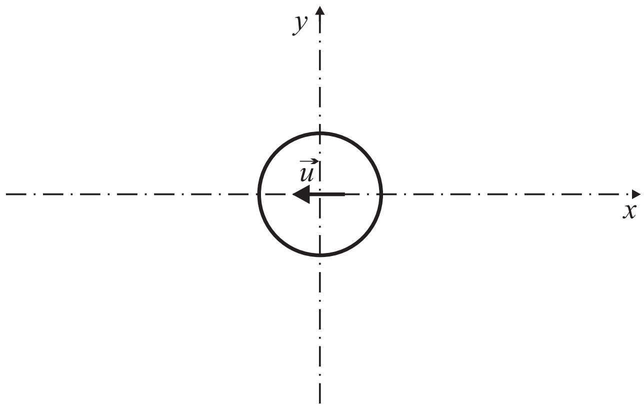
3 Fluid kinematics

3.1 A circular cylinder is moving in motionless environment without friction at constant velocity. Draw the the streamlines and the pathlines for some special fluid particles.



3.2 Determine for the velocity field

$$u = u_0 \ \cos(\omega \ t) \quad v = -v_0 \sin(\omega \ t)$$

with $u_0 \ / \ \omega \ = \ v_0 \ / \ \omega \ = \ 1 \ m$

a) the streamlines for $\omega \ t \ = \ 0 \ , \ \pi \ / \ 2 \ , \ \pi \ / \ 4 \ ,$
b) the pathlines,
c) the pathline for the particle, that is at $t \ = \ 0 \ s$ in $x \ = \ 0 \ m, \ y \ = \ 1 \ m$!

4 Basic equations for fluids

<u>vektors, tensors, operators</u>

1.) Scalars (Tensors of rank 0)

e.g. pressure p, density ρ, temperature T

$\longrightarrow$ real number (+ physical unit)

2.) Vectors (Tensors of rank 1)

e.g. Position vector $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \cdot \vec{i} + y \cdot \vec{j} + z \cdot \vec{k}$,

Velocity vector $\vec{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = u \cdot \vec{i} + v \cdot \vec{j} + w \cdot \vec{k}$

$\longrightarrow$ n dimensions

3.) Dyads (Tensors of rank 2)

e.g. Stress tensor τ , $(\vec{v}\,\vec{v})$

$\longrightarrow$ n dimensions ($n \times n$ Matrix)

Nabla - Operator $\vec{\nabla} \; = \; \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

Gradient $grad\ p \; = \; \vec{\nabla}\, p \; = \; \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right)$

Divergence $div\ \vec{v} \; = \; \vec{\nabla} \cdot \vec{v} \; = \; \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} \; = \; \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

Rotation $\vec{\nabla} \times \vec{v} \; = \; \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \; = \; \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}$

Partial derivatives

Total differential of a function $f = f\,(x,\ y,\ z)$

$$df = \frac{\partial f}{\partial x} \cdot dx + \frac{\partial f}{\partial y} \cdot dy + \frac{\partial f}{\partial z} \cdot dz$$

The total differential describes the increas of a function, e. g. for the velocity $\vec{v} = \vec{v}\,(t,\ x,\ y,\ z)$

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t} \cdot dt + \frac{\partial \vec{v}}{\partial x} \cdot dx + \frac{\partial \vec{v}}{\partial y} \cdot dy + \frac{\partial \vec{v}}{\partial z} \cdot dz$$

$$\Longrightarrow \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \vec{v}}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial \vec{v}}{\partial z} \cdot \frac{dz}{dt}$$

$$\Longrightarrow \underbrace{\frac{d\vec{v}}{dt}}_{\text{substantial}} = \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{local}} + \underbrace{u \cdot \frac{\partial \vec{v}}{\partial x} + v \cdot \frac{\partial \vec{v}}{\partial y} + w \cdot \frac{\partial \vec{v}}{\partial z}}_{\text{convective acceleration}} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \cdot \vec{v}$$

4.1

a) Proof the following identities

1) $$\vec{\nabla} \times \vec{\nabla} p \;=\; 0$$

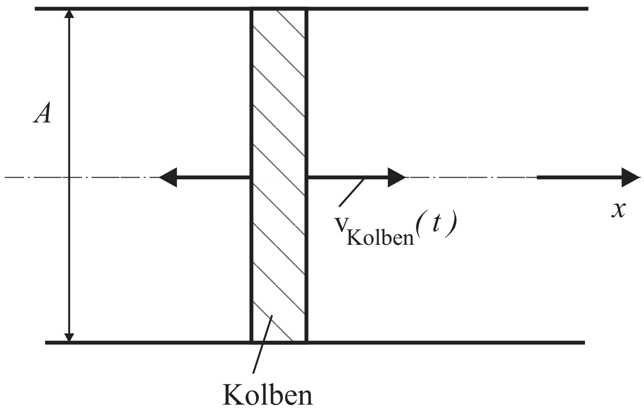
2) $$\vec{\nabla} \times \vec{\nabla}^2 \vec{v} \;=\; \vec{\nabla}^2 (\vec{\nabla} \times \vec{v})$$

3) $$(\vec{v} \cdot \vec{\nabla}) \vec{v} \;=\; \vec{\nabla} \frac{\vec{v}^{\,2}}{2} \;-\; \vec{v} \times (\vec{\nabla} \times \vec{v})$$

b) Formulate the conservation law of mass and momentum for a three-dimensional, incompressible and unsteady flow in vector notation.

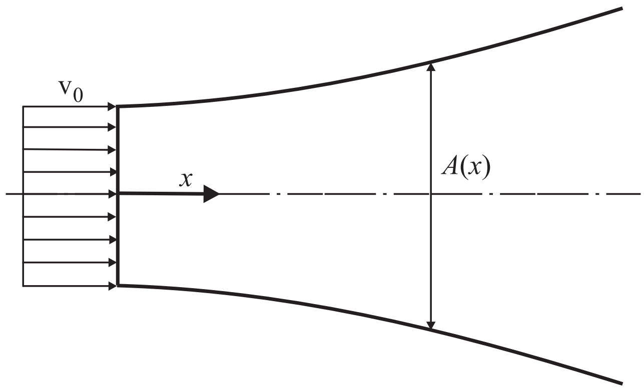
4.2

a) A piston is moving in a tube of infinite length and with constant cross section A with the velocity $v_{\text{piston}}(t)$. The density of the fluid is constant.



Determine the substantial acceleration in the tube.

b) A fluid of constant density flows into a diffusor with the constant velocity $v = v_0$. The cross section of the diffusor is $A(x)$. Determine the substantial acceleration of the fluid along the axis x.



4.3 The following continuity equation is formulated in cartesian coordinates.

$$\frac{d\rho}{dt} + \rho \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \quad \text{(Eq. 1)}$$

with $v = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

Transform the equation into
a) cylindrical coordinates

Hint: $x = r \cdot \cos\phi \qquad r = \sqrt{x^2 + y^2}$

$y = r \cdot \sin\phi \qquad \phi = \arctan\left(\frac{y}{x}\right)$

$z = z \qquad z = z$

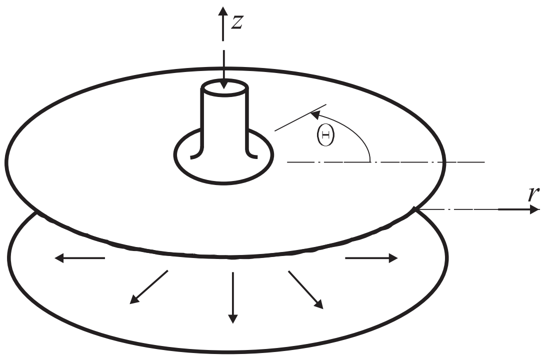
b) sperical coordinates

Hint: $x = r \cdot \sin\Theta \cdot \cos\phi \qquad r = +\sqrt{x^2 + y^2 + z^2}$

$y = r \cdot \sin\Theta \cdot \sin\phi \qquad \Theta = \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right)$

$z = r \cdot \cos\Theta \qquad \phi = \arctan\left(\frac{y}{x}\right)$

4.4 An incompressible fluid with the viscosity η is flowing laminar and steady between two parallel plates



The flow is radial from inside to outside. The determining differential equations in cylindrical coordinates are

$$\frac{1}{r}\frac{\partial(\rho\ r\ v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho\ v_\Theta)}{\partial \Theta} + \frac{\partial(\rho\ v_z)}{\partial z} = 0$$

$$\rho\left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\Theta}{r}\frac{\partial v_r}{\partial \Theta} - \frac{v_\Theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \eta \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r\ v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \Theta^2} - \frac{2}{r^2}\frac{\partial v_\Theta}{\partial \Theta} + \frac{\partial^2 v_r}{\partial z^2} \right)$$

Simplify the equations for the flow problem described.

4.5 The Navier-Stokes equations for rotationally symmetric, unsteady, incompressible flows in cylindrical coordinates read:

$$\frac{1}{r}\,\frac{\partial(r\cdot v_r)}{\partial r} \;+\; \frac{\partial v_z}{\partial z} \;=\; 0$$

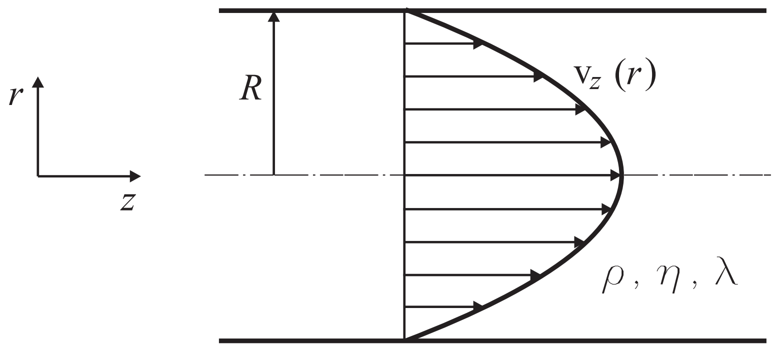
$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\,\frac{\partial v_r}{\partial r} + v_z\,\frac{\partial v_r}{\partial z}\right) \;=\; -\,\frac{\partial p}{\partial r} + \eta\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{\partial}{\partial r}(r\;v_r)\right) + \frac{\partial^2\,v_r}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\,\frac{\partial v_z}{\partial r} + v_z\,\frac{\partial v_z}{\partial z}\right) \;=\; -\,\frac{\partial p}{\partial z} + \eta\left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial v_z}{\partial r}\right) + \frac{\partial^2\,v_z}{\partial z^2}\right]$$

$$\rho\;c_p\left(\frac{\partial T}{\partial t} + v_r\,\frac{\partial T}{\partial r} + v_z\,\frac{\partial T}{\partial z}\right) \;=$$

$$\frac{dp}{dt} + \lambda\left[\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) + \frac{\partial^2\,T}{\partial z^2}\right] + 2\eta\left\{\left(\frac{\partial\,v_r}{\partial r}\right)^2 + \left(\frac{v_r}{r}\right)^2 + \left(\frac{\partial\,v_z}{\partial z}\right)^2\right\} + \eta\left(\frac{\partial\,v_z}{\partial r} + \frac{\partial\,v_r}{\partial z}\right)^2$$

Consider a steady, laminar, fully developed, incompressible duct flow with temporal and spatial variable temperature distribution.



Simplify the equations for the case described.

4.6 The Navier-Stokes equations for unsteady, incompressible flows in graviational field read:

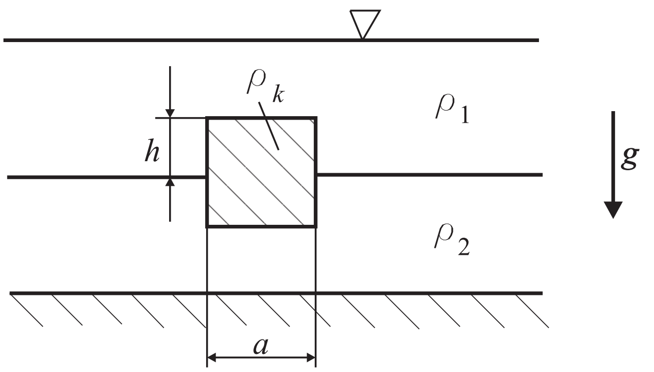
$$\nabla\cdot\vec{v} \;=\; 0$$

$$\rho\frac{d\vec{v}}{dt} \;=\; -\,\nabla p \;+\; \eta\nabla^2\vec{v} \;+\; \rho\vec{g}$$

Formulate the equations for a steady, frictionless, twodimensional flow in a cartesian coordinate system $(x,\;y)$.

