u_\infty^2}{gl} = \frac{u_\infty'^2}{gl'} \quad \rightarrow \quad u'_\infty = u_\infty \sqrt{\frac{l'}{l}} = \frac{u_\infty}{\sqrt{10}}$$

$$Re = Re' \rightarrow \frac{\rho u_\infty l}{\eta} = \frac{u_\infty l}{\nu} = \frac{u'_\infty l'}{\nu'} \quad \rightarrow \quad \nu' = \nu \frac{u'_\infty}{u_\infty} \frac{l'}{l} \frac{\nu}{10\sqrt{10}}$$

Example 2

- Power of the engine:

$$c_D = \frac{F/A}{\rho/2 \cdot u_\infty^2} \hat{=} \frac{\text{friction}}{\text{dynamic pressure}} \qquad (A = lH)$$

$$c_D = c'_D \quad \rightarrow \quad \frac{F}{\rho/2 \cdot u_\infty^2 A} = \frac{F'}{\rho'/2 \cdot u'^2_\infty A'}$$

$$\rightarrow F = F' \frac{\rho}{\rho'} \frac{u_\infty^2}{u'^2_\infty} \frac{A}{A'} = 100 F' \frac{\rho}{\rho'} \frac{u_\infty^2}{u'^2_\infty}$$

$$P = F \cdot u_\infty = 100 F' \frac{\rho}{\rho'} \frac{u_\infty^3}{u'^2_\infty}$$

Example 3

- In a gas flow the heat transfer is determined from the viscous effects and from heat conduction. The influencing quantities are the heat conductivity λ [kg m/s³K], the dynamic viscosity and the reference values for the temperature, the velocity, and the length. The physical relationship can be described with the energy equation:

$$\lambda \frac{\partial^2 T}{\partial y^2} + \eta \left(\frac{\partial u}{\partial y} \right)^2 = 0$$

- Determine the dimensionless parameters of the problem
  - with the method of differential equations
  - with the Π-Theorem

- Expand the resulting parameter with the specific heat capacity c_p and formulate the new coefficient as a product of three different parameters.

- Hint:
  - The material quantities are constant
  - The fourth basic dimension is the temperature.

Example 3

- energy equation :

$$\lambda \frac{\partial^2 T}{\partial y^2} + \eta \left(\frac{\partial u}{\partial y} \right)^2 = 0$$

- energy equation with reference values:

$$\frac{\lambda T_R}{l^2} \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \eta \frac{u_R^2}{l^2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 = 0 \quad : \quad \frac{\lambda T_R}{l^2}$$

$$\Pi = \frac{\eta u_R^2}{\lambda T_R}$$

- Π-Theorem, Physical quantities:
  - Heat conductivity: $\lambda \quad \left[\frac{kgm}{s^3 K} \right]$
  - dynamic viscosity: $\eta \quad \left[\frac{kg}{ms} \right]$
  - Temperature: $T_R \quad [K]$
  - Velocity: $u_R \quad \left[\frac{m}{s} \right]$
  - Length: $l_R \quad [m]$

Example 3

- Number of similarity numbers:
  - Number of physical quantities: $k = 5$
  - Number of basic dimensions (m, s, kg, K): $r = 4$
  - Number of similarity numbers: $m = k - r = 1$

- recurring variables: η, T_R, U_R, l_R $\quad \Pi = \lambda^a \, \eta^b \, T_R^c \, u_R^d \, l_R^e$ choose b=1

$$T_R \quad [K] \; [\quad \begin{array}{lcl} kg & : & 0 = a + 1 \\ m & : & 0 = a - 1 + d + e \\ s & : & 0 = -3a - 1 - d \\ K & : & 0 = -a + c \end{array} \;\rightarrow\; \begin{array}{rcl} a = c & = & -1 \\ d & = & 2 \\ e & = & 0 \end{array} \;\rightarrow\; \Pi = \frac{\eta u_R^2}{\lambda T_R}$$

- Similarity number expressed by well-known similarity numbers:

$$c_p = \frac{\gamma \; R}{\gamma \; - \; 1} \qquad \Pi = \frac{\eta u_R^2}{\lambda T_R} = \frac{\eta \; c_p}{\lambda} \; \frac{u_R^2}{c_p \; T_R} = \frac{\eta \; c_p}{\lambda} \; \frac{u_R^2}{\gamma \; R \; T_R} (\gamma \; - \; 1)$$

$$\Pi = Pr \cdot Ma^2 (\gamma \; - \; 1)$$

Example 3

- The laminar boundary layer flow on a flat plate, neglecting the viscous heat, can be described with the continuity, the momentum, and the energy equation in the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \eta \frac{\partial^2 u}{\partial y^2}$$

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \lambda \frac{\partial^2 T}{\partial y^2}$$

- Determine the dimensionless parameters of the problem

- Reformulate the resulting parameters by using well-known parameters of fluid mechanics.

Assuming constant material quantities the flow field is independent of the temperature field. Both distributions can be computed separately.

- Specify the assumptions to determine the temperature distribution in the boundary layer directly from the velocity distribution. Compare the differential equations and assume that the velocity distribution is already known.

Example 3

- Method of der Differential equations
  - Dimensionless parameters:

$$\overline{u} = \frac{u}{u_\infty}; \quad \overline{v} = \frac{v}{u_\infty}; \quad \overline{\rho} = \frac{\rho}{\rho_\infty}; \quad \overline{x} = \frac{x}{L}; \quad \overline{y} = \frac{y}{L};$$

$$\overline{\eta} = \frac{\eta}{\eta_\infty}; \quad \overline{c_p} = \frac{c_p}{c_{p_\infty}}; \quad \overline{T} = \frac{T}{T_\infty}; \quad \overline{\lambda} = \frac{\lambda}{\lambda_\infty}$$

  - continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \rightarrow \quad \frac{u_\infty}{L}\left(\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}}\right) = 0$$

  - Momentum equation:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \eta\frac{\partial^2 u}{\partial y^2}$$

$$\rightarrow \quad \rho_\infty \frac{u_{\infty^2}}{L} \overline{\rho}\left(\overline{u}\,\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\,\frac{\partial \overline{u}}{\partial \overline{y}}\right) = \eta_\infty\,\overline{\eta}\,\frac{u_\infty}{L^2}\,\frac{\partial^2 \overline{u}}{\partial \overline{y^2}}$$

Example 3

- Momentum equation:

$$\bar{\rho}\left(\bar{u}\,\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \bar{\eta}\,\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\underbrace{\left(\frac{\eta_\infty\; u_\infty\; L}{L^2\;\rho_\infty\; u_\infty^2}\right)}_{\Pi_1} \rightarrow \Pi_1 = \frac{\eta_\infty}{L\;\rho_\infty\; u_\infty} = \frac{1}{Re}$$

- Energy equation:

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \lambda\frac{\partial^2 T}{\partial y^2} \rightarrow$$

$$\frac{\rho_\infty\; c_{p_\infty}\; u_\infty\; T_\infty}{L}\;\bar{\rho}\;\bar{c}_p\left(\bar{u}\,\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{T}}{\partial \bar{y}}\right) = \frac{\lambda_\infty\; T_\infty}{L^2}\;\bar{\lambda}\;\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}$$

$$\bar{\rho}\;\bar{c}_p\left(\bar{u}\,\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{T}}{\partial \bar{y}}\right) = \underbrace{\frac{\lambda_\infty\; T_\infty\; L}{L^2\;\rho_\infty\; c_{p_\infty}\; u_\infty\; T_\infty}}_{\Pi_2}\left(\bar{\lambda}\;\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\right)$$

$$\Pi_2 = \frac{\lambda_\infty}{L\;\rho_\infty\; c_{p_\infty}\; u_\infty}\;\frac{\eta_\infty}{\eta_\infty} = \frac{1}{Pr}\cdot\frac{1}{Re}$$

Example 3

- Dimensionless equations with constant material properties:

$$\overline{\rho} = \overline{c_p} = \overline{\lambda} = \overline{\eta} = 1$$

$$\frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{v}}{\partial \overline{y}} = 0$$

$$\overline{\rho}\left(\overline{u}\,\frac{\partial \overline{u}}{\partial \overline{x}} + \overline{v}\,\frac{\partial \overline{u}}{\partial \overline{y}}\right) = \frac{1}{Re}\,\overline{\eta}\left(\frac{\partial^2 \overline{u}}{\partial \overline{y^2}}\right)$$

$$\overline{\rho}\;\overline{c_p}\left(\overline{u}\,\frac{\partial \overline{T}}{\partial \overline{x}} + \overline{v}\,\frac{\partial \overline{T}}{\partial \overline{y}}\right) = \frac{1}{Pr}\cdot\frac{1}{Re}\,\overline{\lambda}\left(\frac{\partial^2 \overline{T}}{\partial \overline{y}^2}\right)$$

- Comparison between momentum and energy equation:

  By replacing T with u and proposing Pr = 1, the energy and the momentum equation are identical

$$\frac{\eta_\infty\; c_{p_\infty}}{\lambda_\infty} = 1$$

Biological & Medical Fluid Mechanics (BMF/BME)
06: friction

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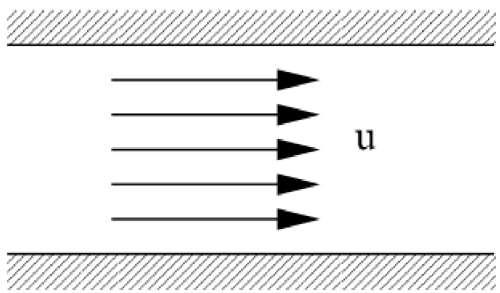
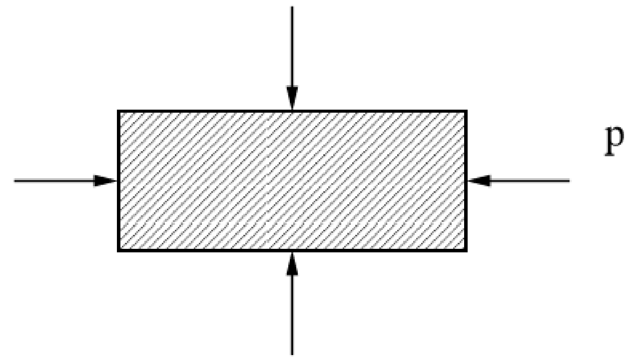
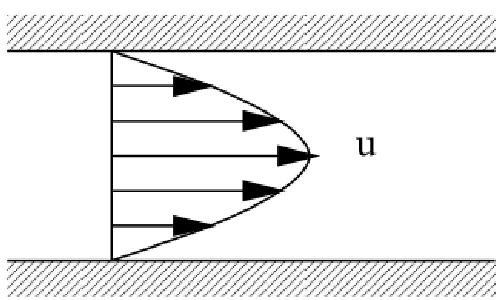
Flows with friction

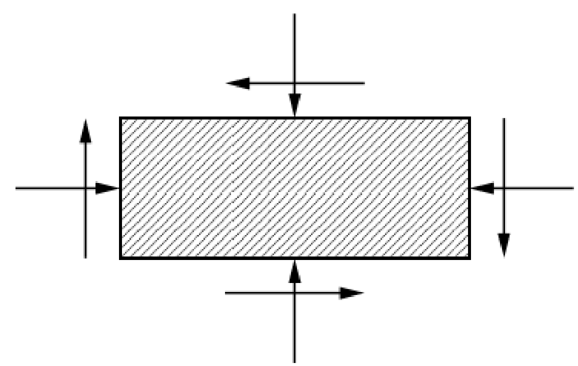
- Up to now: frictionless flows
  → only normal forces → pressure





- Now: flows with friction
  → normal and tangential forces





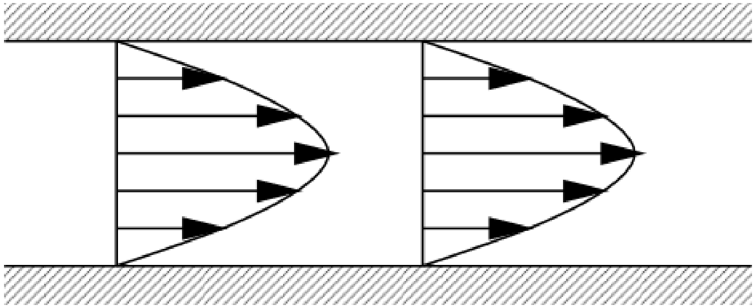
Simplications

- Steady flow $\partial/\partial t = 0$
- Fully developed flow
- Laminar flow
- Imcompressible flow

Fully developed → The velocity profiles does not change along the axis

$$\longrightarrow \frac{\partial u}{\partial s} = 0; \ \frac{\partial^2 u}{\partial s^2} = 0; \ \frac{\partial v}{\partial s} = 0; \ \frac{\partial^2 v}{\partial s^2} = 0$$

→ parallel flow



Simplifications

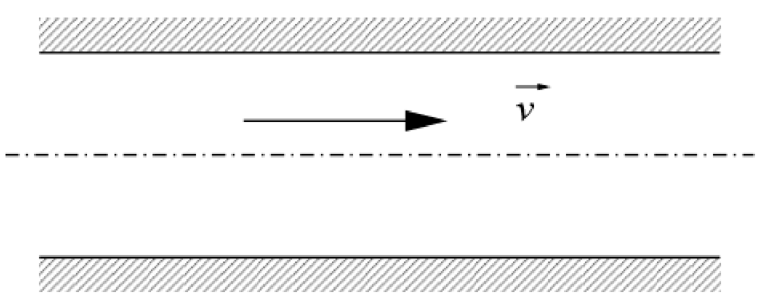
- Continuity equation for incompressible flows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 : \quad \frac{\partial u}{\partial x} = 0 \rightarrow \frac{\partial v}{\partial y} = 0$$

$\rightarrow v = 0$ → at the wall; → on the axis

$\rightarrow v = 0$ in the entire flow field

- Example: flow between parallel walls (pipe, plate)



$$\vec{v} = \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$\longrightarrow v = 0$ everywhere

$\longrightarrow$ parallel flow

Momentum equation with friction

- x-direction:

$$\frac{dI_x}{dt} = \underbrace{\int_A \rho v_x \, (\vec{v} \cdot \vec{n}) dA}_{=0} = \sum F_{A_x} = \underbrace{F_{p_x}}_{\text{pressure}} + \underbrace{F_{R_x}}_{\text{friction}}$$

$$\Rightarrow \sum F_{A_x} = F_{p_x} + F_{R_x} = 0 \qquad \text{Balance of forces}$$

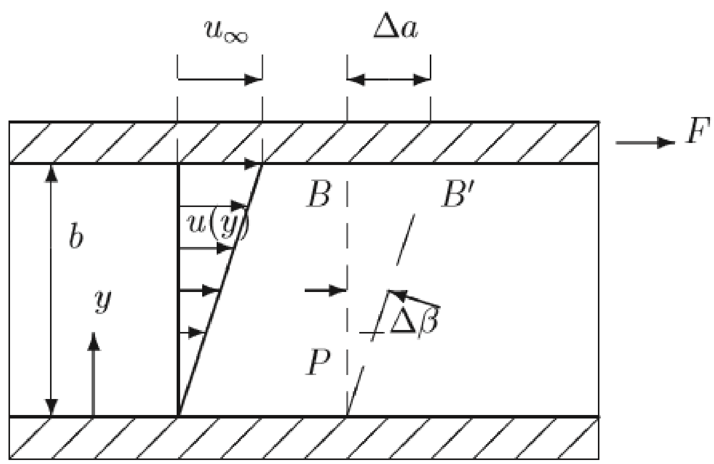
→ friction forces are balanced by pressure force

- y-direction: (volume forces neglected)

$$\frac{dI_y}{dt} = \int_A \rho v_y \, (\vec{v} \cdot \vec{n}) dA = 0 = \sum F_{A_y} = F_{p_y} + F_{R_y}$$

Friction forces

- Experiment: Water between two plates:



$$u(y) = u_\infty \frac{y}{b}$$ (linear velocity profile)

$$\Rightarrow \quad \frac{du}{dy} = \frac{u_\infty}{b} = \text{const.}$$ (in this special case)

Boundary conditions: $u(0) = 0, u(b) = u_\infty$ (no slip-condition)

$$\tan\Delta\beta \underbrace{\approx}_{\Delta\beta \text{ small}} \Delta\beta = \frac{\Delta a}{b}; \qquad \Delta a = u_\infty \Delta t$$

$$\Delta\beta = \frac{u_\infty \Delta t}{b} \qquad \Longrightarrow \Delta\beta = f(F, \Delta t)$$

$$\dot{\gamma} = \lim_{\Delta t \to 0} \frac{\Delta\beta}{\Delta t} = \frac{u_\infty}{b} = \frac{du}{dy} \qquad \text{shear rate}$$

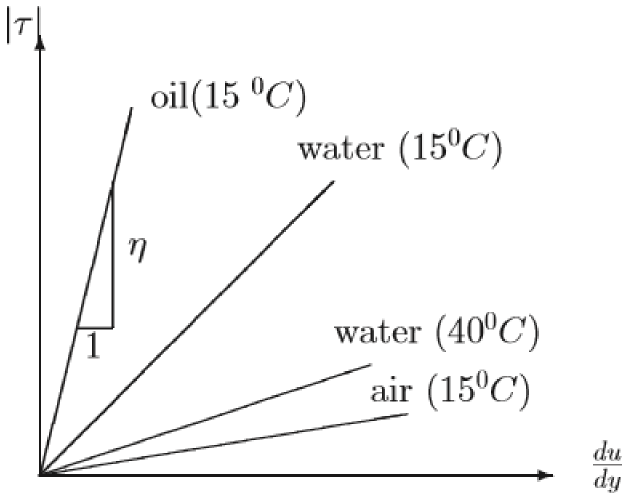
$$\tau = \frac{F_R}{A} = f(\dot{\gamma}) = f\left(\frac{du}{dy}\right) \qquad \text{shear stress}$$

$\Rightarrow$ for ordinary fluids (water, oil, air, ...):

$$\tau = \underbrace{-\eta}_{\text{viscosity}} \frac{du}{dy} \qquad \text{Newtonian fluids}$$

Newtonian fluids

- $\eta = \eta(T, p) \approx \eta(T)$ (weak dependence on p)
- $\tau = f\left(\frac{du}{dy}\right) = -\eta \frac{du}{dy}$ → linear dependence with slope η



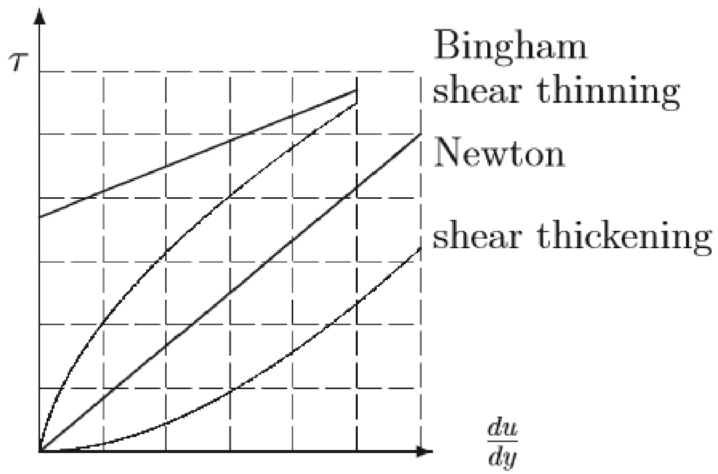
$$\boxed{\tau = -\eta \frac{du}{dy}}$$

$[\eta] = \left[\frac{Ns}{m^2}\right]$ dynamic viscosity

gases: $\eta \uparrow$ with $T \uparrow$

liquids: $\eta \downarrow$ with $T \uparrow$

- $\tau = f\left(\frac{du}{dy}\right)$ → nonlinear dependence!



$\boxed{\eta = f\left(\frac{du}{dy}\right)}$

shear thickening: $\eta \uparrow$ with $\frac{du}{dy} \uparrow$ (e.g. quicksand)

shear thinning: $\eta \downarrow$ with $\frac{du}{dy} \uparrow$ (e.g. latex paint)

Bingham fluid: solid if $\tau < \tau_0$ ($u = 0$ for $\tau < \tau_0$)
fluid if $\tau > \tau_0$ ($u \neq 0$ for $\tau > \tau_0$) (e.g. toothpaste)

Summary friction forces

- Friction forces react to movements and accelerations
- The higher the viscosity the higher the friction force
- The tangential forces depend strongly on the velocity gradient
- The friciton model depends on the fluid
- “Ordinary fluids” (water, oil, air, ..): Newtonian fluids

$$\tau = -\eta \frac{du}{dy}$$

- Blood is a Non-Newtonian fluid!
  → But under certain conditions (e.g. blood flow in big arteries) the Newtonian model could be a good approximation

Equilibrium of forces

- y-direction:

$$F_{p_y} + F_{R_y} = 0$$

$$F_{R_y} \sim \frac{dv_y}{dx} \quad \Rightarrow \quad F_{R_y} = 0 \quad \text{as } v_y = 0$$

$$\Rightarrow \quad F_{p_y} = 0 \quad \Rightarrow \quad p(y) = \text{const.} \quad \rightarrow \quad \frac{\partial p}{\partial y} = 0$$

→ without volume forces!

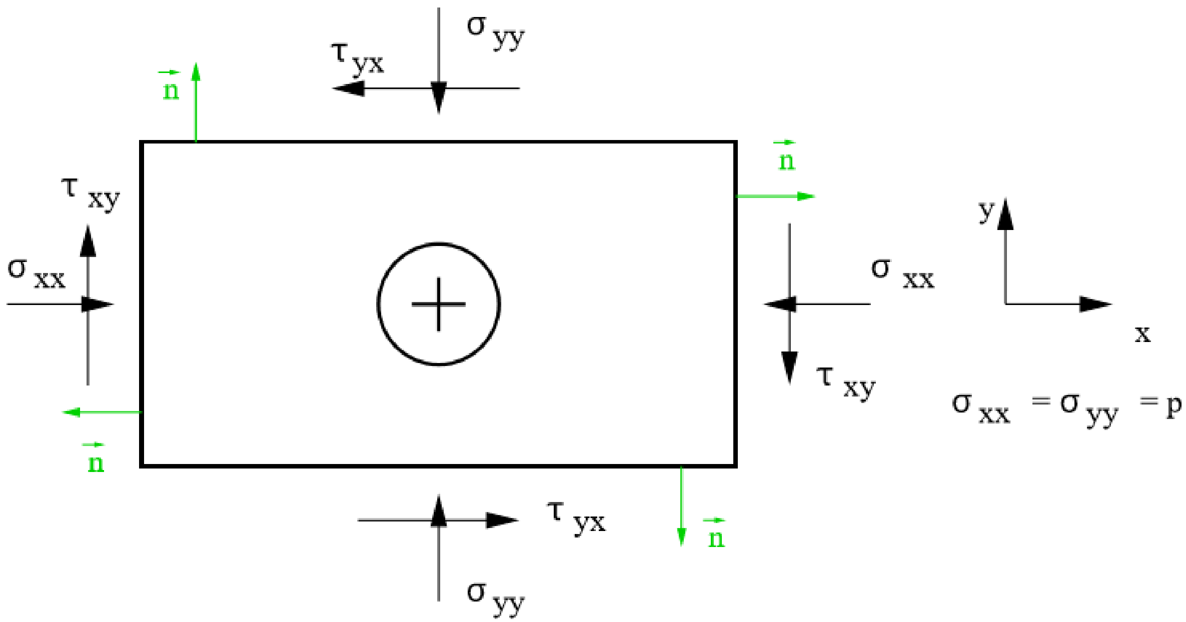
- x-direction:

$$F_{p_x} + F_{R_x} = 0; \qquad F_{R_x} \neq 0$$

driving mechanisms:

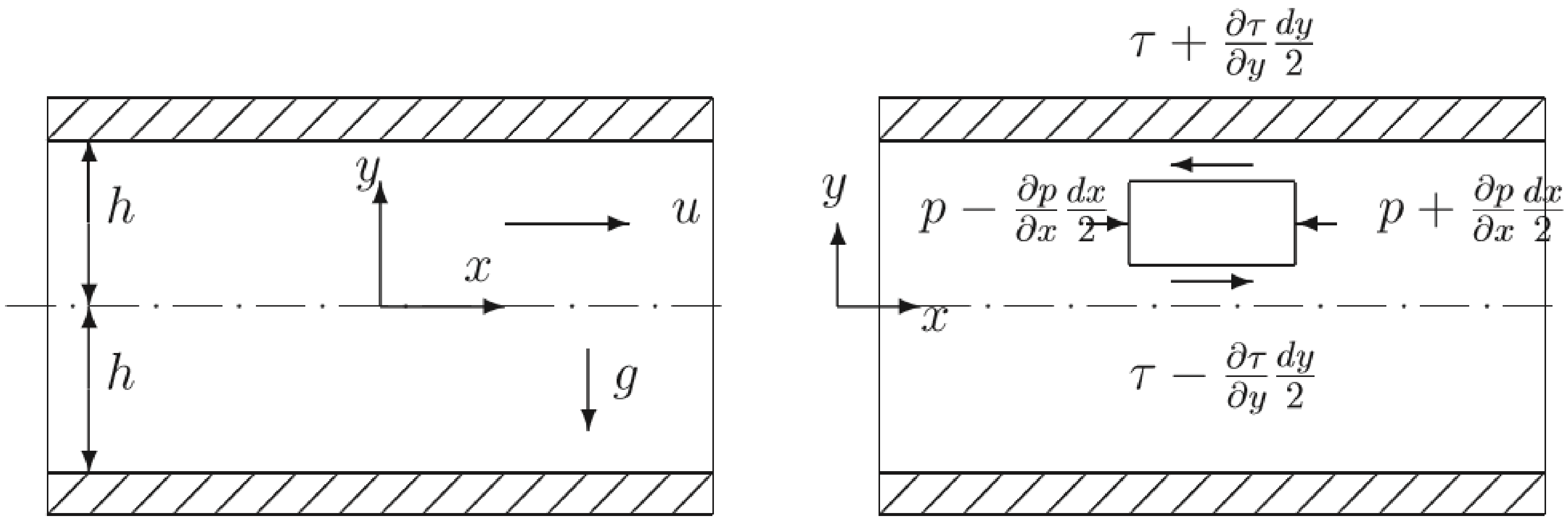
- Pressure gradient Δp in pipes or between plates
- Moving walls u_W (Couette flow, no slip condition)
- Gravitation $\vec{g}$ (oil films with free surface)

Equilibrium of forces



- Convention: sign of forces
  - Positive normal stress (=pressure), if $\vec{F}_p$ is contrary to the normal vector
  - Positive tangential stress (=friction) points at the coordinate direction, if the normal vector points against the coordinate direction

Steady flow between parallel plates



- Balance of forces in x-direction:

$$p(x)dy - p(x+dx)dy + \tau(y)dx - \tau(y+dy)dx = 0$$

$$\Rightarrow p - (p + \frac{\partial p}{\partial x}dx)dy + \tau - (\tau + \frac{\partial \tau}{\partial y}dy)dx = 0$$

$$-\frac{\partial p}{\partial x}dx\,dy - \frac{\partial \tau}{\partial y}dy\,dx = 0$$

$$-\frac{\partial p}{\partial x} - \frac{\partial \tau}{\partial y} = 0$$

Steady flow between parallel plates

- Newton: $\tau = -\frac{\partial u}{\partial y} \qquad \Rightarrow \qquad -\frac{\partial p}{\partial x} + \eta \frac{\partial^2 u}{\partial y^2} = 0$
- Y-direction: $p dx - (p + \frac{\partial p}{\partial y} dy) dx - \rho g dx\, dy = 0$

$$\Rightarrow \quad -\frac{\partial p}{\partial y} - \rho g = 0 \Rightarrow \quad p = -\rho g y + f_1(x)$$

Hydrostatics

- Velocity profile u(y): 1st integration of $\frac{\partial^2 u}{\partial y^2} = \frac{1}{\eta} \frac{\partial p}{\partial x}$

with $u = u(y) \quad \Rightarrow \quad \frac{\partial u}{\partial y} = \frac{du}{dy} \quad \text{and} \quad \frac{\partial p}{\partial x} \neq f(y):$

$$\Rightarrow \quad \frac{du}{dy} = \frac{1}{\eta} \frac{\partial p}{\partial x} y + c_1$$

- 2nd integration: $u(y) = \frac{1}{2\eta} \frac{\partial p}{\partial x} y^2 + c_1 y + c_2$

Steady flow between parallel plates

- Boundary conditions:

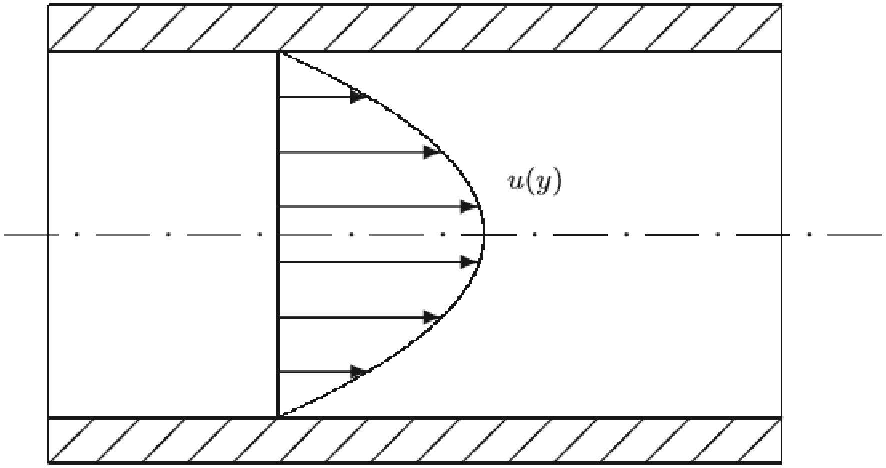
$$y = \pm h: \qquad u = 0$$

$$\Rightarrow 0 \quad = \quad \frac{1}{\eta}\frac{\partial p}{\partial x}\frac{h^2}{2} + c_1 h + c_2$$

$$0 \quad = \quad \frac{1}{\eta}\frac{\partial p}{\partial x}\frac{h^2}{2} - c_1 h + c_2$$

$$\Rightarrow c_1 = 0; \qquad c_2 = -\frac{1}{\eta}\frac{\partial p}{\partial x}\frac{h^2}{2}$$

$$\Rightarrow u(y) = \frac{1}{2\eta}\frac{\partial p}{\partial x}(y^2 - h^2)$$



- Volume flux per unit width:

$$q = \frac{\dot{V}}{b} = \int_{-h}^{h} u(y)dy = \int_{-h}^{h} \frac{1}{2\eta}\frac{\partial p}{\partial x}(y^2 - h^2)dy$$

$$= \frac{1}{2\eta}\frac{\partial p}{\partial x}\left[\frac{y^3}{3} - h^2 y\right]_{-h}^{h} = -\frac{2h^3}{3\eta}\frac{\partial p}{\partial x}$$

with $\quad -\frac{\partial p}{\partial x} = \frac{\Delta p}{l} \quad \Rightarrow \quad q = \frac{2h^3 \Delta p}{3\eta l}$

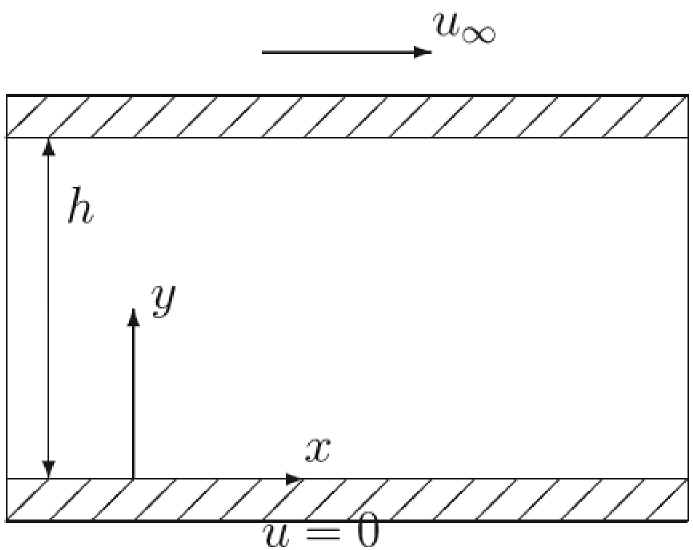
$$\overline{u} = \frac{q}{2h} = \frac{h^2 \Delta p}{3\eta l}; \quad u_{\text{max}} = u(y=0) = -\frac{h^2}{2\eta}\frac{\partial p}{\partial x} = \frac{3}{2}\overline{u}$$

- Pressure distribution: if $\frac{\partial p}{\partial x}, \eta, h$ are known, for $p_0 = p(x=0)$:

$$f_1(x) = \frac{\partial p}{\partial x}x + p_0 \Rightarrow p = -\rho g y + \frac{\partial p}{\partial x}x + p_0$$ (laminar flow)

Couette flow

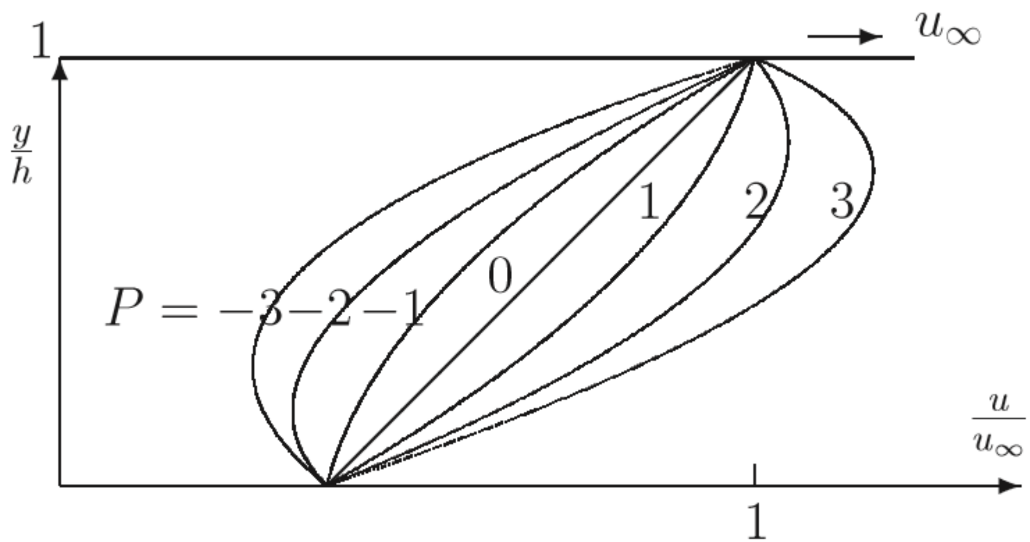
- Changed boundary conditions:



$$
\begin{aligned}
y = 0: \qquad u &= 0 \\
y = h: \qquad u &= u_\infty \\
c_2 &= 0 \\
c_1 &= \frac{u_\infty}{h} - \frac{1}{2\eta}\frac{\partial p}{\partial x} h
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad u(y) &= u_\infty \frac{y}{h} + \frac{1}{2\eta}\frac{\partial p}{\partial x}(y^2 - hy) \\
\frac{u(y)}{u_\infty} &= \frac{y}{h} \underbrace{- \frac{1}{2\eta u_\infty}\frac{\partial p}{\partial x}}_{P} \frac{y}{h}\left(1 - \frac{y}{h}\right) \\
&= \frac{y}{h} + P\frac{y}{h}\left(1 - \frac{y}{h}\right)
\end{aligned}
$$

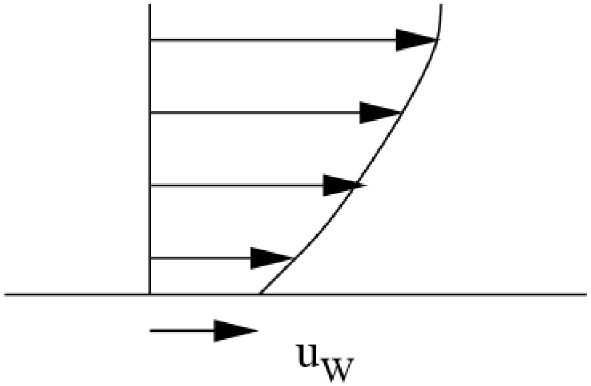
Couette flow



$$\frac{\partial p}{\partial x} = 0 \quad \Rightarrow \quad P = 0 \Rightarrow \quad u = u_\infty \frac{y}{h}$$

Boundary conditions

- Wall



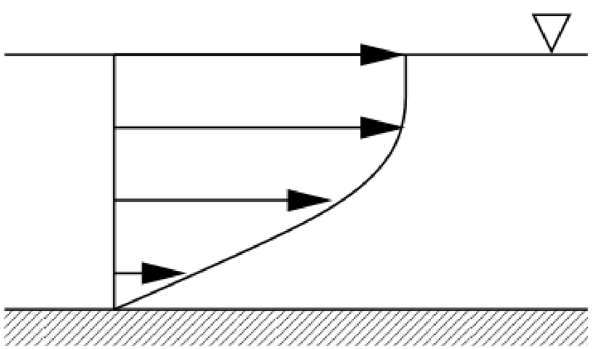
No slip condition

$\rightarrow u = u_W$

$v = 0$

but $\tau \neq 0$ is unknown

- Free surface



Ambient pressure

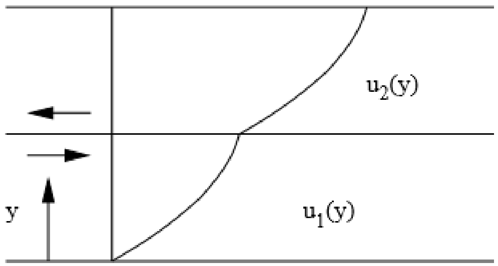
$\tau \approx 0$

$\tau = -\eta \dfrac{du}{dy} = 0$ friction between air and fluid can be neglected

$$\frac{\partial p}{\partial x} = \frac{\partial^2 u}{\partial y^2} = 0$$

Boundary conditions

- Limiting surface between two fluids

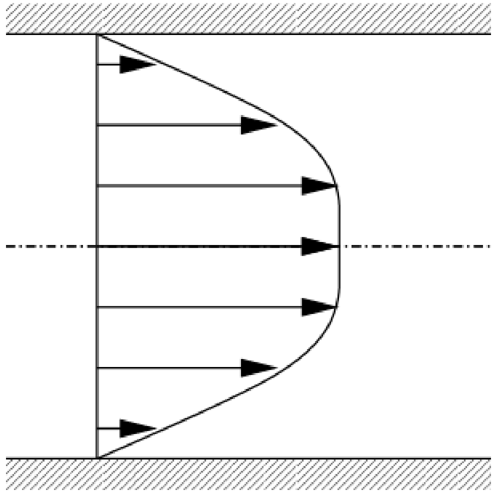


on the contact surface

$$u_1 = u_2$$

$$\tau_1 = \tau_2$$

- Symmetry



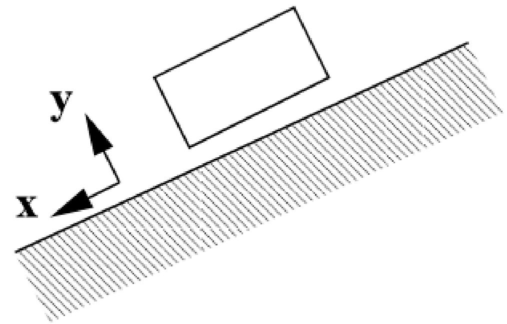
on the axis

$$\tau = 0$$

$$\frac{du}{dy} = 0$$

Method for solving typical laminar flow problems

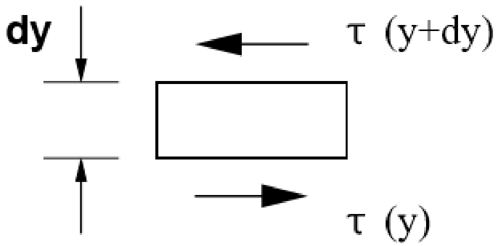
1. Choose an applicable coordinate system