5 Hydrostatics

5.1 A cube swims in two fluids that are arranged in layers.

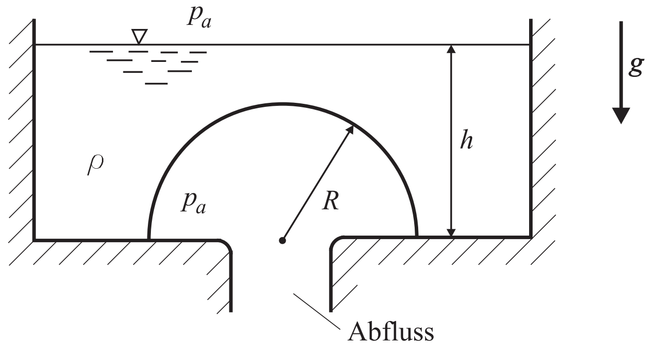


$\rho_1 = 850\ kg/m^3$ $\quad$ $\rho_2 = 1000\ kg/m^3$ $\quad$ $\rho_K = 900\ kg/m^3$ $\quad$ $a = 0,1\ m$

Determine the height h!

5.2 A container is filled with a fluid of the density ρ. The drain of the container, filled up to a height h, is closed with a hollow hemisphere (radius R, weight G).

Given: $h,\ \rho,\ R,\ G,\ g$



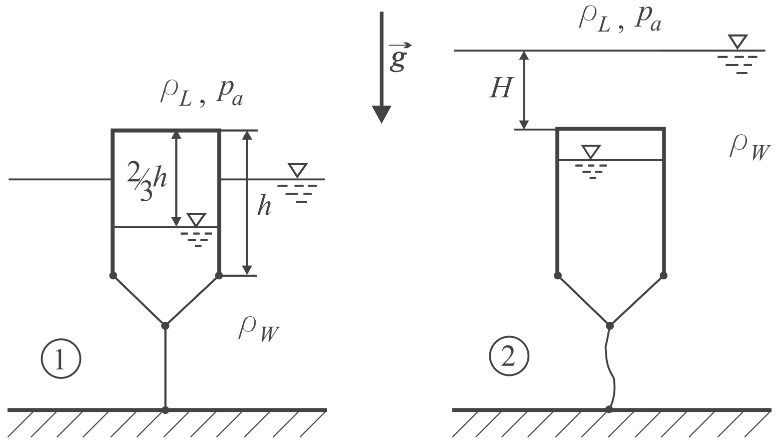
Determine the necessary force F to open the drain.

Hint: Volume of a sphere: $V_k \;=\; \dfrac{4}{3}\,\pi\ R^3$

5.3 A buoy with the mass m_B, which is open on the lower side, is fixed with a cable at the bottom of a sea, and sticks out with a third of its height if the cable is not strained (Sketch 1). With increasing water level the buoy is drawn under the water surface (Sketch 2) and sinks when the immersion depth is H.

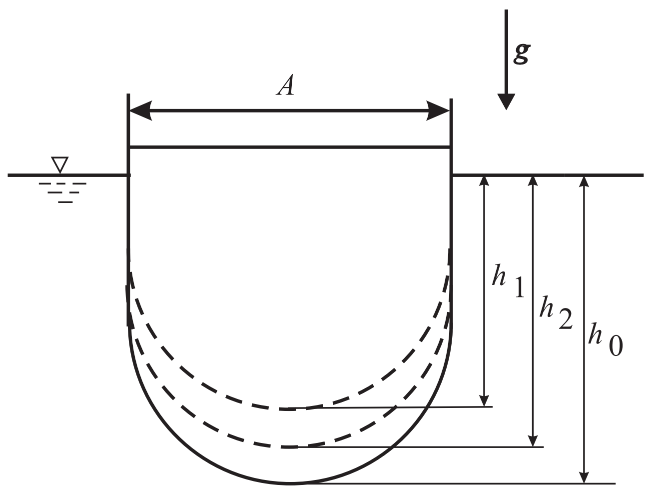
Determine H!

Given: $m_B,\ p_a,\ \rho_W,\ h,\ g,\ \rho_L\ <<\ \rho_W$



Hint: Assume, that the temperature inside of the buoy is constant and the enclosed air can be treated as a perfect gas. Neglect the weight of the cable.

5.4 A ship with vertical side walls has a weight of G_0 Its draught is h_0 and it displaces the volume τ_0 in sea water. At the entrance of an estuary the weight is decreased by ΔG in order to avoid the ship runs aground. Now the draught is h_1 and the displaced volume is τ_1. The density of sea water is ρ_M, the density of the rivers water is ρ_F.



$G_0 = 1,1 \cdot 10^9\ N \quad \Delta G = 10^8\ N \quad h_0 = 11\ m \quad h_1 = 10,5\ m$
$\rho_M = 1,025 \cdot 10^3\ kg/m^3 \quad \rho_F = 10^3\ kg/m^3 \quad g = 10\ m/s^2$

Determine

a) the volume τ_0,

b) the deck area A,

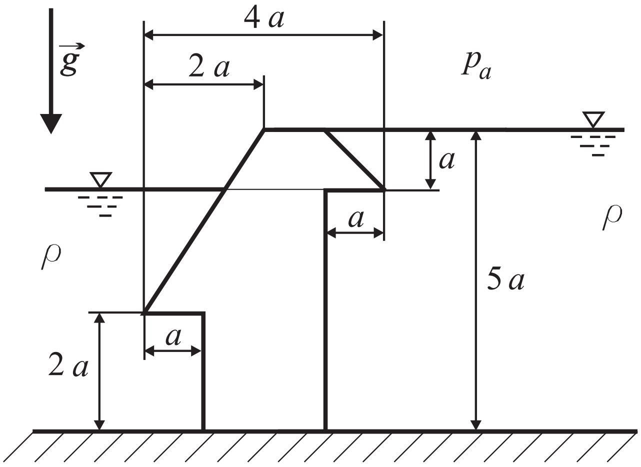
c) the difference $\tau_2 - \tau_1$ between the displaced volumes in fresh water and sea water,

d) the draught h_2 in fresh water!

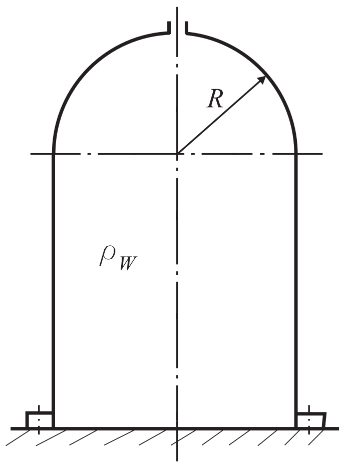
5.5 The sketched weir of length L separates two basins of tdifferent depth.

Determine the force of water onto the weir.

Given: $\rho,\ g,\ L,\ a$



5.6 A boiler with a small hole at the top is filled with water and is screwed on a plate.



$R = 1\ m \qquad \rho_W = 10^3\ kg/m^3 \qquad g = 10\ m/s^2$

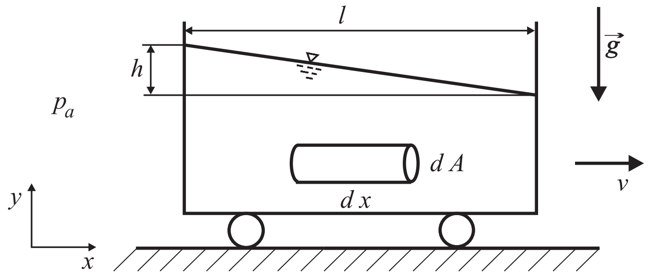
Determine the force on the screws by neglecting the boilers weight.

5.7 A vehicle filled with water is moving under constant acceleration a_x in x-direction.

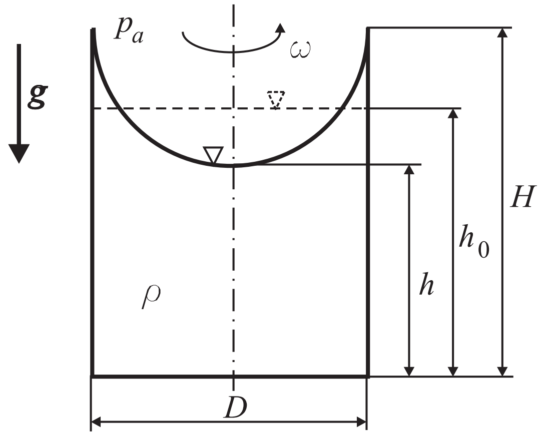
1. Determine the pressure gradient $\partial p/\partial x$ by using the balance of forces at the plotted volume element of length dx and of an area dA.

2. Determine the vehicle length l.

Given: $h,\ \rho_W,\ g,\ a_x$



5.8 A liquid rotates in an open cylindrical jar with a constant angular speed which is such that the liquid just reaches the upper border. Under quiescent condition the liquid fills the jar up to a height of h_0.



$D = 0,5\ m \qquad h_0 = 0,7\ m \qquad H = 1\ m \qquad \rho = 10^3\ kg/m^3 \qquad p_a = 10^5\ N/m^2 \qquad g = 10\ m/s^2$

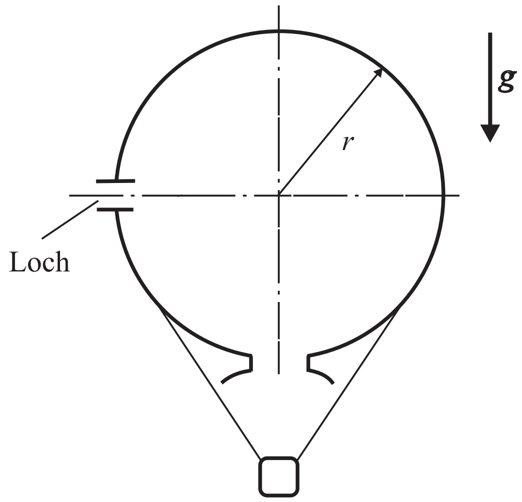
Determine
a) the height h and the angular speed ω,
b) the pressure distribution along the side wall and the bottom!

Hint:

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \qquad \frac{\partial p}{\partial z} = -\rho g \qquad\qquad dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial z}dz$$

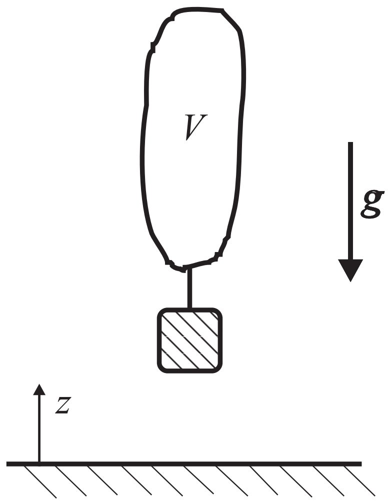
5.9 A spherical, open, and rigid gas balloon is designed for a ceiling of $H = 10km$ in isothermal atmosphere (temperature T_0).

$H = 10^4\ m, \qquad R_L = 287\ \frac{Nm}{kg\ K}, \qquad g = 10\ \frac{m}{s^2}, \qquad T_0 = 287\ K$



What is the ceiling h, if a hole is in the envelope (see sketch)?

5.10 A wheather balloon with mass m and initial volume V_0 ascends in an isothermal atmosphere. Its envelope is loose up to the achievement of the maximal volume V_1.