(x along the stream lines)
sketch an infinitesimal element

2. Sketch all forces and stresses

3. Formulate the equilibrium of forces in the direction of streamlines

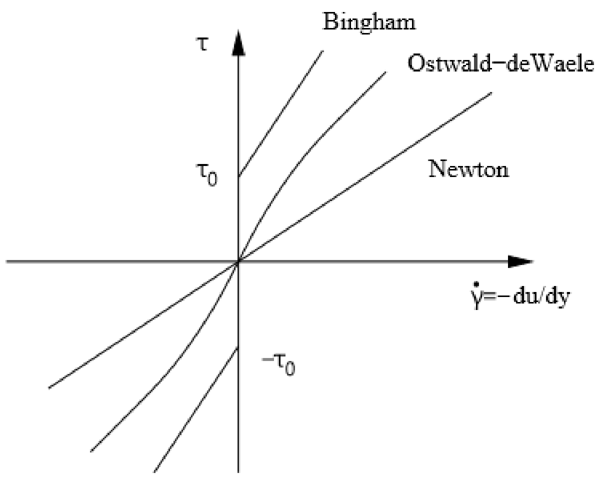


Taylor expansion

$$\tau(y+dy) = \tau(y) + \frac{\partial \tau}{\partial y} dy + \ldots$$

4. Integrate the differential equation (1st integration)
→ distribution of the shear stress

5. Introduce a model for τ as a function of u



Bingham: $\tau = -\eta \dfrac{\partial u}{\partial y} \pm \tau_0$

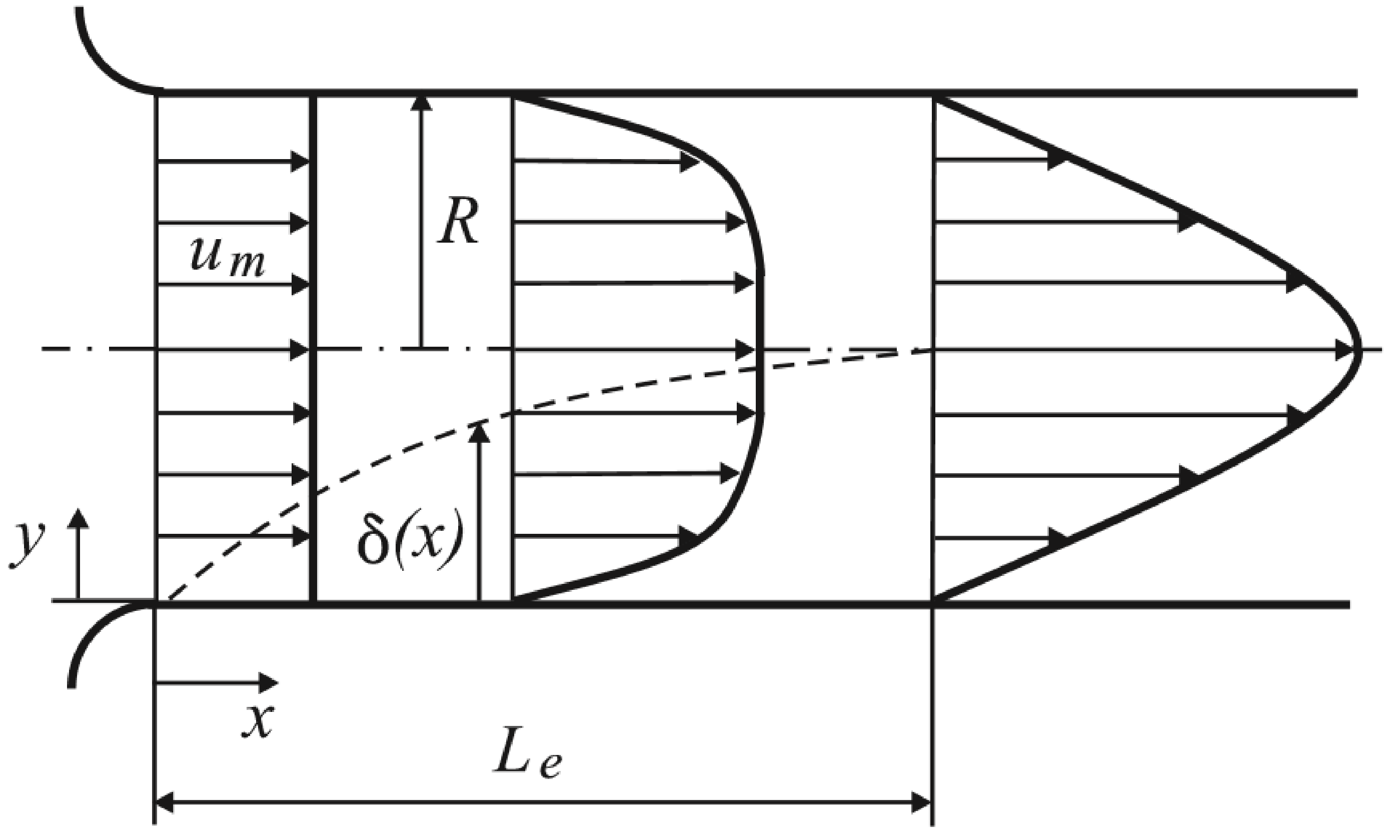
Ostwald-de Waele: $\tau = -C\left|\dfrac{\partial u}{\partial y}\right|^{n-1} \dfrac{\partial u}{\partial y}$

Newton: $\tau = -\eta \dfrac{\partial u}{\partial y}$

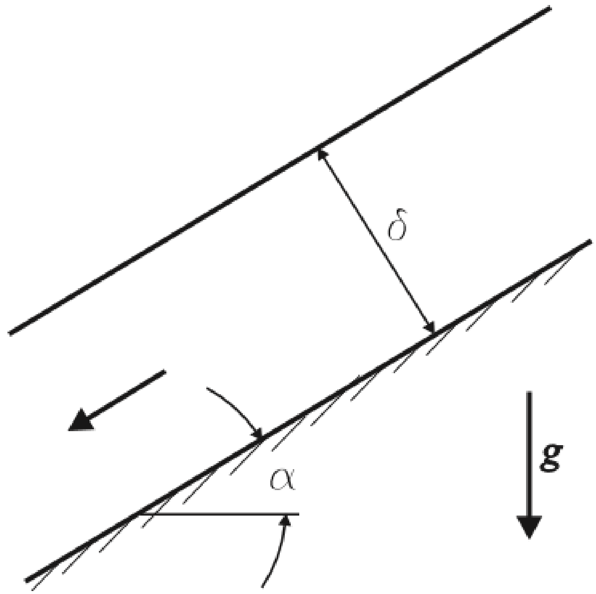
6. Integrate the differential equation (2nd integration)
→ velocity profile

7. Use boundary conditions for the unknown constants of the integration

Entrance flow region



- An oil film of constant thickness δ and width B is flowing on an inclined plate.



$$\delta = 3 \cdot 10^{-3}\ m \quad B = 1\ m \quad \alpha = 30^o \quad \rho = 800\ kg/m^3$$

$$\eta = 30 \cdot 10^{-3}\ Ns/m^2 \quad g = 10\ \frac{m}{s^2}$$

Calculate the volume flux.

Example

- An oil film of constant thickness and width → fully developed flow

$\rightarrow dI_x/dt = 0 \rightarrow$ Equilibrium of forces

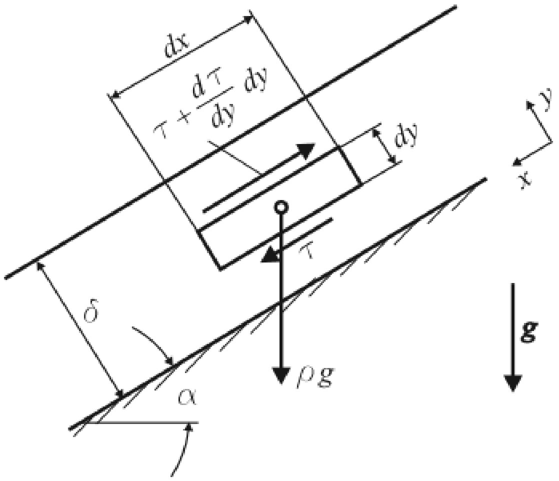
$\rightarrow \partial u/\partial x = 0 \rightarrow u = u(y),\ u(y=0) = 0$ No-slip condition

Continuity: $\dfrac{du}{dx} + \dfrac{dv}{dy} = 0 \ \wedge \ v(y=0) = 0 \ \rightarrow v \equiv 0$ anywhere

$$\dot{Q} = \int_A \vec{v} \cdot \vec{n} dA = \int_0^\delta u(y) B dy$$

Example

- Equilibrium of forces for an infinitesimal element



$$\frac{dI_x}{dt} = 0 = pBdy - (p + \frac{\partial p}{\partial x}dx)Bdy$$

$$+\tau Bdx - (\tau + \frac{\partial \tau}{\partial y}dy)Bdx$$

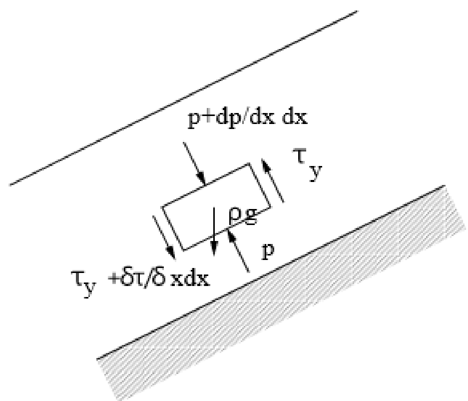
$$+\rho g \ \sin\alpha \ B \ dxdy$$

$$= \ -\frac{\partial p}{\partial x} \ dxdyB \ - \frac{\partial \tau}{\partial y} \ dy \ Bdx \ + \rho g \sin\alpha \ Bdxdy \ = \ 0$$

$$\partial p / \partial x = ?$$

Example

- Momentum equation: equilibrium of forces in y-direction



Fully developed flow $\rightarrow v = \ const \ = 0$

$$\rightarrow \tau_y = -\eta \frac{\partial v}{\partial x} = 0$$

$$\rightarrow \frac{\partial \tau_y}{\partial x} = 0$$

$$0 = \frac{dI_y}{dt} = p \ Bdx - \ (p + \frac{\partial p}{\partial y} dy) Bdx \ - \ \rho g \ \cos\alpha \ Bdxdy$$

$$\rightarrow \ \frac{\partial p}{\partial y} \ = \ \rho g \ \cos\alpha \ = \ \text{const} \ \neq f(y)$$

$$\rightarrow \ p(x, y) = \rho g y \ \cos\alpha + C(x)$$

Boundary condition: $p(x, y = \delta) = p_a = \text{const}$

Example

$\rightarrow \quad C(x) \neq f(x)$

$\rightarrow \quad p \neq f(x) \rightarrow \quad \frac{\partial p}{\partial x} = 0$ for free surfaces

$$\frac{\partial \tau}{\partial y} = \rho g \sin \alpha = \frac{d\tau}{dy}$$

1st integration: $\tau(y) = \rho g \sin \alpha y + C_1$

B.C.: $\tau(y = \delta) = 0 \longrightarrow C_1 = -\rho g \delta \sin \alpha$

Newtonian fluid: $\tau = -\eta \frac{du}{dy}$

$$\longrightarrow \frac{du}{dy} = -\frac{\tau}{\eta} = \frac{\rho g \sin \alpha}{\eta}(\delta - y)$$

Example

2 nd integration: $$u(y) = \frac{\rho g \sin\alpha}{\eta}(\delta y - \frac{1}{2}y^2 + C_2)$$

B.C.: $u(y=0) = 0 \longrightarrow C_2 = 0$

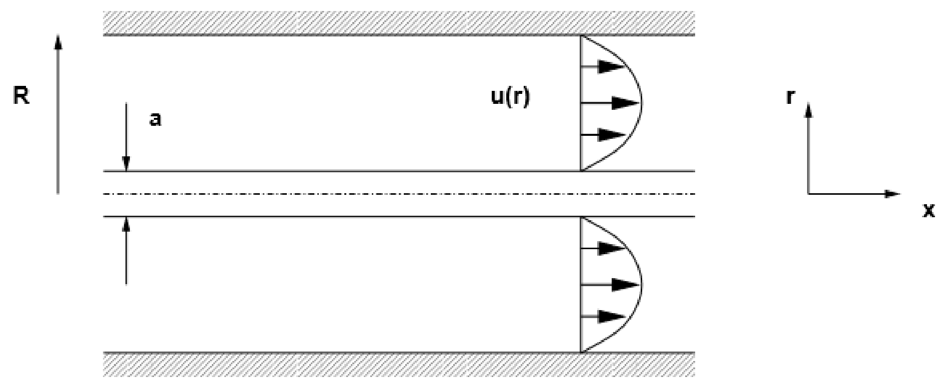
$$\boxed{u(y) = \frac{\rho g \sin\alpha}{\eta}(\delta y - \frac{1}{2}y^2)}$$

$$\dot{Q} = \int\limits_0^\delta u(y) B dy = B\left[\frac{\rho g \sin\alpha}{\eta}\left(\frac{\delta}{2}y^2 - \frac{1}{6}y^3\right)\right]_0^\delta = 1.2 \cdot 10^{-3}\frac{m^3}{s}$$

Example 2

- Fully developed flow of a Newtonian fluid between two coaxial cylinders

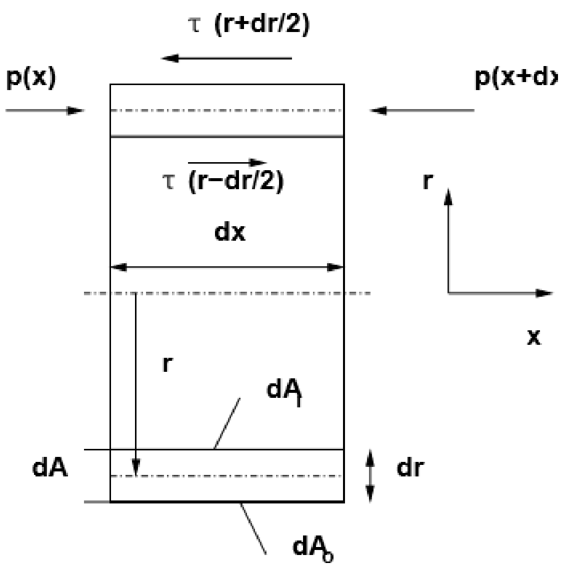
- Given: $R,\ a,\ \eta,\ \dfrac{dp}{dx}$



a) Derive the differential equation for the shear stress distribution $\tau(r)$ and the velocity $u(r)$. Integrate the equations.

b) What is the velocity of the inner cylinder $u_{c,i}$ for the case that the flow does not impose any force on it in x-direction?

a) $\tau(r)$, $u(r)$?



$$dA = 2\pi r dr$$

$$dA_i = 2\pi(r - \frac{dr}{2})dx$$

$$dA_o = 2\pi(r + \frac{dr}{2})dx$$

$$\tau(r \pm \frac{dr}{2}) = \tau(r) \pm \frac{1}{2}\frac{\partial \tau}{\partial r}dr + \ldots$$

Example 2

- Equilibrium of forces $\sum F_x = 0$

$$0 = \; pdA - (p + \frac{\partial p}{\partial x}dx)dA + (\tau - \frac{1}{2}\frac{\partial \tau}{\partial r}dr)dA_i \; - (\tau + \frac{1}{2}\frac{\partial \tau}{\partial r}dr)dA_o$$

$$= \; 0 - \frac{\partial p}{\partial x}dx2\pi r dr + (\tau - \frac{1}{2}\frac{\partial \tau}{\partial r}dr)(2\pi(r - \frac{dr}{2})dx)$$

$$-(\tau + \frac{1}{2}\frac{\partial \tau}{\partial r}dr)(2\pi(r + \frac{dr}{2})dx)$$

Example 2

$$= -\frac{\partial p}{\partial x}dx2\pi rdr + 2\pi dx\Big[\underbrace{\tau r} - \tau\frac{dr}{2} - \frac{1}{2}\frac{\partial\tau}{\partial r}rdr + \underbrace{\frac{1}{4}\frac{\partial\tau}{\partial r}drdr}$$

$$-(\underbrace{\tau r} + \tau\frac{dr}{2} + \frac{1}{2}\frac{\partial\tau}{\partial r}rdr + \underbrace{\frac{1}{4}\frac{\partial\tau}{\partial r}drdr})\Big]$$

$$= -\frac{\partial p}{\partial x}2\pi rdrdx - \tau dr2\pi dx - \frac{\partial\tau}{\partial r}drr2\pi dx$$

$$\longrightarrow -\frac{\partial p}{\partial x} - \frac{\tau}{r} - \frac{\partial\tau}{\partial r} = -\frac{dp}{dx} - \frac{1}{r}\frac{d(\tau r)}{dr} = 0$$

Example 2

Newtonian fluid: $\tau = -\eta \dfrac{du}{dr}$

$$\boxed{\frac{dp}{dx} - \frac{\eta}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0}$$

b) $u_{c,i} = ?$

Boundary conditions

- $u(r = R) = 0$, no-slip condition

  Friction: $F_R(r = a) = 0 \to \tau(r = a) = 0 : (F_R = \tau A) \to \dfrac{du}{dr}\,|_{r=a} = 0$

- Fully developed flow $\to \dfrac{\partial p}{\partial x} \neq f(r) \to \dfrac{r}{\eta}\dfrac{dp}{dx} = \dfrac{d}{dr}\left(r\dfrac{du}{dr}\right)$

Example 2

1 st integration: $$\frac{1}{2\eta}\frac{dp}{dx}r^2 + C_1 = r\frac{du}{dr}$$

B.C.: $$\frac{du}{dr}\,|_{r=a} = 0 \to C_1 = -\frac{a^2}{2\eta}\frac{dp}{dx}$$

$$r\frac{du}{dr} = \frac{1}{2\eta}\frac{dp}{dx}(r^2 - a^2) \to \frac{du}{dr} = \frac{1}{2\eta}\frac{dp}{dx}(r - \frac{a^2}{r})$$

2 nd integration: $$u(r) = \frac{1}{2\eta}\frac{dp}{dx}(\frac{1}{2}r^2 - a^2\ln r) + C_2$$

Example 2

B.C.: $u(r=R)=0$

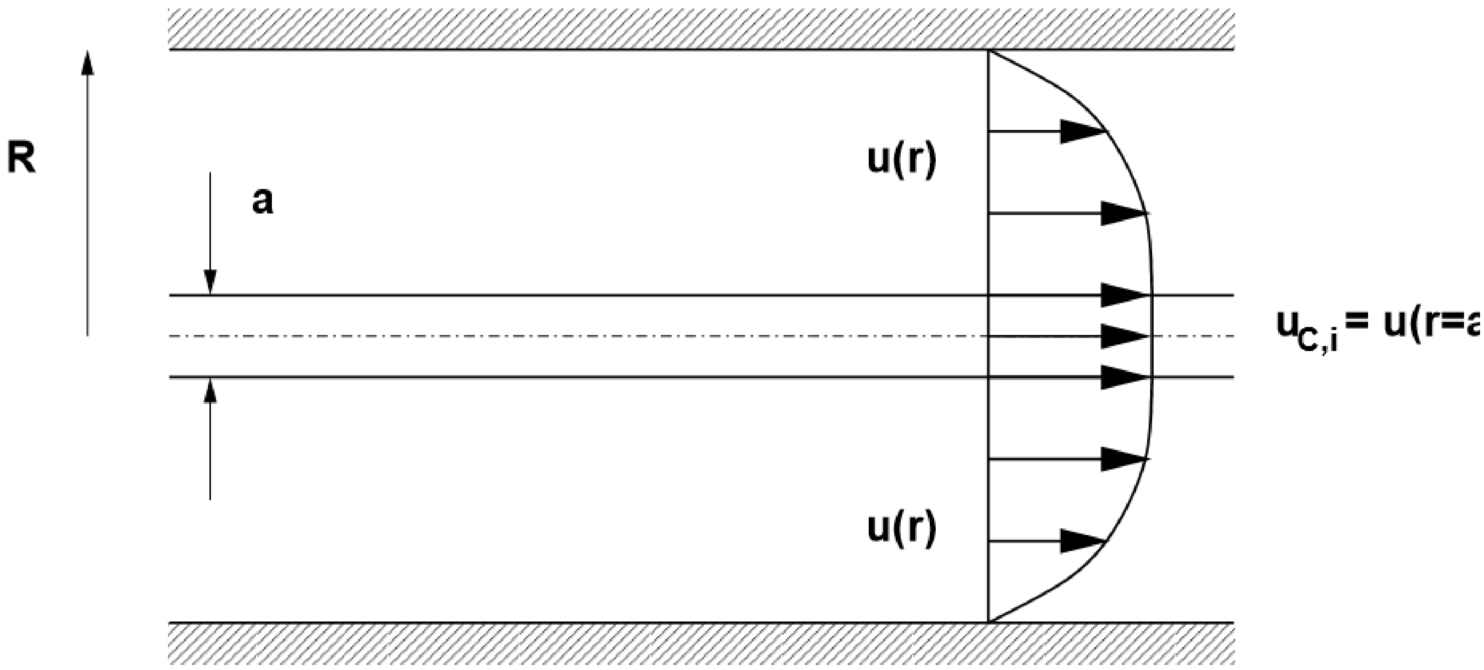
$$\rightarrow C_2 = -\frac{1}{2\eta}\frac{dp}{dx}\left(\frac{1}{2}R^2 - a^2\ln R\right)$$

$$\rightarrow u(r) = \frac{1}{2\eta}\frac{dp}{dx}\left[\frac{1}{2}\left(r^2 - R^2\right) - a^2\ln r + a^2\ln R\right]$$

$$= \frac{1}{2\eta}\frac{dp}{dx}\left(\frac{r^2 - R^2}{2} + a^2\ln\frac{R}{a}\right)$$

$$u_{C,i} = u(a) = \frac{1}{2\eta}\frac{dp}{dx}\left(\frac{a^2 - R^2}{2} + a^2\ln\frac{R}{a}\right)$$

Example 2



$\longrightarrow$ Bingham fluid: same behaviour!

Biological & Medical Fluid Mechanics (BMF/BME)
07: turbulent flows

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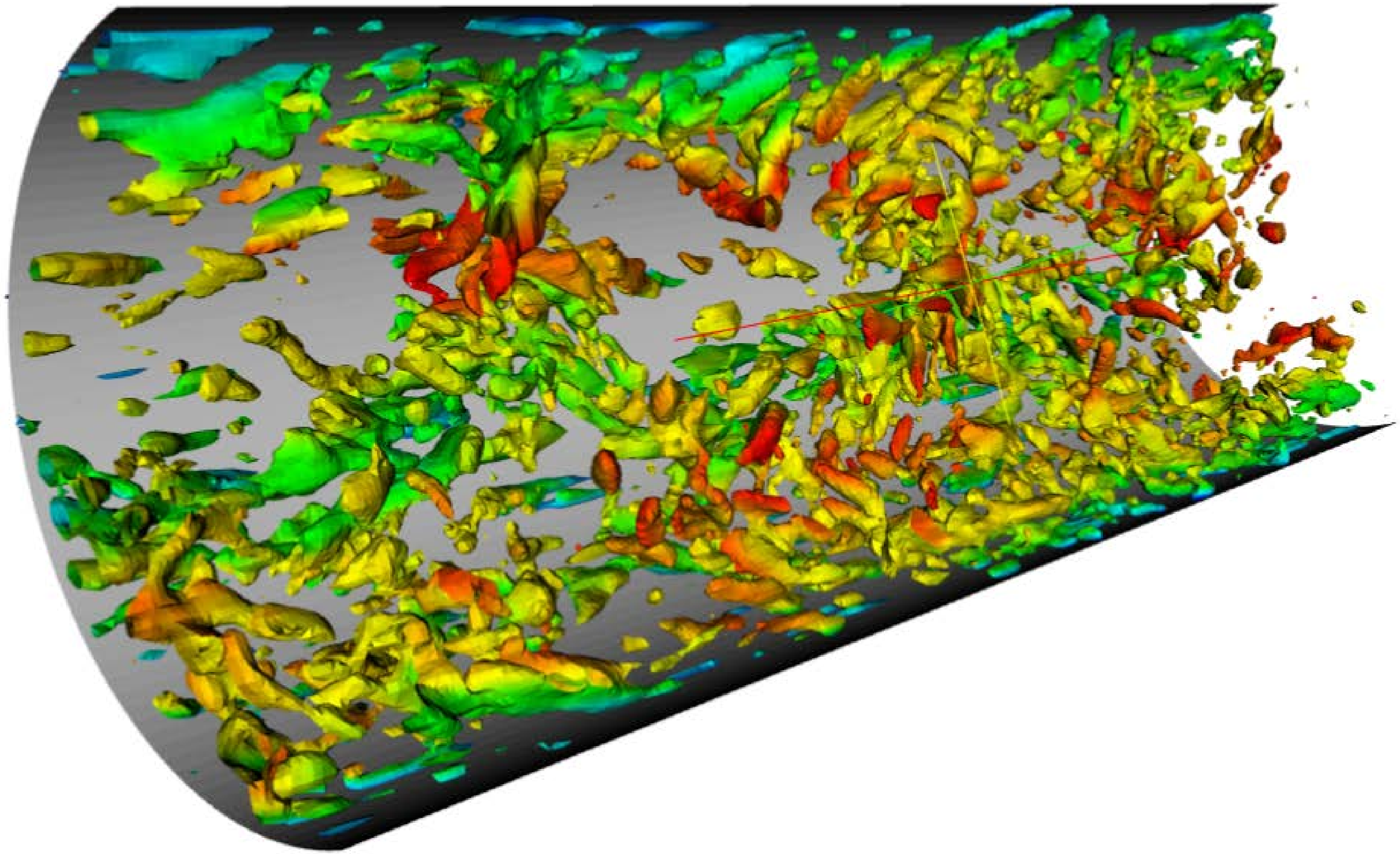
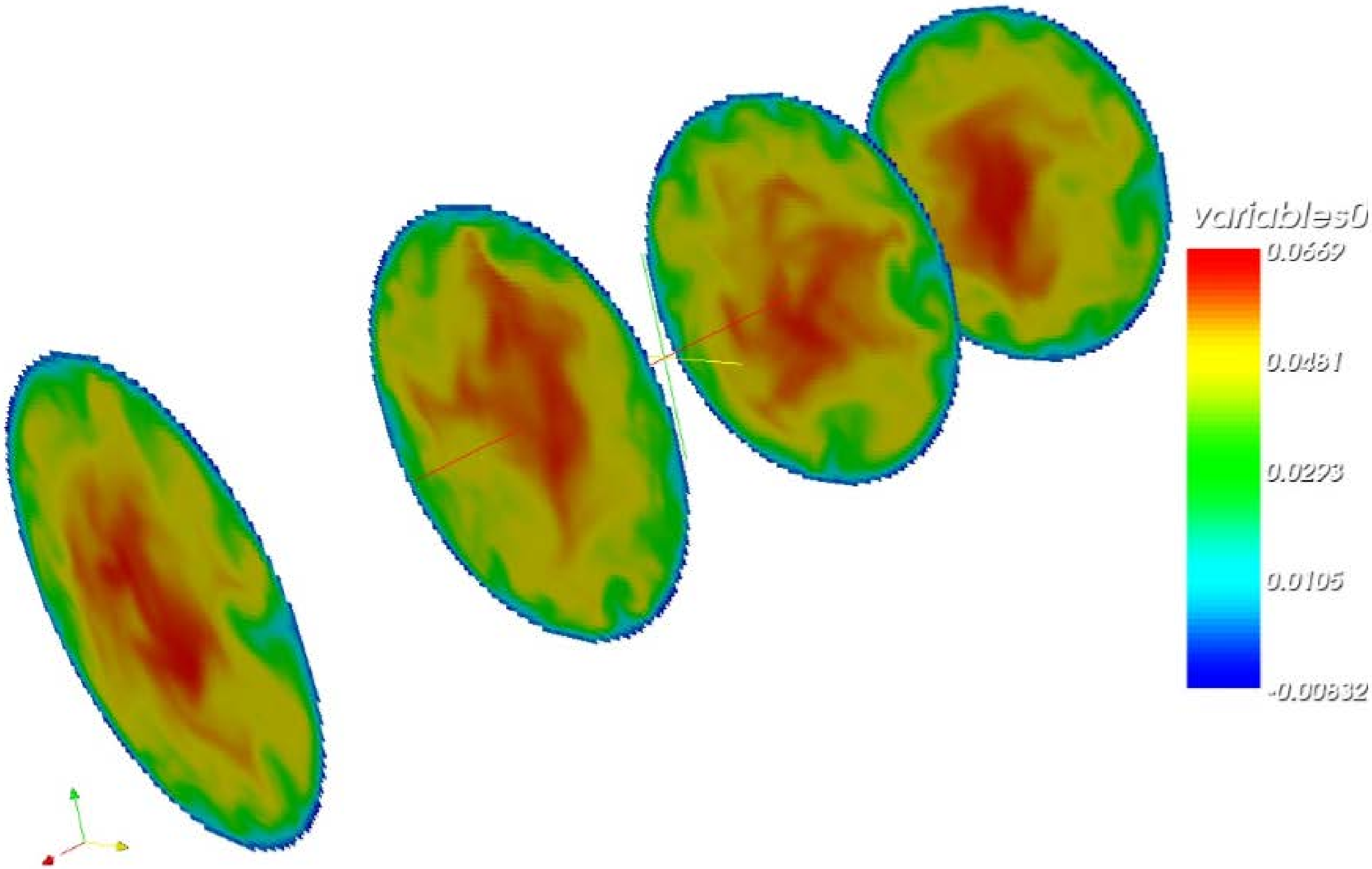
Turbulent flows