$p_0 = 10^5\ N/m^2 \quad \rho_0 = 1,27\ kg/m^3 \quad m = 2,5\ kg \quad V_0 = 2,8\ m^3 \quad V_1 = 10\ m^3$
$R = 287\ Nm/kgK \quad g = 10\ m/s^2$

a) What is the necessary force to hold down the balloon before launch?
b) In what altitude the balloon reaches its amximum volume V_1?
c) What ceiling reaches the balloon?

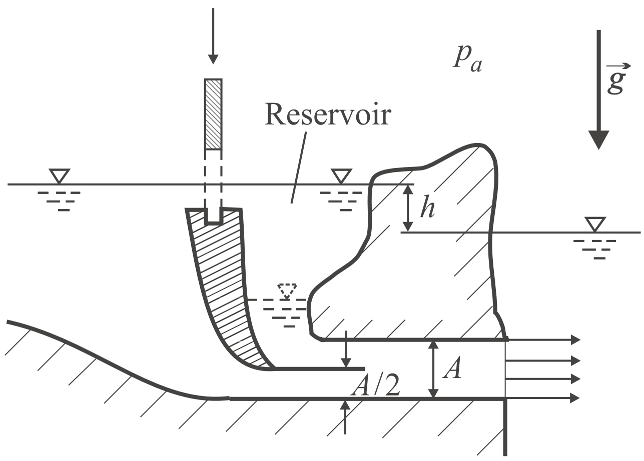
6 Continuity and Bernoulli's equation

6.1 A large lake is connected via overflow to a reservoir and via pipe system with a lower lake (see sketch).

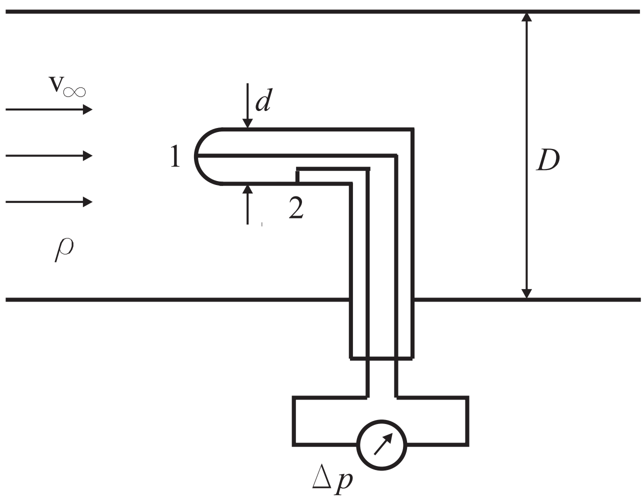
Determine the total volume flux $\dot{V}$ that flows into the lower lake when the overflow is opened.

Hint: Neglect the altitude difference in the horizontal pipes and neglect friction!

Given: $\rho,\ g,\ h,\ A$

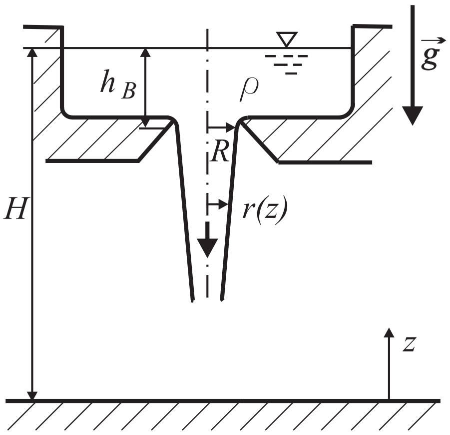


6.2 In order to determine the velocity in a duct flow the pressure difference Δp is measured. Through strong obstruction the pressure difference deviates from the dynamic pressure of the undisturbed incoming flow.



Outline the distribution of $v_\infty \,/\, \sqrt{\dfrac{2\Delta p}{\rho}}$ in terms of $\dfrac{d}{D}$ for frictionless flow.

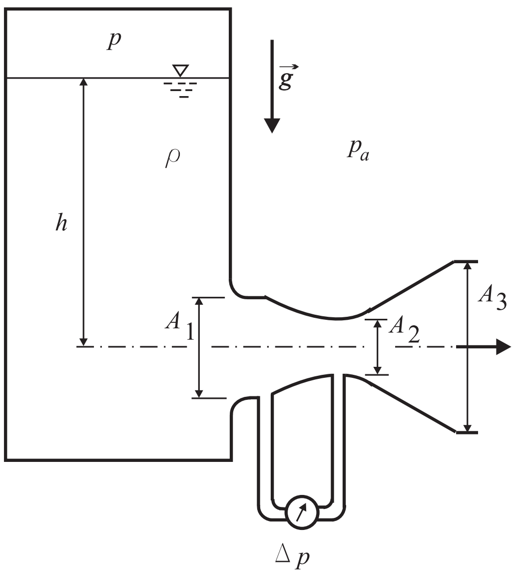
6.3 An incompressible fluid of density ρ is flowing stationarily from a large tank through a well rounded exit of radius R into the surrounding.



Given: $H,\ h_B,\ R,\ \rho,\ g$

Determine the radius $r(z)$ of the jet as a function of the altitude z.

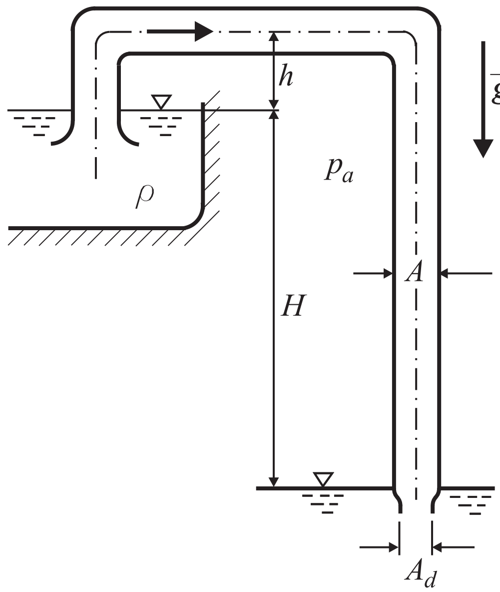
6.4 Water flows from a large pressurized tank into the open air. The pressure difference Δp is measured between the cross sections A_1 and A_2.



$A_1 = 0,3\ m^2, \qquad A_2 = 0,1\ m^2, \qquad A_3 = 0,2\ m^2, \qquad h = 1\ m,$
$\rho\ = 10^3\ kg/m^3, \qquad p_a = 10^5\ N/m^2, \qquad \Delta p = 0,64 \cdot 10^5\ N/m^2, \qquad g = 10\ m/s^2$

Compute the
a) velocities $v_1,\ v_2,\ v_3$,
b) pressures $p_1,\ p_2,\ p_3$ and the pressure p above the water surface!

6.5 Two large basins located one upon the other are connected with a duct.



$A = 1\ m^2, \qquad A_d = 0,1\ m^2, \qquad h = 5\ m, \qquad H = 80\ m,$
$p_a = 10^5\ N/m^2, \qquad \rho\ = 10^3\ kg/m^3, \qquad g = 10\ m/s^2$

a) Determine the volume rate!
b) Outline the distribution of static pressure in the duct!
c) At what exit cross section bubbles are produced, when the vapour pressure is
$p_D = 0,025 \cdot 10^5\ N/m^2$?

6.6 The flap at the exit of the water pipe (constant width B) of a large container is opened abruptly. The appearing flow is without any losses.

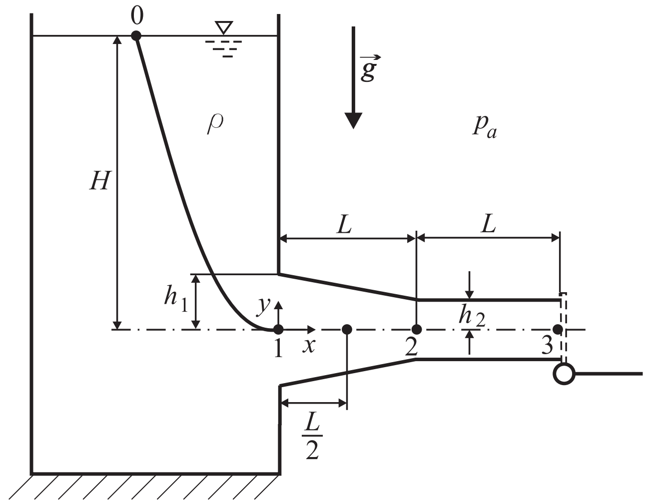
Given: $H,\ h_1,\ h_2,\ g,\ L \quad ; \quad L >> h_1$

Determine

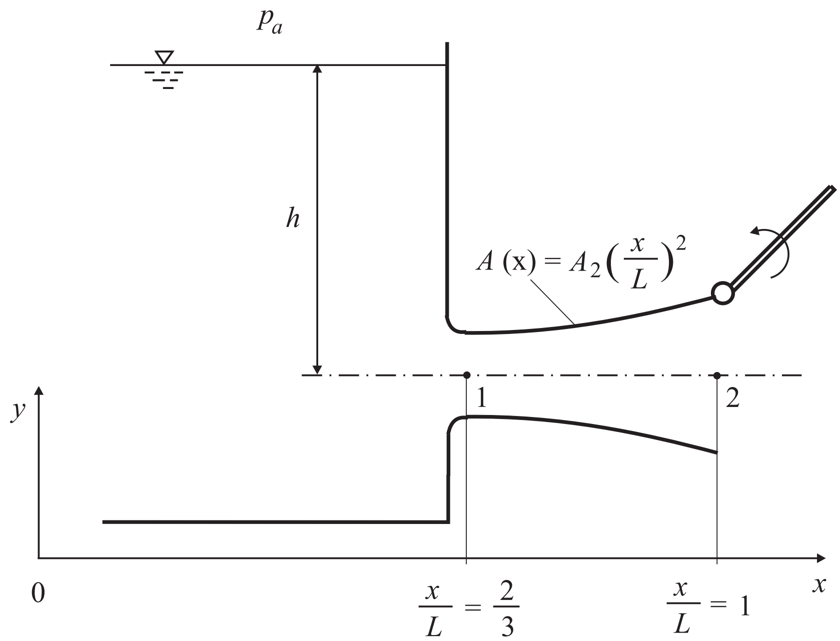
a) the differential equation for the exit velocity v_3
b) - the local acceleration
- the convective acceleration
- the substantial acceleration

at $x = \dfrac{L}{2}$ when the exit velocity reaches half of its asymptotic final value!

Hint: The computation of $v(t)$ is not necessary for solving this problem.



6.7 The exit of a large container is build as diffusor. The flap at the end of the diffusor is opened at $t = 0$ abruptly.



Given: $L,\ p_a,\ p_2 \approx p_a,\ A_2,\ h$

Determine

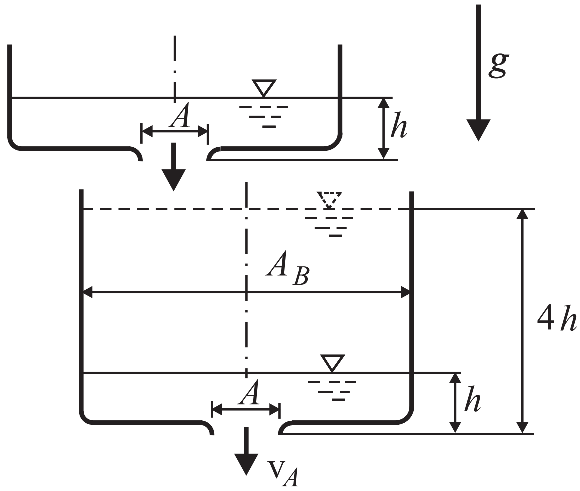
a) the acceleration $\dfrac{dv_2}{dt}$ in point "2" immediately after the opening,

b) the position x in the diffusor, where the the pressure has its maximum, when v_2 reaches half of its asymptotic final value.

6.8 Water flows from a large reservoir into a lower lying basin whose drain orifice is decreased to one third of its origin size abruptly.