- Laminar and turbulent flows

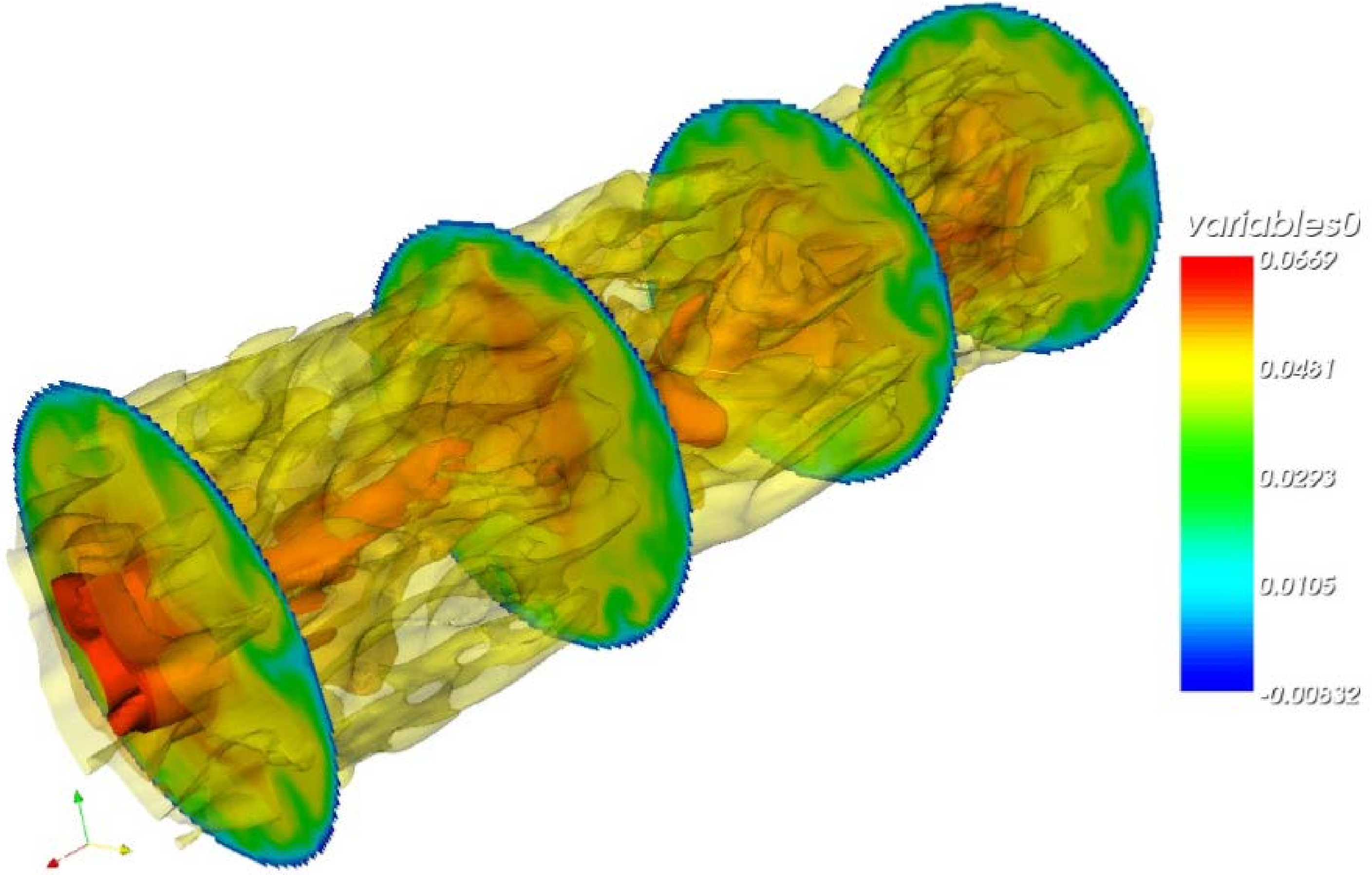
Turbulent pipe flow



Turbulent pipe flow



Turbulent pipe flow



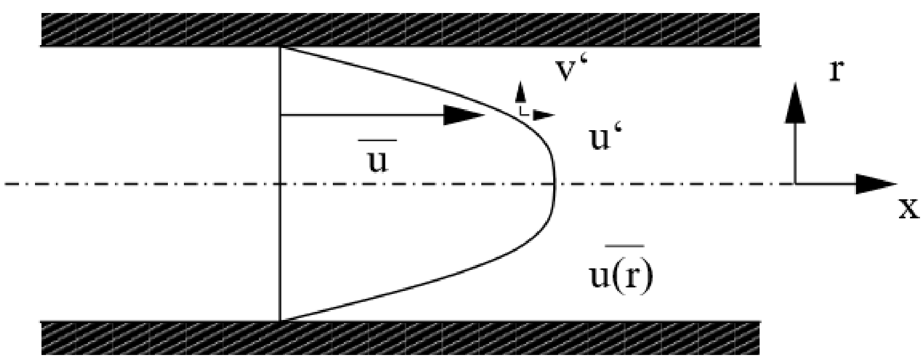
Turbulent flows

- Reynolds averaging: The turbulent velocity $\vec{v}$ is split into two parts:

  - Mean value $\vec{\bar{v}}$
  - Velocity fluctuation $\vec{v}'$

$$\underset{\text{total vector}}{\vec{v}} = \underset{\text{time average}}{\vec{\bar{v}}} + \underset{\text{fluctuation}}{\vec{v}'}$$

- Example: Pipe



fully turbulent

symmetrical flow

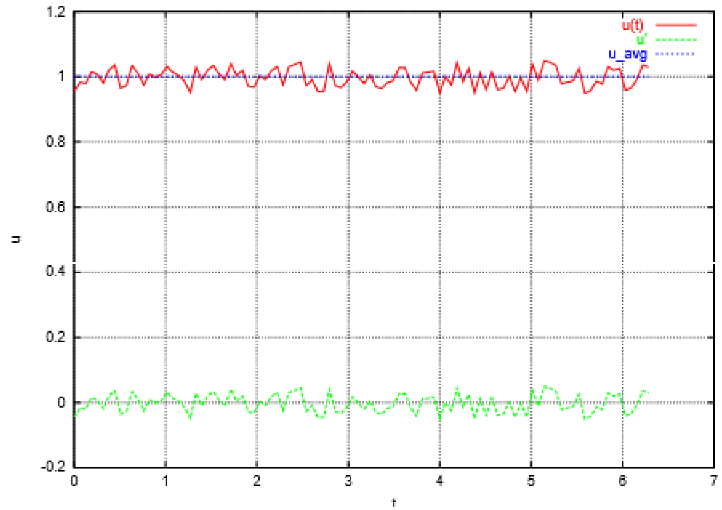
Turbulent flows

$$u(r,\Phi,x,t) = \overline{u}(r) + u'(r,\Phi,x,t)$$

$$v(r,\Phi,x,t) = \qquad v'(r,\Phi,x,t)$$

- Definition: $$\overline{u} = \frac{1}{T}\int_T u(x,y,z,t)dt$$

$$\rightarrow \overline{u} = \overline{u}(x,y,z) \neq f(t) \qquad u' = u - \overline{u}$$



Characteristics of turbulent flows

- Chaotic, stochastic property changes
- Rapid variation of pressure and velocity in time and space
- Laminar flow at low Reynolds numbers, turbulent flow at high Reynolds numbers
- Increased diffusion due to turbulent fluctuations
  → higher mixing
  → increased heat transfer
- Additional (turbulent) shear stresses
  → higher pressure losses (pipe flow)
  → increased boundary layer skin friction

Computational rules

$\overline{f'} = 0$ Mean value of the fluctuation

$\overline{\overline{f}} = \underbrace{\overline{f}}_{\text{const.} \neq f(t)}$ Mean value of the mean value

$\overline{\frac{\partial f}{\partial x}} = \frac{\partial \overline{f}}{\partial x}$ Mean value of the derivative

$$\overline{f+g} = \frac{1}{T}\int_T (f+g)dt = \frac{1}{T}\int_T f\ dt + \frac{1}{T}\int_T g\ dt = \overline{f} + \overline{g}$$

$$\overline{f\ \overline{g}} = \overline{f}\ \overline{g} \quad \overline{g} \neq \overline{g}(t) \rightarrow \frac{1}{T}\int_T f\ \overline{g}\ dt = \frac{1}{T}\overline{g}\int_T f\ dt = \overline{f}\ \overline{g}$$

Computational rules

$$\overline{f\ g} = \frac{1}{T}\int\limits_T fg\ dt = \frac{1}{T}\int\limits_T (\overline{f} + f')(\overline{g} + g')\ dt$$

$$= \frac{1}{T}\int\limits_T (\overline{f}\overline{g} + f'\overline{g} + \overline{f}g' + f'g')dt$$

$$= \overline{f}\overline{g} + \overline{g}\underbrace{\frac{1}{T}\int\limits_T f'\ dt}_{=0} + \overline{f}\underbrace{\frac{1}{T}\int\limits_T g' dt}_{=0} + \overline{f\ g'}$$

(linear velocity profile)

$$= \overline{f}\overline{g} + \underline{\underline{\overline{f'\ g'}}}$$

(usually ≠ 0, e.g. $f = g \rightarrow \overline{f'^2} \neq 0$

Level of turbulence

Turbulent intensity)

$$\mathrm{Tu} = \frac{1}{\underline{\underline{u_\infty}}}\sqrt{\frac{1}{3}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})}$$

Momentum equation

- Convective terms in the momentum equation for three-dimensional, incompressible and unsteady flow:

$$\frac{\partial v_k v_j}{\partial x_k} \quad \text{e.g.} \quad \frac{\partial uv}{\partial x} \; ; \; \frac{\partial vw}{\partial y}$$

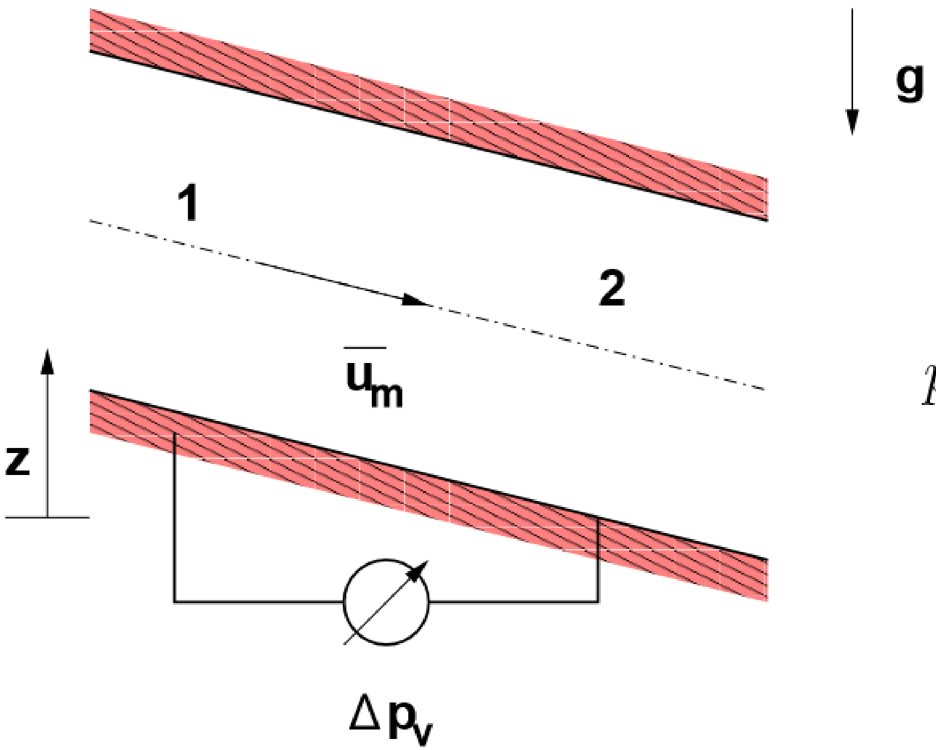
- Mean value of the convective terms:

$$\rightarrow \overline{\frac{\partial v_k v_j}{\partial x_k}} = \frac{\partial}{\partial x_k}\left(\overline{v_k}\;\overline{v_j} + \underbrace{\overline{v'_k v'_j}}_{\text{additional term}}\right)$$

$$-\rho\overline{v'_k v'_j} \qquad \text{turbulent shear stress tensor}$$

Bernoulli equation (Energy equation)

- Pipe flow with total pressure loss:



$$p_{01} = p_{02} + \underbrace{\Delta p_v}_{\text{Total pressure loss}}$$

Bernoulli equation (Energy equation)

$$p_1 + \frac{\rho}{2}\overline{u}_{m1}^2 + \rho g z_1 = p_2 + \frac{\rho}{2}\overline{u}_{m2}^2 + \rho g z_2 + \Delta p_v$$

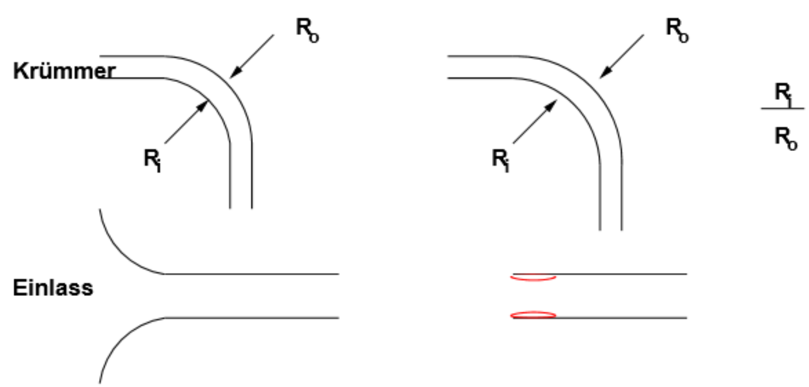
$$\text{with} \quad \Delta p_v = \sum_i (\zeta_i + \lambda_i \frac{L_i}{D_i}) \frac{\rho}{2}\overline{u}_{mi}^2$$

$\zeta_i \triangleq$ pressure loss coefficient for inlets, ellbows, ...

$\lambda_i \triangleq$ loss coefficient for straight pipes

$\overline{u}_{mi} \triangleq$ bulk mean velocity

For most geometries, $\zeta = \zeta(\mathrm{Re},\ \mathrm{geometry})$ is determined in experiments and listed in tables



Pressure loss coefficients and reference velocity

- Pressure loss coefficients for pipes (smooth pipes)

$$Re = \frac{\overline{u}_m \rho D}{\eta}$$

- Laminar ($Re \leq 2.300$)

$$\lambda = \frac{C}{Re}$$

$C = 64$ for circular cross-sections (Hagen-Poiseuielle)

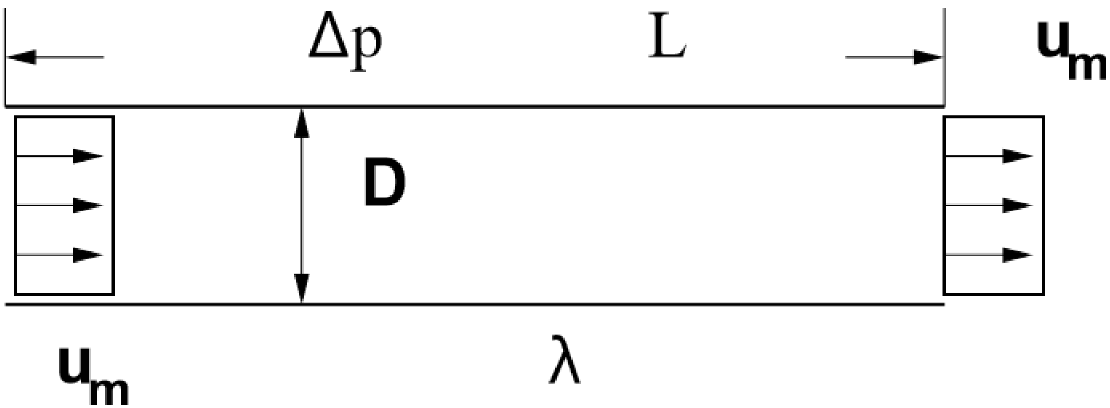
- Turbulent ($2.300 \leq Re \leq 10^5$)

$$\lambda = \frac{0.316}{\sqrt[4]{Re}}$$ Blasius

$$\frac{1}{\sqrt{\lambda}} = 2\log(Re\sqrt{\lambda}) - 0.8$$ Prandtl, iterative solution

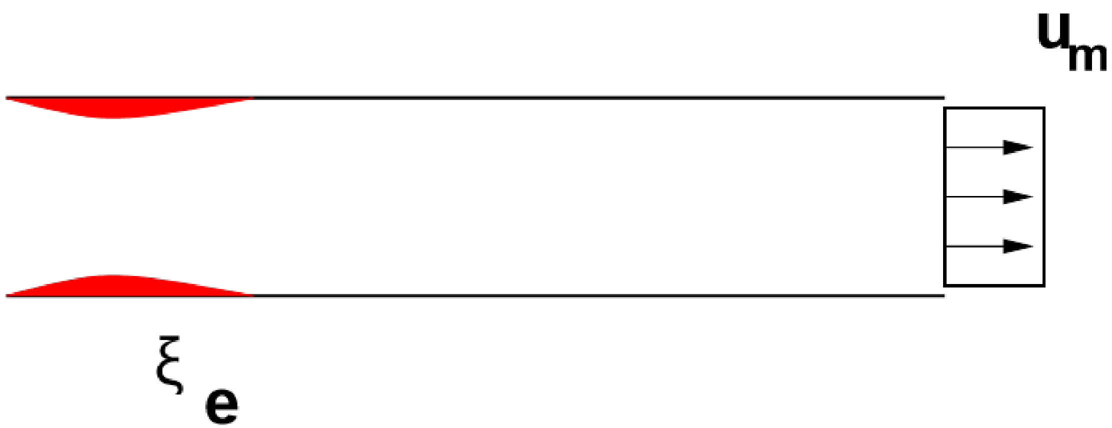
Pressure loss coefficients and reference velocity

- Viscous effects in pipes: bulk mean pipe velocity



$$\Delta p_v = \lambda \frac{L}{D} \frac{\rho}{2} \overline{u}_m^2$$

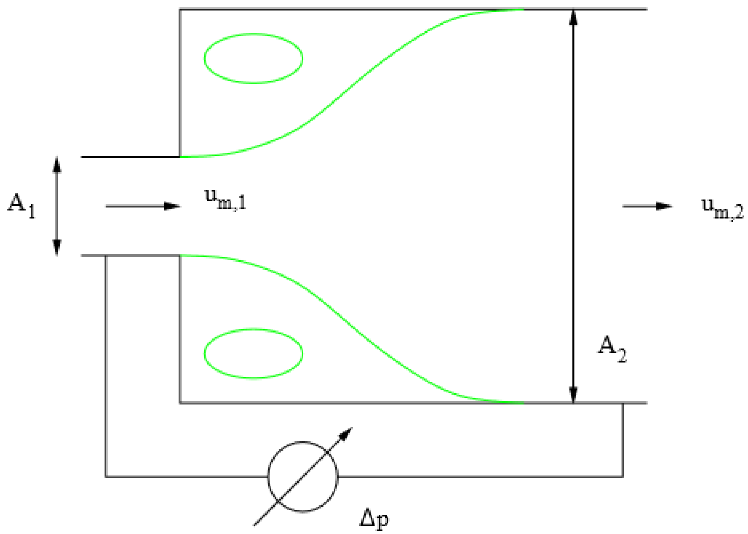
- Inlets: bulk mean pipe velocity



$$\Delta p_v = \zeta_e \frac{\rho}{2} \overline{u}_m^2$$

Pressure loss coefficients and reference velocity

- Unsteady change of cross section: bulk mean velocity at the inlet



Carnot equation

$$\zeta_e = \frac{\Delta p}{\frac{\rho}{2}\overline{u}_{m1}^2} = (1 - \frac{A_1}{A_2})^2$$

$$\Delta p_v = \zeta_e \frac{\rho}{2}\overline{u}_{m1}^2$$

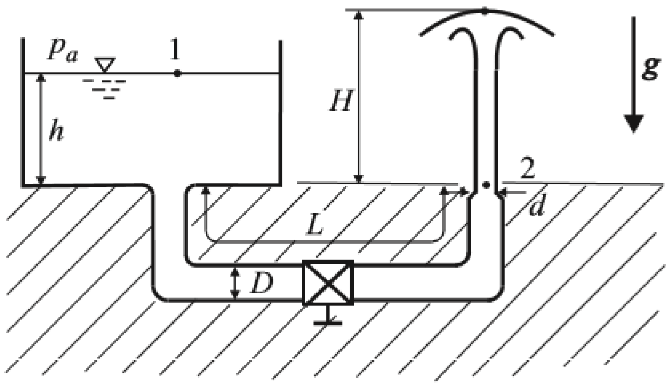
- Laminar flow, inlet, circular pipes:

$$\rightarrow 1.12 \leq \zeta_e \leq 1.45 \quad \text{from experimental data}$$

Example

- A fountain is supplied by a large tank and is connected to this tank using a pipe system. This system consist of four straight pipes with an overall length of L, two ellbows and a valve.

- Given: $h = 10\ m, D = 0.05\ m, L = 4\ m, \zeta_K = 0.25, \zeta_V = 0.025$



- Determine the volume flux and the height H for a flow including losses and a flow without losses for a) $d = D/2$ and b) $d = D$.

Example

- Bernoulli:

$$p_{01} \quad = \quad p_{02} \quad + \quad \Delta p_v$$

available total energy in '1' | remaining total energy in '2' | energy transformed to inner energy $\rightarrow$ total pressure loss

$$p_0 = p + \frac{\rho}{2}u^2 + \rho g z$$

$$\Delta p_v = \sum_i (\zeta_i + \lambda_i \frac{L_i}{D_i}) \frac{\rho}{2} \overline{u}_{mi}^2$$

- Remarks:
  - “0”: total
  - “1”: surface of the fluid of the tank
  - “2”: nozzle exit

Example

- Bernoulli from “d” → “H” ($u_H = 0$)

$$p_a + \frac{\rho}{2} u_{md}^2 = p_a + \rho g H$$

$$\rightarrow H = \frac{u_{md}^2}{2g} \rightarrow \quad \text{unknown} \quad u_{md} \, ?$$

- Extended Bernoulli

$$p_{01} = p_a + \rho g h = p_a + \frac{\rho}{2} u_{md}^2 + \frac{\rho}{2} u_{mD}^2 \underbrace{(2\zeta_K + \zeta_v + \lambda \frac{L}{D})}_{K}$$

bulk mean nozzle velocity

bulk mean pipe velocity

- Continuity

$$u_{mD} A_D = u_{md} A_d \rightarrow u_{mD} = u_{md} \left(\frac{d}{D} \right)^2$$

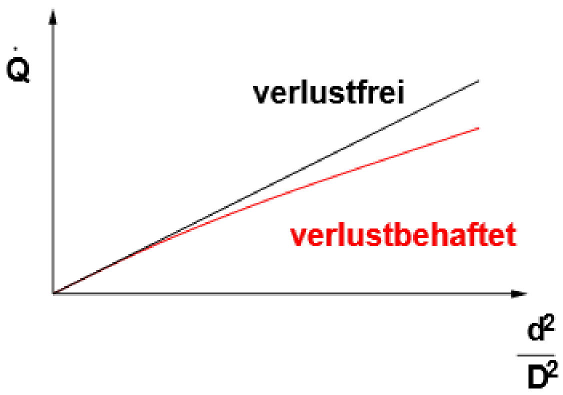
Example

lossfree	with losses
$\zeta_K = \zeta_v = \lambda = 0$	$\rho g h = \frac{\rho}{2} u_{md}^2 + \rho 2 u_{md}^2 \left(\frac{d}{D}\right)^4 K$
$u_{md} = \sqrt{2gh}$	$u_{md} = \sqrt{\frac{2gh}{1 + \left(\frac{d}{D}\right)^4 K}}$

$$Volume\ flux \quad \dot{Q} = \frac{\pi}{4} d^2 u_{md}$$

lossfree	with losses
$\dot{Q} = \frac{\pi}{4}\sqrt{2gh} D^2 \frac{d^2}{D^2}$	$\dot{Q} = \frac{\pi}{4}\sqrt{2gh} D^2 \frac{d^2}{D^2} \frac{\left(\frac{d}{D}\right)^2}{\sqrt{1 + \left(\frac{d}{D}\right)^4 K}}$
$\dot{Q} \sim \left(\frac{d}{D}\right)^2$	

Example



no losses	with losses
$H = h$	$H = \frac{h}{1+\left(\frac{d}{D}\right)^4 K}$
	influence of $\frac{d}{D}$
	$\frac{d}{D} \downarrow \rightarrow H \uparrow$

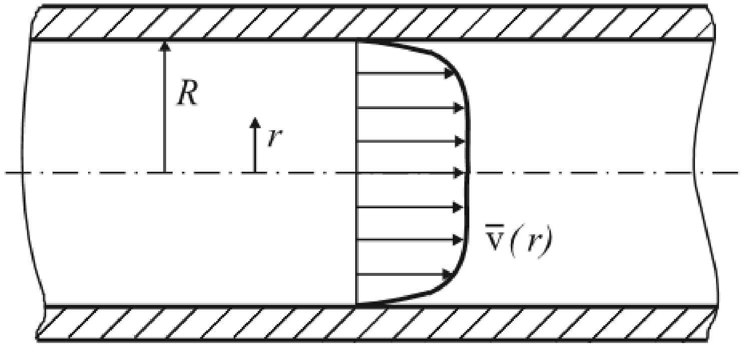
- Ceiling of the fountain: $H = \frac{u_{md}^2}{2g}$

Example 2

- The velocity profile of the fully developed flow in a pipe with a smooth surface can be approximated with the following potential law:

$$\frac{\overline{v}(r)}{\overline{v}_{max}} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}, \text{ with } n = n(Re)$$

Re	n
$1 \cdot 10^5$	7
$6 \cdot 10^5$	8
$1.2 \cdot 10^6$	9
$2 \cdot 10^6$	10



Example 2

a) Use the continuity equation to compute the relation between the bulk mean bulk mean velocity $\overline{v}_m$ and the maximum velocity $\overline{v}_{max}$, i.e.

$$\frac{\overline{v}_m}{\overline{v}_{max}} = f(n)$$

b) Determine the position $\frac{r_m}{R}$, where $\overline{v}(r/R) = v_m$.

c) How can the results of a) and b) be used to measure the volume flux?

a) The ratio between the average and the maximum velocity is

$$\frac{\overline{v}_m}{\overline{v}_{max}} = 2 \int_0^1 \xi(1 - \xi)^{\frac{1}{n}}\, d\xi = \frac{2n^2}{(n+1)(2n+1)} \quad \text{with} \quad \xi = \frac{r}{R}$$

Example 2

b) The integral is solved using partial integration, and the result of this integration can be used to compute the distance r_m/R using the following relationship:

$$\frac{r_m}{R} = 1 - \left(\frac{\overline{v}_m}{\overline{v}_{max}}\right)^n$$

Re	n	$\overline{v}_m/\overline{v}_{max}$	r_m/R
$1 \cdot 10^5$	7	0.8166	0.7577
$6 \cdot 10^5$	8	0.8366	0.76
$1.2 \cdot 10^6$	9	0.8526	0.762
$2 \cdot 10^6$	10	0.8658	0.7633

c) Measuring $\overline{v}(r)$ at a distance $R - r_m$ from the wall, and with the known $\overline{v}_{max}$ the average velocity can be determined, and the volume flux $\dot{V} = v_m \; \pi \; R^2$ can be computed.

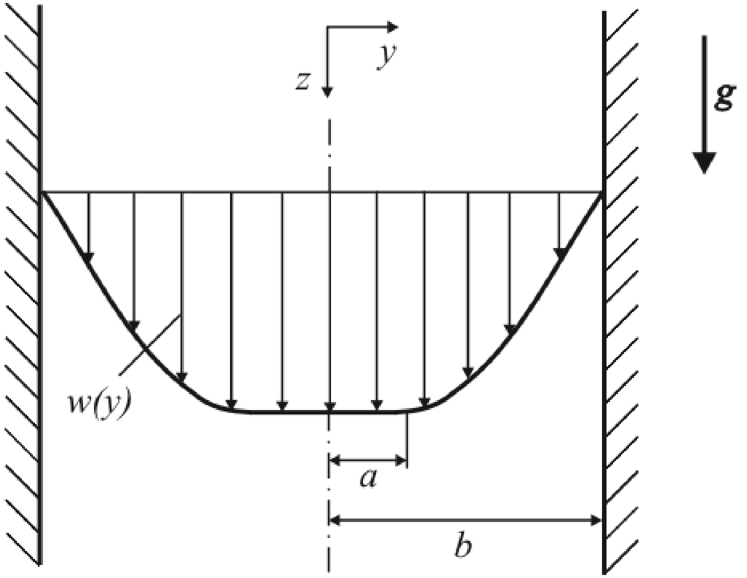
Example 3

- A Bingham fluid flows into the direction of gravity between two inifinite, parallel, vertical plates.