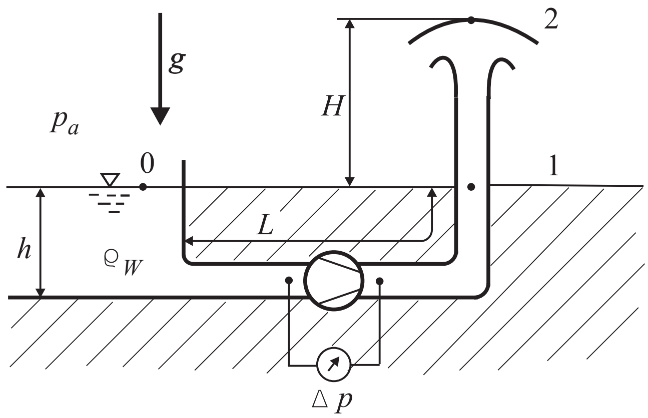
Determine the time interval in that the water surface quadrupels from its origin altitude h.

$A = 0,03\ m^2, \qquad A_B = 1\ m^2, \qquad h = 5\ m, \qquad g = 10\ m/s^2, \qquad \left(\frac{dh}{dt}\right)^2 << (v_A)^2$



$$\text{Hint}: \int \frac{dx}{a - \sqrt{x}} = 2\left[(a - \sqrt{x}) - a\ \ln(a - \sqrt{x})\right] + C$$

6.9 A spring well fountain is feeded from a large container. When the pump is switched on a pressure difference of Δp is created. After a time of 7. 6 seconds the velocity at the end of the duct reaches 99,9 % of its asymptotic value ($v_{(t \to \infty)}$).



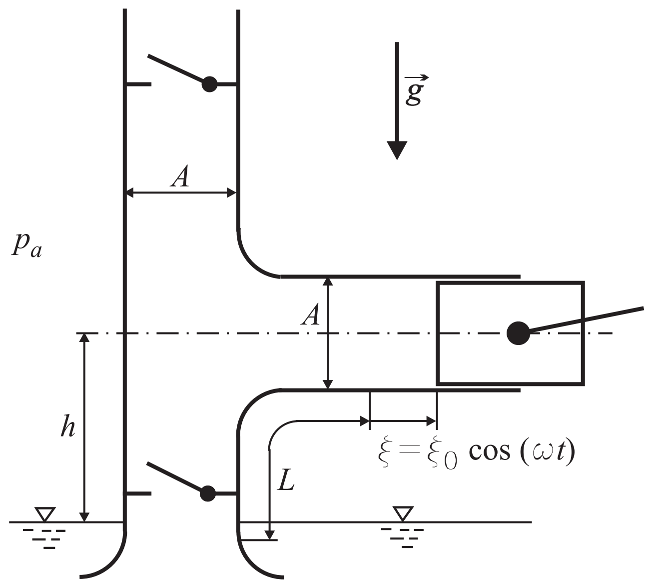
$T_{99,9} = 7.6\ s, \qquad L = 10\ m, \qquad g = 10\ m/s^2$
At "1" the fountain is a free jet.

a) Determine the altitude h of the fountain for $t \to \infty$.
b) At what time $t_{1/2}$ it reaches 50 % of this altitude?

Hint: Neglect the friction losses!.

$$\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \ln \frac{a+x}{a-x}$$

6.10 A piston pump surges water with a vapour pressure p_D from a reservoir. The piston has the frequency ω and the stroke ξ_0. The pump has a constant cross-section A and the pistons position is in the altitude h above the water surface. The length of the inlet is L.



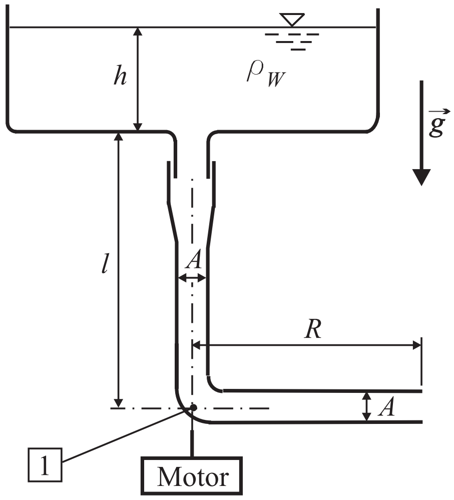
Given: $p_a,\ L,\ \xi_0,\ h,\ A,\ \rho,\ p_D,\ g$

a) Determine the nondimensional pressure $p_K\ /\ (\rho L^2 \omega^2) = f(\omega)$ at the piston head in terms of the piston frequency.
b) Determine the critical pump frequency for $\xi_0 << L$ when the vapour pressure p_D is just reached at the piston head.
c) Determine the average displaced volume per time $\dot{V}$.

6.11 Water flows under the influence of gravity from a large reservoir through a convoluted duct into open air. At $t = 0$ the convoluted duct is accelerated by an engine up to the angular speed ω_r.

a) At what time ΔT the flow reaches 50 % of the velocity which is at $t \rightarrow \infty$ in the vertical part of the duct?
b) Determine the pressure at point $\boxed{1}$ for $t \rightarrow \infty$ in the duct that rotates with $\omega = \omega_r$.

Given: $h,\ l,\ R,\ A,\ \omega_r,\ \rho_W,\ p_a,\ \vec{g}$



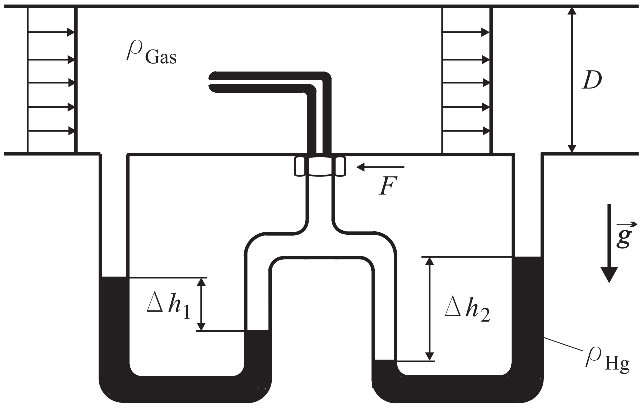
Hint:

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \frac{a+x}{a-x} + C \qquad ; \qquad \frac{\partial p}{\partial r} = \rho\, \omega^2\, r$$

7 Momentum and momentum of momentum equation

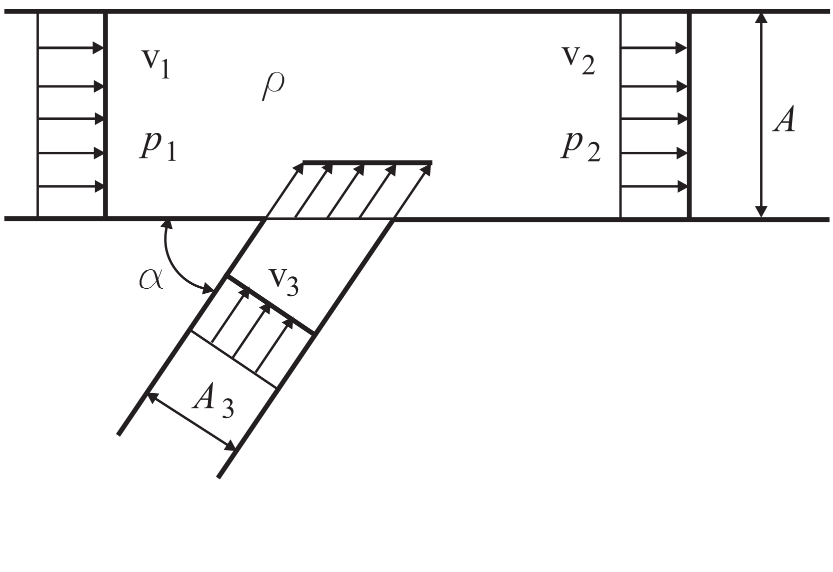
7.1 A pitot tube in a gas pipeline is connected via two u-tubes with a pressure hole (see sketch). Determine the force F on the fitting of the pitot tube. Assume an incompresible fluid! Neglect the friction in the pipeline!

Given: $\rho_{H_q} \gg \rho_{Gas},\ g,\ \Delta h_1,\ \Delta h_2,\ D$

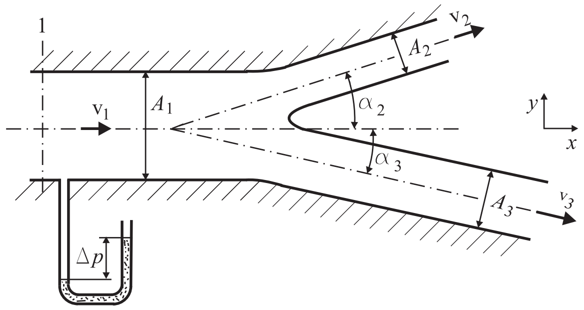


7.2 Determine the pressure difference $\Delta p = p_2 - p_1$ in the plotted bifurcation by neglecting the friction.

Given: $v_1,\ v_2,\ A_3 = \dfrac{1}{4}A,\ \alpha,\ \rho = konst.$



7.3 Water is flowing through a bifurcation into open air stationarily. The pressure in the incoming tube is Δp higher than in the surrounding.

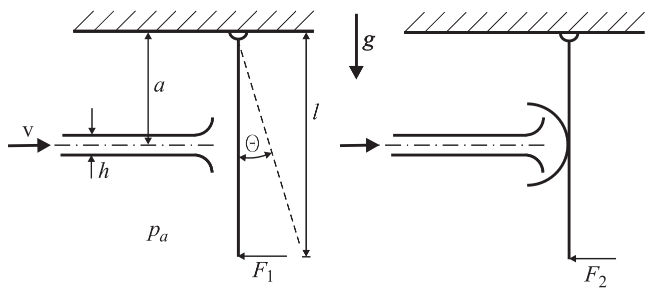


$A_1 = 0,2\ m^2$ $\quad A_2 = 0,03\ m^2$ $\quad A_3 = 0,07\ m^2$ $\quad \alpha_2 = 30$ $\quad \alpha_3 = 20$
$\Delta p = 10^4 N/m^2$ $\quad \rho = 10^3\ kg/m^3$

Determine

a) the velocities $v_1,\ v_2,\ v_3$,
b) the force F_s in section 1,
c) the angle α_3, when F_{sy} vanishes.

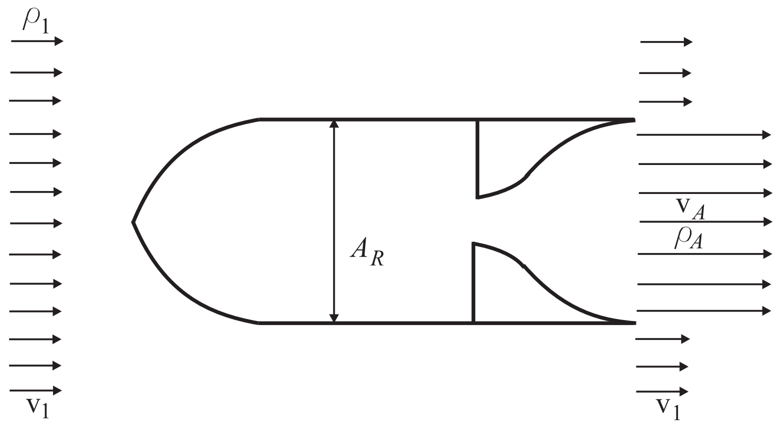
7.4 A flat plate of constant thickness with the mass m and the length l is hung at a hinge and is passed by a planar water jet of height h and width B with the velocity v. The flow in the jet is without any losses.



a) Determine the force F_1, at the lower end of the plate, that is necessary to fix the plate in a vertical position.
b) Determine the necessary forceF_2, when a deflector blade is mounted on the plate.
c) Compute the rotation angle θ for the steady state if the plate is swinging undisturbedly..

Given: $m,\ a,\ l,\ h,\ B,\ v,\ \rho,\ g,\ p_a$
Hint: The volume forces are neglectable.

7.5 A rocket is moving at constant velocity. The passing air is displaced radially. Inside the jet the velocity is v_A, outside v_1.

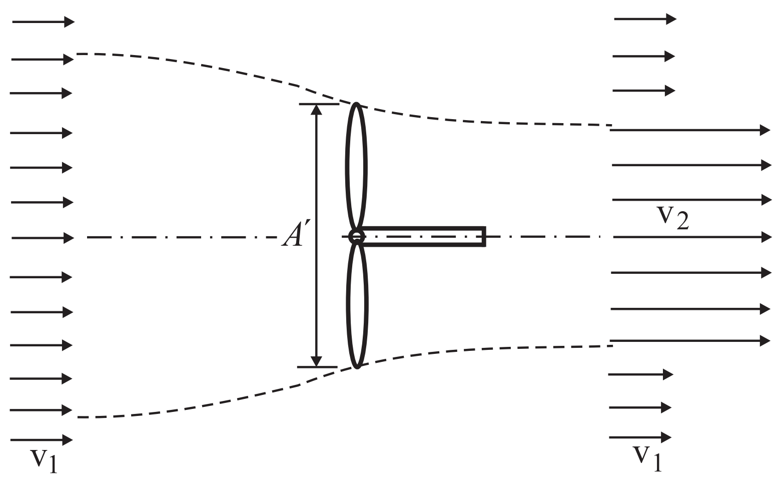


Given: $v_1,\ v_A,\ \rho_1,\ \rho_A,\ A_R$

Determine

a) the displaced air mass,
b) the thrust and the engine power.

7.6 A propeller is passed with constant velocity v_1. At a certain distance downstream of the propeller the velocity inside the jet is v_2, and v_1 outside the jet.

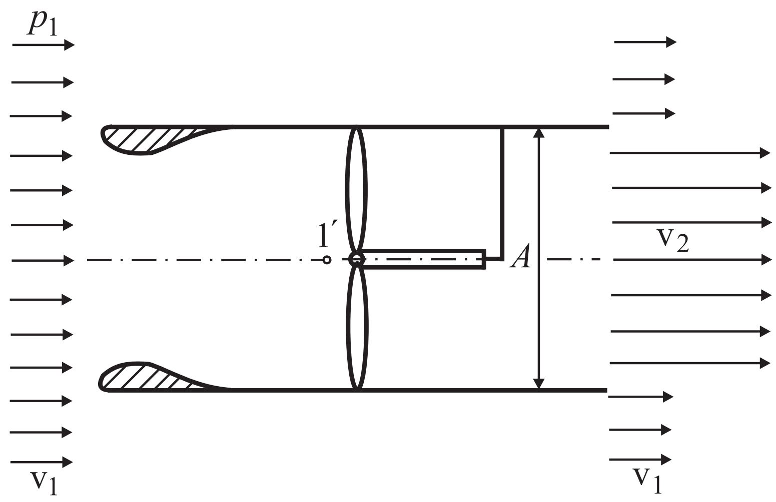


$A' = 7,06\ m^2 \qquad v_1 = 5\ m/s \qquad v_2 = 8\ m/s \qquad \rho = 10^3\ kg/m^3$

Determine

a) the velocity v' in the propeller plane,
b) the efficiency.

7.7 The shell of a propeller is passed with constant velocity. The inlet is well rounded.



$A = 1\ m^2$ $\qquad v_1 = 10\ m/s$ $\qquad p_1 = 1,345 \cdot 10^5\ N/m^2$ $\qquad p_{1'} = 10^5\ N/m^2$ $\qquad \rho = 10^3\ kg/m^3$

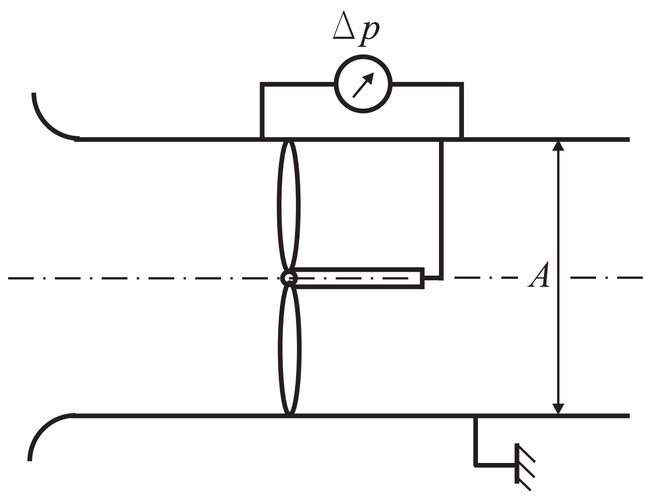
a) Outline the distribution of the static pressure along the axis

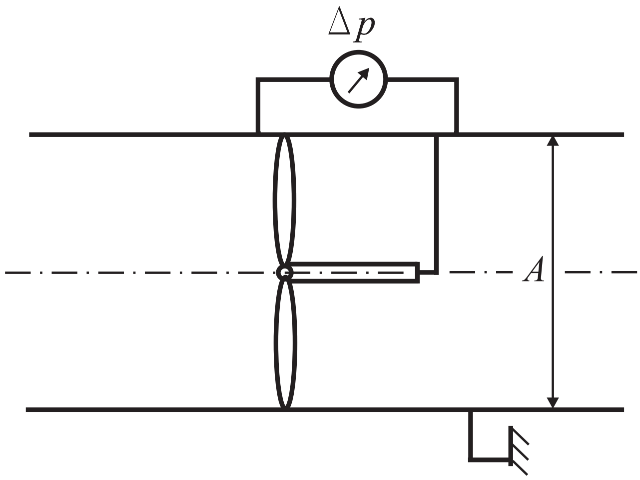
and determine

b) the mass flux,
c) the thrust,
d) the power that the propeller emits to the flow.

7.8 Two fans sucking air from the surrounding differ in their inlets

Given: $\rho,\ A,\ \Delta p$

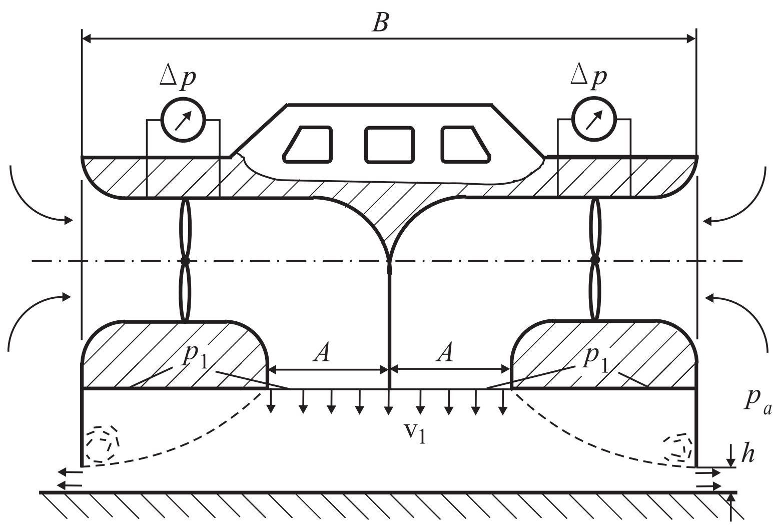




Compute

a) the volume flux,
b) the power of the fans,
c) the force on the fitting.

7.9 An air cushion vehicle of weight G, width B and length L is hovering above the ground in steady state.



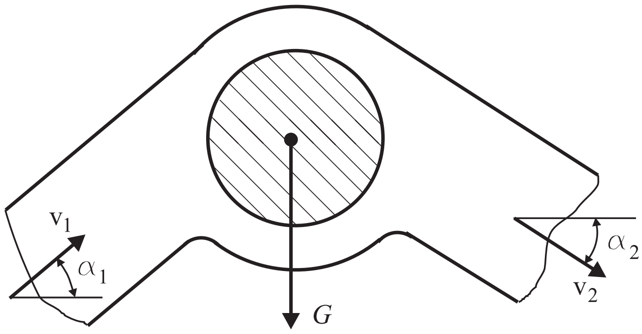
Compute by neglecting the friction and the compressibility

a) the pressure p_1 and the volume flux $\dot{V}$ that is flowing through the hover craft,
b) the power that is emitted from the fans to the flow and the power losses throught the accumulation of vortices.

Given: $\Delta p,\ \rho,\ G,\ A, \qquad BL = 20\ A, \qquad h(B+L) = A$

Hint: Neglect the differences between the geodetic altitudes.

7.10 A ball of weight G is passed frictionless by a fluid jet (e. g. air jet) and is hovering from the forces of the jet. Tne jet is flowing with the velocity v_1 under the influence of the angle α_1 (see sketch).

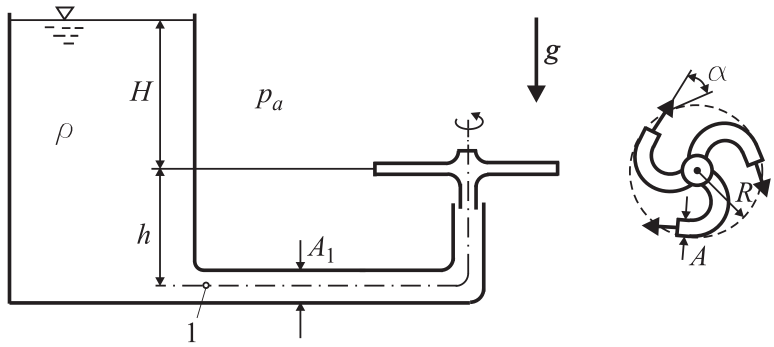


Determine

a) the downstream velocity v_2
b) the angle α_2 of the downstream jet,
c) the mass flux $\dot{m}$, to hang the ball in balance.

Given: $v_1,\ G,\ \alpha_1$.
Hinweis: Neglect the force of gravity of the jet!

7.11 A sprinkler with three arms is supplied by a large tank and rotates with the angular velocity $\omega = const.$. The angle between the outflowing jets and the circumferential direction is α.

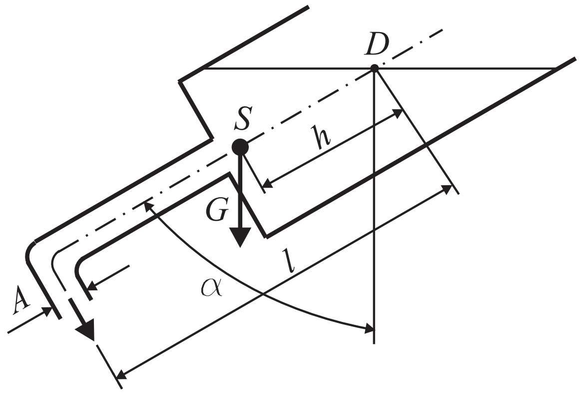


$H = 10\ m$ $\quad R = 0,5\ m$ $\quad h = 1\ m$ $\quad A = 0,5 \cdot 10^{-4}\ m^2$ $\quad A_1 = 1,5.10^{-4}\ m^2$ $\quad \alpha = 30$
$p_a = 10^5\ N/m^2$ $\quad \rho = 10^3\ kg/m^3$ $\quad g = 10\ m/s^2,$ $\quad \omega = 15\ s^{-1}$

Determine

a) the relative exit velocity,
b) the torque and the volume flux,
c) the pressure p_1,
d) the maximum torque.

7.12 A atnk with the weight G is fixed in a rotatable bearing in D. Its drain-pipe has a 90°-bend. The center of gravity of the system has the distance h to point D. What is the angle α between the pipe-axis and the vertical axis, if the water flows without friction?

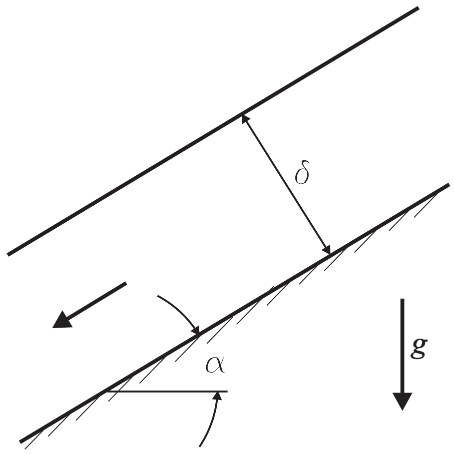


Given: $G,\ l,\ h,\ A,\ \rho$

Hint: The tank is such large that the water surface is not moving.