- Given: $b, \; \rho, \; \eta, \; \tau_0, \; g, \; dp/dz = 0$

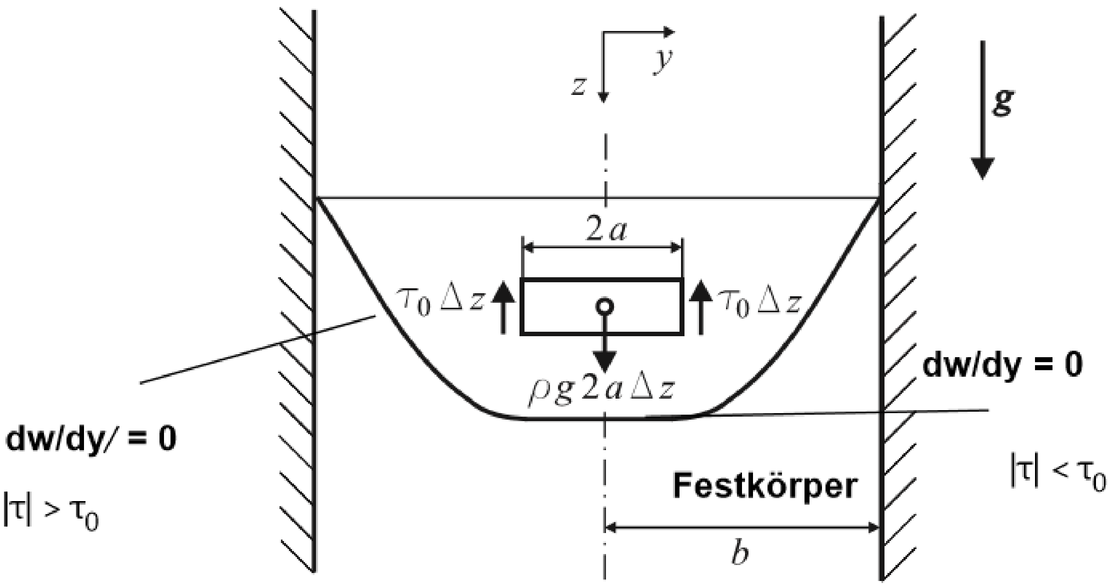
- Determine for a fully developed flow

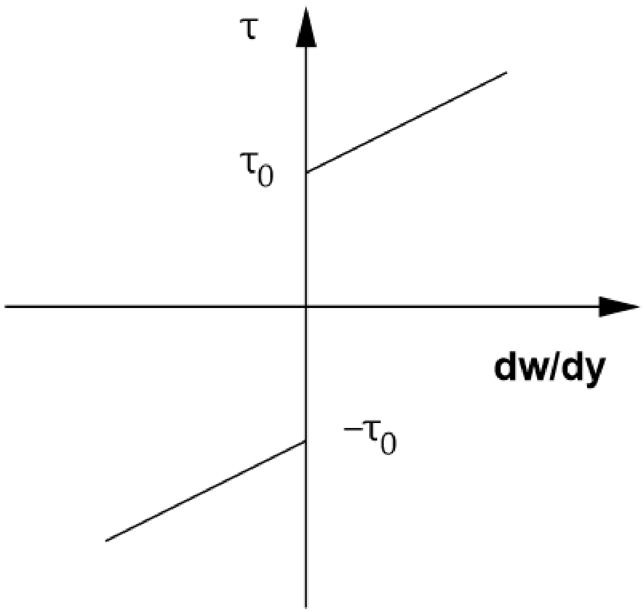
  a) the distance a
  b) the velocity profile $w(y)$



Example 3

a)





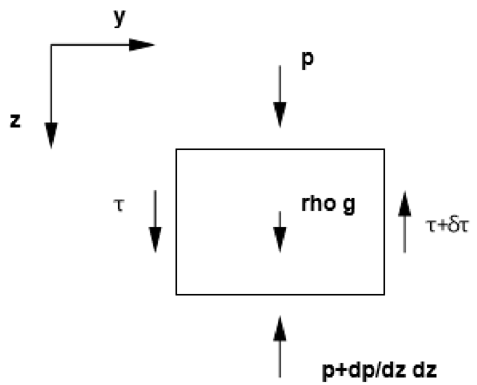
Bingham fluid: $\tau = -\eta \dfrac{\partial w}{\partial y} \pm \tau_0$

If τ exceeds τ_0, the fluid starts to flow.

As long as τ does not exceed τ_0, the fluid acts like a solid body.

Example 3

Infinitesimal element:



Fully developed flow: $\rightarrow \dfrac{\partial w}{\partial z} = 0 \ ; \ \dfrac{\partial}{\partial z} = 0$

Equilibrium of forces: $\tau B dz - \left(\tau + \dfrac{\partial \tau}{\partial y} dy\right) B dz$

$$+pBdy - \left(p + \frac{\partial p}{\partial z} dz\right) Bdy + \rho g B dy dz = 0$$

$$\rightarrow -\frac{\partial \tau}{\partial y} dy B dz + \rho g B dy dz = 0$$

Example 3

Hence:

$$\boxed{\frac{\partial \tau}{\partial y} = \rho g = \frac{d\tau}{dy}}$$

Integration: $d\tau \rho g dy \rightarrow \tau = \rho g y + C_1(z)$

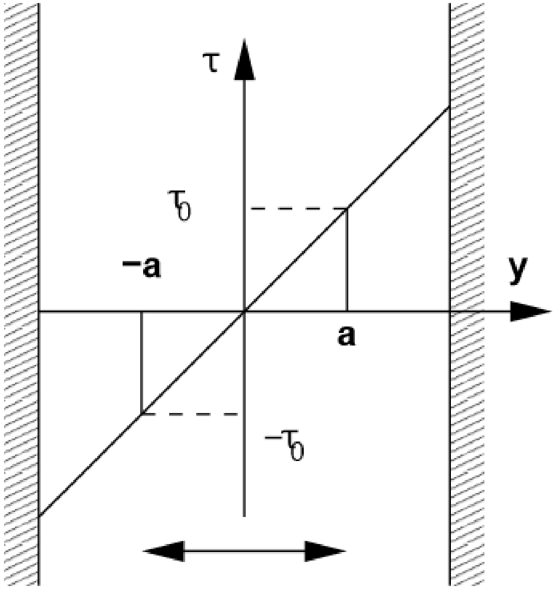
B.C. for $C_1(z)$

Symmetry: $\tau(y=0) = 0 \rightarrow C_1(z) = 0$

$$\boxed{\tau(y) = \rho g y}$$

(does not depend on the fluid)

Example 3



$\tau(y)$ straight line

$$y = |a| \rightarrow |\tau| = \tau_0$$

$$\tau(y = a) = \rho g a = \tau_0$$

$$a = \frac{\tau_0}{\rho g}$$

a) Velocity profile $w(y)$

inner region, solid body: $|y| \leq a \rightarrow \dfrac{dw}{dy} = 0 \quad \rightarrow \quad w(y) = const.$

outer region, flow for $|y| > a \rightarrow \tau = -\dfrac{dw}{dy} \pm \tau_0$

Example 3

Fully developed flow: $$\frac{\partial w}{\partial z} = 0$$

Hence: $$\frac{dw}{dy} = -\frac{\tau \pm \tau_0}{\eta}$$ Symmetry: $$w(y) = w(-y)$$

$$y > 0 : \tau(y) = \rho g y \rightarrow \frac{dw}{dy} = -\frac{\rho g y}{\eta}(+/-)\frac{\tau_0}{\eta}$$

Sign: $$\frac{dw}{dy} < 0 \quad \text{for} \quad y > a$$

$$\frac{dw}{dy} = 0 \quad \text{for} \quad y \leq a$$

Integration: $$w(y) = \frac{1}{\eta}(\tau_0 y - \frac{1}{2}\rho g y^2) + C_2$$

Example 3

B.C.: no-slip condition on the wall

$$\rightarrow y = b \;:\; w = 0$$

Finally:

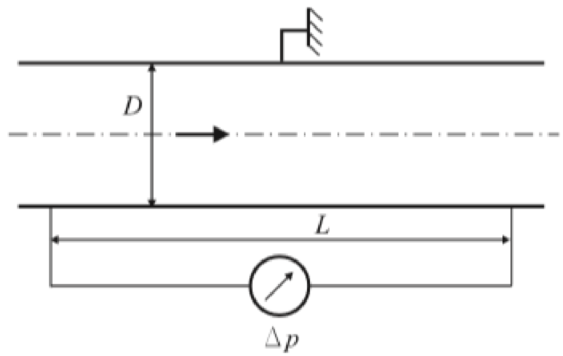
$$w(y) = \frac{\rho g}{2\eta}(b^2 - y^2) - \frac{\tau_0}{\eta}(b - y) \quad y > a$$

$$\frac{\rho g}{2\eta}(b^2 - a^2) - \frac{\tau_0}{\eta}(b - a) \quad y \leq a$$

Example 4

- The pressure decrease Δp along L measured in a fully developed pipe flow with the volume flux $\dot{V}$.
- Given: $\dot{V} = 0,393\ m^3/s \quad L = 100\ m \quad D = 0,5\ m$

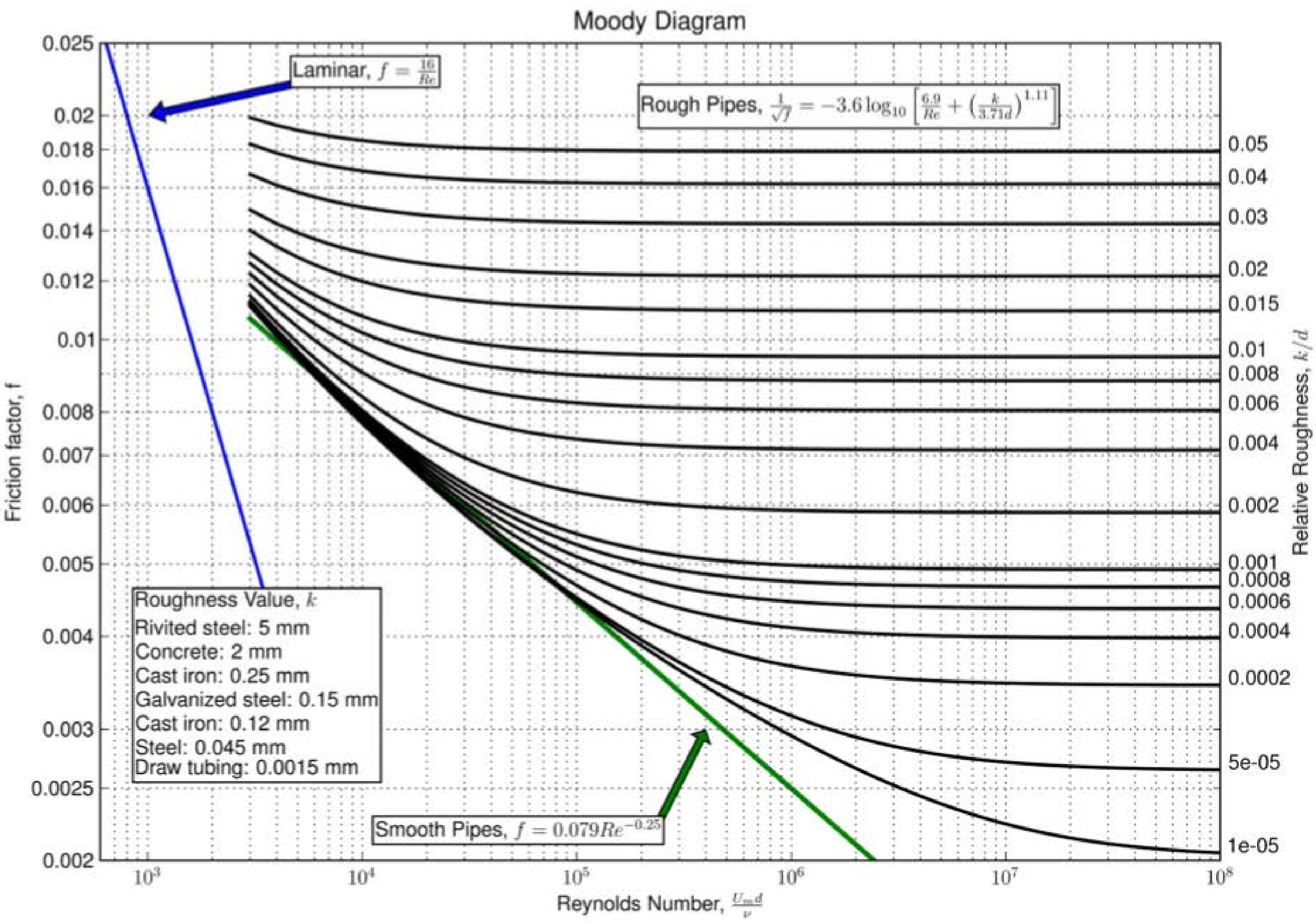
  $\Delta p = 12820\ N/m^2 \quad \rho = 900\ kg/m^3 \quad \eta = 5 \cdot 10^{-3}\ Ns/m^2$



- Determine

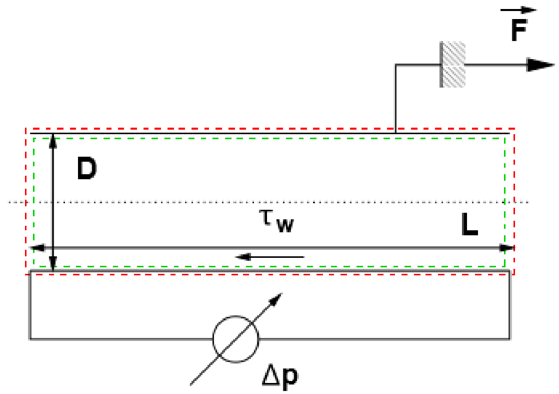
  a) the skin-friction coefficient,
  b) the equivalent roughness of the pipe,
  c) the wall shear stress and the force of the support.
  d) What is the pressure decrease, if the pipe is smooth?

Example 4



Moody Diagram
Laminar, $f = \frac{16}{Re}$
Rough Pipes, $\frac{1}{\sqrt{f}} = -3.6\log_{10}\left[\frac{6.9}{Re} + \left(\frac{k}{3.71d}\right)^{1.11}\right]$
Roughness Value, k
Rivited steel: 5 mm
Concrete: 2 mm
Cast iron: 0.25 mm
Galvanized steel: 0.15 mm
Cast iron: 0.12 mm
Steel: 0.045 mm
Draw tubing: 0.0015 mm
Smooth Pipes, $f = 0.079Re^{-0.25}$
Friction factor, f
0.025
0.02
0.018
0.016
0.014
0.012
0.01
0.008
0.007
0.006
0.005
0.004
0.003
0.0025
0.002
Reynolds Number, $\frac{U_m d}{\nu}$
10^3
10^4
10^5
10^6
10^7
10^8
Relative Roughness, k/d
0.05
0.04
0.03
0.02
0.015
0.01
0.008
0.006
0.004
0.002
0.001
0.0008
0.0006
0.0004
0.0002
5e-05
1e-05

Example 4



a)

$$\Delta p = \lambda \frac{L}{D} \frac{\rho}{2} \overline{u_m}^2$$

$$\dot{Q} = \frac{\pi}{4} D^2 \overline{u_m}^2$$

$$\Longrightarrow \quad \lambda = \frac{\pi^2 \Delta p D^5}{8 \rho L \dot{Q}^2} = 0.0356$$

Example 4

b)

$$\mathrm{Re} = \frac{\rho \overline{u_m} D}{\eta} = 1.8 \cdot 10^5$$

$$\rightarrow \text{Moody-Diagram: } \frac{k_s}{D} = 0.0083$$

$$\rightarrow k_s = 4.2 \text{ mm}$$

c) Momentum equation for the inner control surface:

$$\Delta p \frac{\pi}{4} D^2 - \tau_w \pi D L = 0$$

$$\rightarrow \tau_w = \Delta p \frac{D}{4L} = 16 \ \frac{\mathrm{N}}{\mathrm{m}^2}$$

Example 4

c) Momentum equation for the outer control surface:

$$F = -\Delta p \frac{\pi D^2}{4} = -2517\mathrm{N}$$

d)

$$\lambda = 0.016 \rightarrow \Delta p = 5.8 \cdot 10^3 \, \frac{\mathrm{N}}{\mathrm{m}^2}$$

Moody Diagramm