9 laminar viscous flows

9.1 An oil film of constant thickness and width is flowing on an inclined plane.
$\delta = 3 \cdot 10^{-3}\ m \quad B = 1\ m \quad \alpha = 30^o \quad \rho = 800\ kg/m^3 \quad \eta = 30 \cdot 10^{-3}\ Ns/m^2 \quad g = 10\ \dfrac{m}{s^2}$



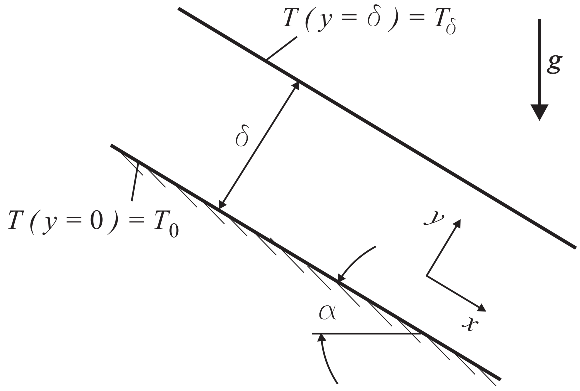
Calculate the volume flux.

9.2 A viscous oil film is flowing on an inclined plane (angle α) under the influence of gravity.The wall temperature of the plane is T_0 and the temperature at the surface of the oil is T_δ. The temperature distribution of the oil is linearly in terms of y and constant in x-direction. The thickness of the film is δ and is constant. The kinematic vviscosity ν is constant and the density in the important temperature range is Temperaturbereich mit

$$\rho = \rho_0 \cdot e^{-\frac{T}{T_0}}$$

The density is independent of the pressure

Given: $g,\ \delta,\ \alpha,\ \nu,\ \rho_0 = \rho(y = 0),\ T_0 = T(y = 0),\ T_\delta = T(y = \delta)$



a) Compute the shear stress $\tau(y)$.
b) Compute the velocity $u(y)$.
c) Show that $\dfrac{\partial p}{\partial x} = 0$ by using the momentum equation in y-direction.

9.3 A car is in a wind tunnel with the velocity u_∞. To simulate the relative motion between the vehicle and the roadway a bend-conveyor with the velocity u_∞ is mounted unde the fixed car (see sketch). Between the lower side of the vehicle and the bend-conveyor a gap flow is formed. The flow shall be analyzed for

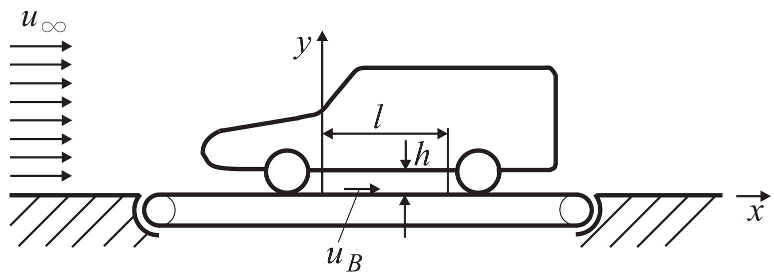
1. for the non-moving bend-conveyor ($u_B = 0$) and
2. for the moving bend-conveyor ($u_B = u_\infty$)

Assume, the flow is fully devloped in both cases. The following relation is valid:

$$\frac{dp}{dx} = \eta \frac{d^2u}{dy^2} \quad \text{mit} \quad \frac{dp}{dx} = konst < 0$$

a) Compute and sketch the velocity profiles $u(y)$ in the gap for the non-moving and the moving bend-conveyor.
b) Compute the friction per unit width, that the flow affects on the lower side of the car over the length l for the non-moving and the moving conveyor.

Given: $\dfrac{dp}{dx},\ h,\ l,\ \eta,\ u_\infty$

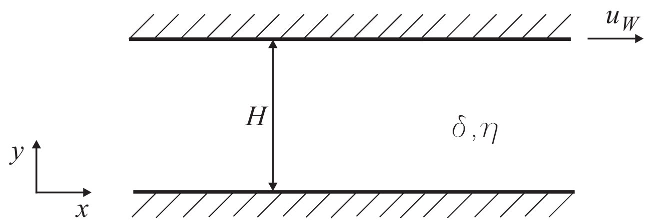


9.4 A Newtonian fluid is flowing between two horizontal plates. The upper one is moving at a velocity u_w. The lower oneis standing still. The pressure is decreasing linearly in x-direction.

Given: $H,\ u_w,\ \rho,\ \eta,\ dp/dx$

Determine for a fully developed laminar flow

a) the velocity distribution,



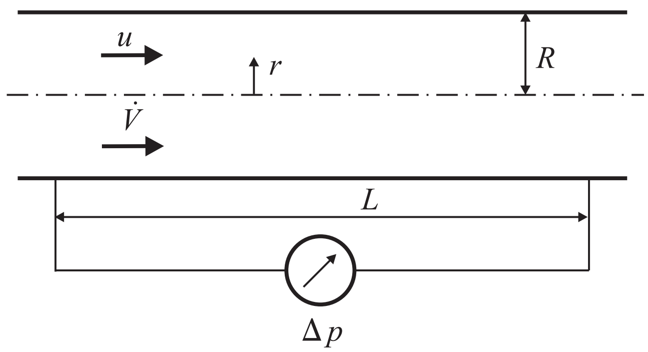
b) the relation between the shear stresses at $y = 0$ and $y = H$,
c) the volume flux for a width of B,
d) the maximum velocity for $u_w = 0$,
e) the momentum flux for $u_w = 0$,
f) the wall shear stress in non-dimensional form for $u_w = 0$.
g) Outline the distribution of the velocity and the shear stress for $u_w > 0, u_w = 0$, and $u_w < 0$.

9.5 An incompressible fluid is flowing through a pipe with radius R. The volume flux is $\dot{V}$. The shear stresses are depicted with the model of Ostwald-de-Waele

$$\tau = -\eta_{OdW} \left|\frac{du}{dr}\right| \frac{du}{dr}$$

Given: $\dot{V},\ L,\ R,\ \eta_{OdW}$

The pressure decreases within a length of L with Δp.



Compute the pressure decrease Δp for a fully developed pipe flow.

Hinteis: The equation of motion for a fully developed flow

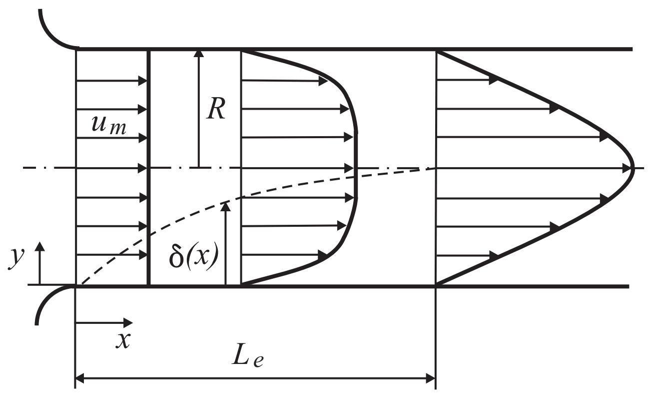
$$r\frac{dp}{dx} + \frac{d(\tau r)}{dr} = 0$$

9.6 The velocity distribution of a laminar pipe flow can be described in the inlet with the following approximation

$$\frac{u}{u_m} = \frac{f\left(\frac{y}{\delta}\right)}{1 - \frac{2}{3}\frac{\delta}{R} + \frac{1}{6}\left(\frac{\delta}{R}\right)^2}$$

$$f(\frac{y}{\delta}) = \begin{cases} 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 & 0 \leq y \leq \delta(x) \\ 1 & \delta(x) \leq y \leq R \end{cases}$$

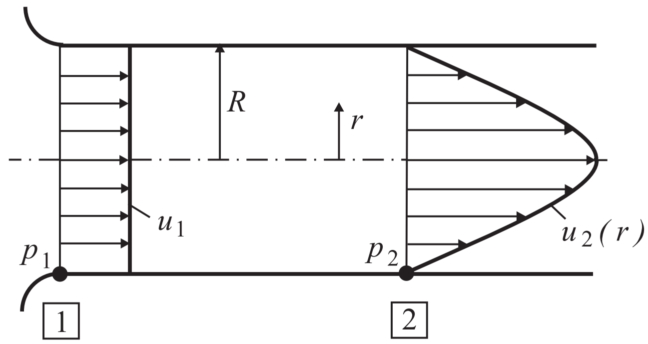
Given: $u_m,\ R,\ \rho,\ \eta$



Determine in the inlet cross section, at the end of the inlet section and for $\delta/R = 0,5$

a) the momentum flux,
b) the wall shear stress.

9.7 The velocity u_1 in the inlet cross-section (1) of a circular pipe with radius R is constant. In the cross-section (2) the velocity of a fully developed laminar pipe flow can be written as parabola of the form $u_2(r) = u_{2_{max}}\left[1 - \left(\frac{r}{R}\right)^2\right]$.



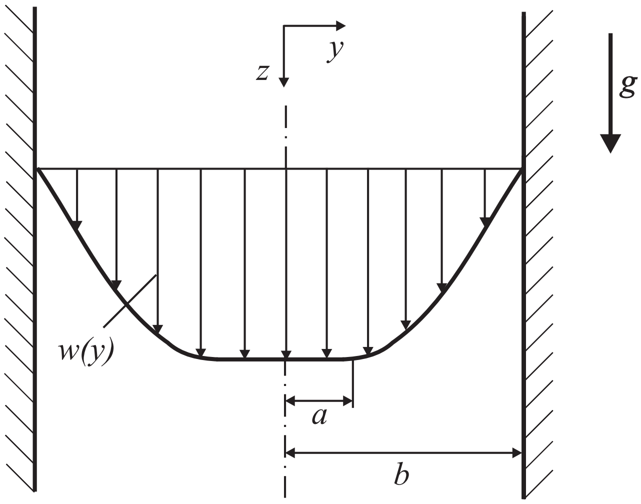
Given: $\rho,\ R,\ u_1$

Determine

a) the pressure loss $(p_1 - p_2)$,
b) the coefficient of the pressure loss $\zeta_E = (p_1 - p_2)/\frac{\rho}{2}\bar{u}^2$

Hint: Neglect the wall friction.

9.8 A Bingham fluid is flowing between two infinite parallel plates under the influence of gravity.



Given: $b,\ \rho,\ \eta,\ \tau_0,\ g,\ dp/dz = 0$

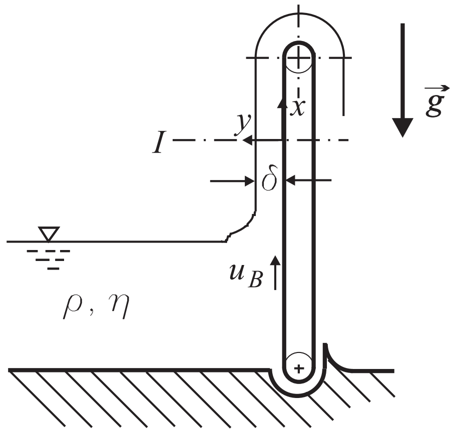
Assume a fully developed flow and determine

a) the distance a,
b) the velocity distribution.

9.9 A Bingham fluid is in a basin. A vertical bend-conveyor, moving with the velocity u_B is used to transport the fluid in another basin.

Given: $u_B,\ \delta,\ g,\ \varrho,\ \eta,\ \tau_0 = \varrho g \frac{\delta}{2}$

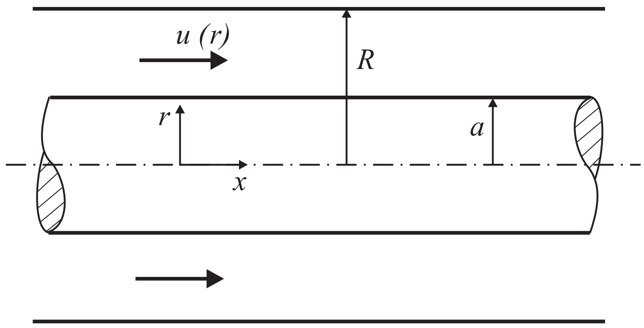
$$\tau = \begin{cases} +\tau_0 - \eta \dfrac{du}{dy} & \text{für} \quad \dfrac{du}{dy} < 0 \\ -\tau_0 - \eta \dfrac{du}{dy} & \text{für} \quad \dfrac{du}{dy} > 0 \end{cases}$$



a) Determine and sketch the distribution of the shear stress in section I.
b) Determine and sketch the distribution of the velocity in section I.
c) What is the minmum velocity $u_{B,min}$ for conveying mass?

Hint: The flow at the conveyor is fully developed and laminar. The thickness δ of the fluid is assumed to be constant.

9.10 A Newtonian fluid is moving between two coaxial cylinders.

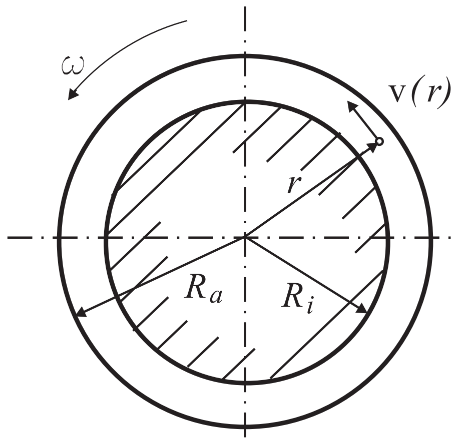


Given: $R,\ a,\ \eta,\ dp/dx$

Bestimmen Sie für eine ausgebildete laminare Strömung

a) the velocity distribution. Outline the result,
b) the relation of the shear stresses for $r = a$ and $r = R$,
c) the average velocity.

9.11 A Couette viscosimeter consists of two concentrical cylinders with the length L. The interstice is filled with a Newtonian fluid. The oute cylinder rotates with the angular velocity ω, and the inner one is standing still. At the inner cylinder the hinge momentum M_z is measured.



$R_a = 0,11\ m \qquad R_i = 0,1\ m \qquad L = 0,1\ m \qquad \omega = 10\ s^{-1} \qquad M_z = 7,246 \cdot 10^{-3}\ Nm$

Determine

a) the velocity distribution,
b) the dynamic viscosity of the fluid.

Hint: The differential equations for the velocity- and the shear stress distribution are:

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rv)\right] = 0 \qquad \tau = -\eta r \frac{d}{dr}\left(\frac{v}{r}\right)$$

10 Turbulent pipe flows

10.1 Proof the following rules:

a) $\overline{\overline{f}} = \overline{f}$

b) $\overline{f + g} = \overline{f} + \overline{g}$

c) $\overline{\overline{f} \cdot g} = \overline{f} \cdot \overline{g}$

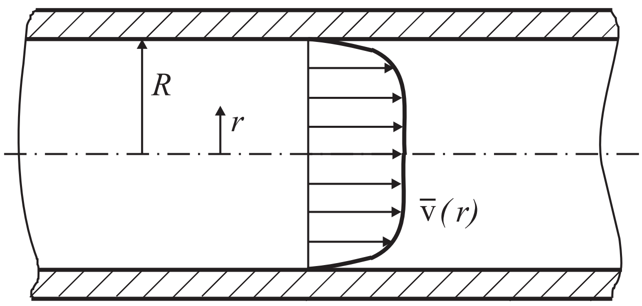
d) $\overline{\dfrac{\partial f}{\partial s}} = \dfrac{\partial \overline{f}}{\partial s}$

e) $\overline{\int\limits_s f \cdot ds} = \int\limits_s \overline{f} \cdot ds.$

10.2 The velocity profile in a fully developed flow in a pipe with a smooth surface can be approximated with the potential law:

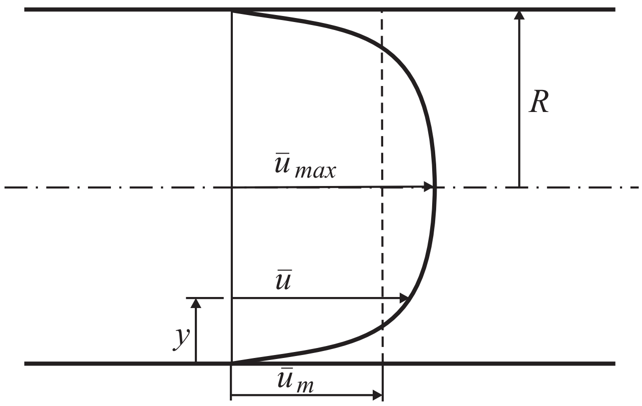
$$\frac{\bar{v}}{\bar{v}_{max}} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}, \text{ mit } \quad n = n(Re).$$

Re	n
$1 \cdot 10^5$	7
$6 \cdot 10^5$	8
$1.2 \cdot 10^6$	9
$2 \cdot 10^6$	10



a) Use the continuity equation to compute the relation between the average velocity $\bar{v}_m$ and the maximum velocity $\bar{v}_{max}$, i. e. $\dfrac{\bar{v}_m}{\bar{v}_{max}} = f(n)$.
b) At what position $\dfrac{r}{R}$ is $\bar{v}(r/R) = \bar{v}_m$?
c) How can the results of a) and b) be used, if the volume flux shall be measured?

10.3 The velocity distribution of a turbulent pipe flow can be approximated with the following law: $\bar{u} \,/\, \bar{u}_{max} = (y \,/\, R)^{1/7}$.



Determine

a) the ratio of the velocities $\bar{u}_m \,/\, \bar{u}_{max}$,
b) the ratio of the momentum fluxes $\dot{I} \,/\, \rho\, \bar{u}_m^2\, \pi\, R^2$.

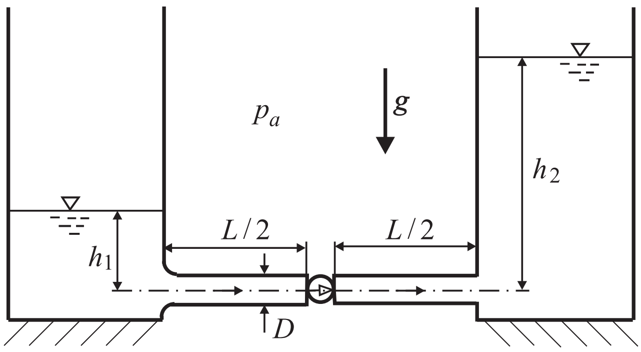
10.4 Water is flowing through a hydrauliccaly smooth pipe.

$D = 0,1\ m \qquad Re = 10^5 \qquad \rho = 10^3\ kg/m^3 \qquad \eta = 10^{-3}\ Ns/m^2$

Determine

a) the wall shear stress,
b) the ratio of the velocities $\bar{u}_m \,/\, \bar{u}_{max}$,
c) the velocity for $\dfrac{y\, u_*}{\nu} = 5$ and for $\dfrac{y\, u_*}{\nu} = 50$,
d) the mixing length for $\dfrac{y\, u_*}{\nu} = 100$.

10.5 Water is pumped through a pipe (roughness k_s) from h_1 to h_2.



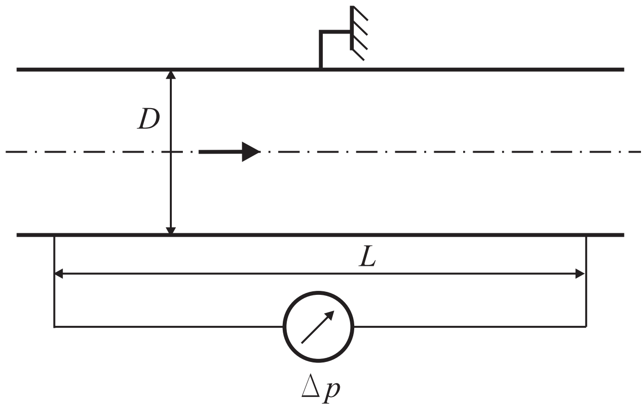
$\dot{V} = 0,63\ m^3/s \qquad h_1 = 10\ m \qquad h_2 = 20\ m \qquad L = 20\ km \qquad D = 1\ m \qquad k_s = 2\ mm$
$\rho = 10^3\ kg/m^3 \qquad \nu = 10^{-6}\ m^2/s \qquad p_a = 10^5\ N/m^2 \qquad g = 10\ m/s^2$

a) Sketch the distribution of the static pressure along the pipe axis.

Determine

b) the pressure at the pump inlet,
c) the pressure at the pump exit,
d) the net power of the pump.

10.6 The pressure decrease Δp along L is measured in a fully developed pipe flow with the volume flux $\dot{V}$.

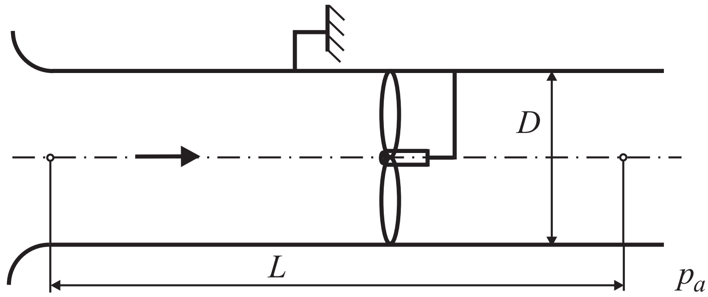


$\dot{V} = 0,393\ m^3/s \quad L = 100\ m \quad D = 0,5\ m \quad \Delta p = 12820\ N/m^2 \quad \rho = 900\ kg/m^3$
$\eta = 5 \cdot 10^{-3}\ Ns/m^2$

Determine

a) the skin-friction coefficient,
b) the equivalent roughness of the pipe,
c) the wall shear stress and the force of the support.
d) What is the pressure decrease, if the pipe is smooth?

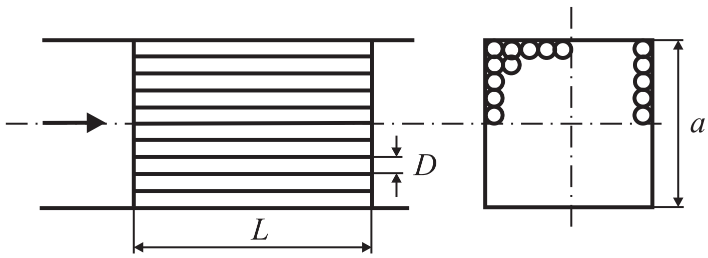
10.7 Air is pumped with a fan through a rough pipe with a well rounded inlet.



$L = 200\ m \quad k_s = 1\ mm$

Determine the ratio of the net power for $D = 0,1\ m$ and $D = 0,2\ m$ for a constant volume flux. Assume a very large Reynolds number.

10.8 Water is pumped through a channel with a quadratic cross-section. At a certain position the water is pumped through a bundle of 100 pipes with the length L.



$\dot{V} = 0,01\ m^3/s \quad L = 0,5\ m \quad a = 0,1\ m \quad D = 0,01\ m \quad \rho = 10^3\ kg/m^3 \quad \eta = 10^{-3}\ Ns/m^2$

Compute the length of the channel that produces the same pressure loss as the pipe bundle.

Hint: The pipes are hydraulically smooth.

11 Similarity theory

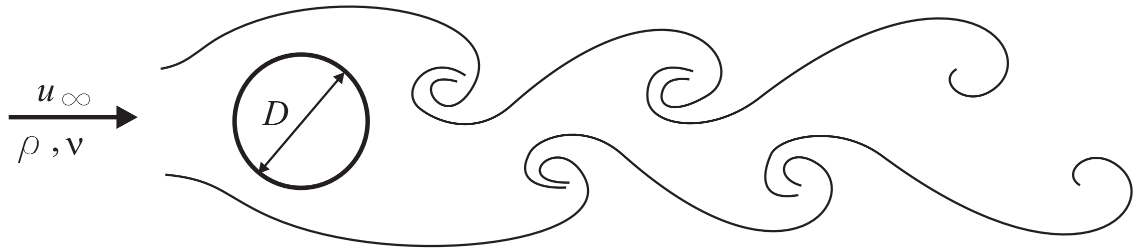
a) dimensional analysis

Frequently used quantities and their dimensions

Quantity	**Dimension**
* Length L	meter, m
* Mass M	kilogram, kg
* Time t	second, s
* Temperature T	Kelvin K $[1K] = [^oC] + 273,16$
Force F	Newton N, $\frac{kg\ m}{s^2}$
Velocity $u, v, w, \vec{u}$	m/s
Acceleration $a, b, \vec{a}$	m/s^2
Density ρ	kg/m^3
Pressure, Stress p, τ, σ	Pascal, $Pa = N/m^2$
Massflux $\dot{m}$	kg/s
Volumeflux $\dot{V}$	m^3/s
Momentum, work, energy M, W, E	Joule, $J = Nm$
Power P	Watt, $W = Nm/s$
dyn. viscosity μ, η	$Ns/m^2 \ = \ \frac{kg}{m \cdot s}$
kin. viscosity ν	m^2/s
spec. heat capacity c_p, c_r	$\frac{J}{kg\ K} = \frac{m^2}{s^2 \cdot K}$
spec. gas constant R	$\frac{J}{kg\ K} = \frac{m^2}{s^2 \cdot K}$

* Basic dimensions

11.1 The wake of a long cylinder with the diameter D is anlyzed experimentally in a wind-tunnel. On certain circumastances a periodic vortex configuration is generated, the Kármán vortex street. The dimensionless parameters of the problem shall be determined. How many variations of parameters are necessary in this investigation to measure the frequency of the vortex street?



11.2 A liquid is flowing steadily through a hydraulical smooth pipe. The flow is laminar and fully developed.

a) Deduce from the Ansatz $\dot{V} = \left(\frac{\Delta p}{L}\right)^{\alpha} \eta^{\beta} D^{\gamma}$ the Hagen-Poisseuille law by using the dimensional analysis!

b) Show, that the skin-friction coefficient in a pipe is inversely proportional to the Reynolds number!

11.3 An upward directed flow develops along a vertical plate with the temperature T_W. At a large distance away from plate the temperature is T_∞. The temperature distribution can be written with the equation

$$F\left(\frac{T - T_\infty}{T_W - T_\infty}, \frac{y}{x^{1/4}}, g, \rho, \eta\right) = 0$$

Deduce, using the dimensional analysis, an expression for the temperature ratio $\frac{T - T_\infty}{T_W - T_\infty}$ in terms of the nondimensional coordinate

$$\mu = f\left(\frac{y}{x^{1/4}}, g, \rho, \eta\right)!$$

11.4 What is the ratio of drag for spheres with different diameter and with the same Reynolds numbers, if one is flown against with air and the other with water and if the drag coefficient is only a function of the Reynolds number?

$$\frac{\rho_L}{\rho_W} = 0,125 \cdot 10^{-2} \qquad \frac{\eta_L}{\eta_W} = 1,875 \cdot 10^{-2}$$

11.5 The necessary power of a car with quadratic surface A to overcome the air drag shall be analyzed experimentally in a windtunnel test. The surface of the model must not exceed A_m from technical reasons.

$$A = 4\ m^2 \qquad A_m = 0,6\ m^2 \qquad v = 30\ m/s$$

a) Choose the wind velocity for the model tests?
b) Determine the power of the car, if the drag force $F'_W = 810N$ is measured at the largest possible model!

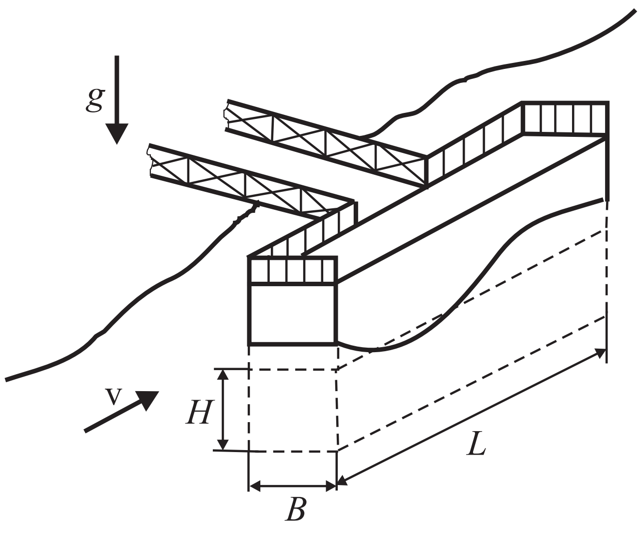
11.6 An axial fan (diameter D, number of revolutions n) shall be designed for air. In a model test with water (reduced scale 1:4) the increase of total pressure $\Delta p'_0$ is measured.

$$\dot{V} = 30\ m^3/s \qquad D = 1\ m \qquad n = 12,5\ s^{-1} \qquad \rho = 1,25\ kg/m^3 \qquad \eta = 1,875 \cdot 10^{-5}\ Ns/m^2$$

$$\rho' = 10^3\ kg/m^3 \qquad \eta' = 10^{-3}\ Ns/m^2 \qquad \Delta p'_0 = 0,3 \cdot 10^5\ N/m^2$$

Determine
a) the volume flux and the number of revolutions in the model test,
b) the increase of the total pressure for the fan,
c) the power and the torque for the model and for the fan.

11.7 A pontoon swims at the bank of a river. A test with a model and the reduced scale 1:16 shall be carried out.



$$L = 3,6\ m \qquad B = 1,2\ m \qquad H = 2,7\ m \qquad v = 3\ m/s$$

$$F'_W = 4\ N \qquad h' = 2,5\ cm \qquad \rho = 10^3\ kg/m^3$$

Determine

a) the flow velocity in the experiment,

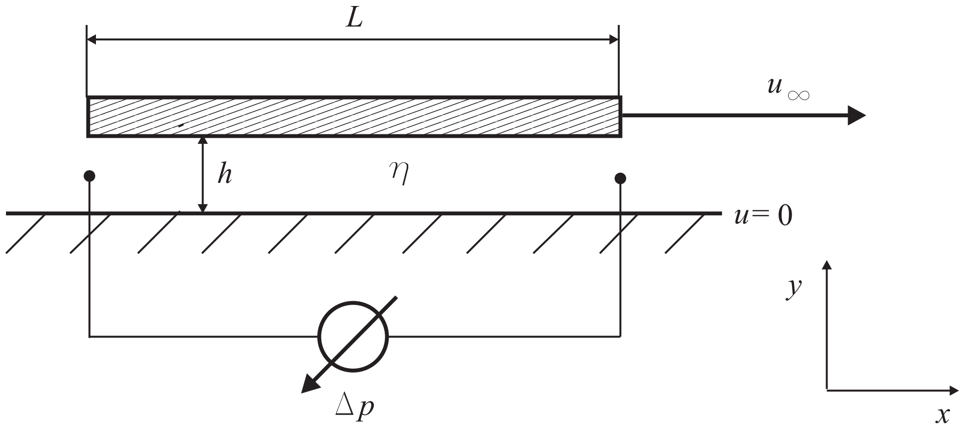
b) the resulting force on the pontoon, if the model force is F'_W,

c) the drag coefficient of the pontoon,

d) the height of the wave h upstream of the pontoon, if the wave height in the model is h'!

11.8 The following Couette flow can be described with a partial differential equation

$$\frac{\partial p}{\partial x} = \eta \frac{\partial^2 u}{\partial y^2}$$



a) Determine the dimensionless parameters of this problem using the method of differential equations.

b) How many parameters are achieved with the Π-theorem?

11.9 Deduce from the momentum equation in x-direction

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \eta \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

the dimensionless parameters!

11.10 In a gas flow the heat transfer is determined from the viscous effects and from heat conduction. The influencing quantities are the heat conductivity $\lambda \left[\frac{kgm}{s^3 K} \right]$, the dynamic viscosity $\eta \left[\frac{kg}{ms} \right]$ and the reference values for the temperature, the velocity, and the length. The physical relationship can be described with the energy equation

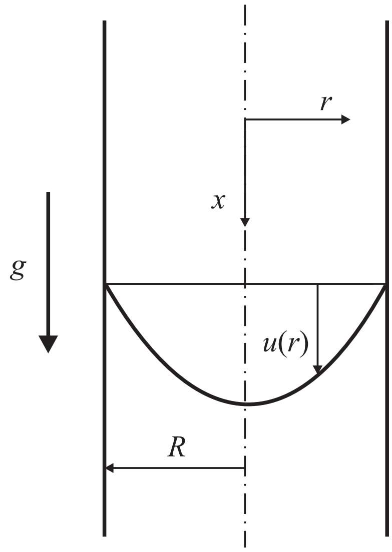
$$\lambda \frac{\partial^2 T}{\partial y^2} + \eta \left(\frac{\partial u}{\partial y} \right)^2 = 0.$$

Deduce the dimensionless parameters of the problem
a) with the method differential equations
b) with the Π-theorem.
c) Expand the resulting parameter with the specific heat capacity c_p and formulate the new coefficient as a product of three different parameters.

Hint: The material quantities are constant. The fourth basic dimension is the temperature.

11.11 The equilibrium of forces in a fully developed laminar pipe flow is described with the differential equation

$$-\frac{\partial p}{\partial x} + \rho g + \frac{\eta}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) = 0.$$



Determine the parameters of the problem
a) with the method of differential equations
b) with the Π-theorem.

Interpret the relationship between the solutions of a) and b)

11.12 The hydrodynamic attributes of a motor ship shall be analyzed with a model in a water cahnnel.

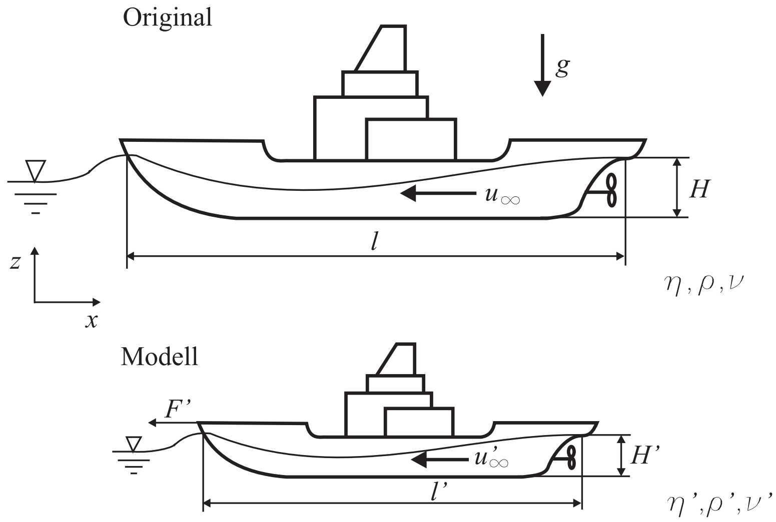
a) Determine the dimensionless parameters of the problem with the method of differential equations using the momentum equation in z-direction, that is decisive for the wave motion.

$$\rho\frac{dw}{dt} = -\frac{\partial p}{\partial z} - \rho g + \eta\nabla^2 w$$

Use only the given quantities.
Given: $l,\ u_\infty,\ \eta,\ \rho,\ g$

b) Compute the velocity u'_∞ and the kinematic viscosity ν' of the model fluid such that the flows are similar.
Given: $u_\infty,\ \nu,\ l/l' = 10$

c) Compute the power of the motor ship at the velocity u_∞.
Given: $l/l' = H/H' = 10,\ u_\infty,\ u'_\infty,\ \rho,\ \rho'$, Drag force in the experiment F′



11.13 The energy equation for steady, compressible flows with constant material quantities is

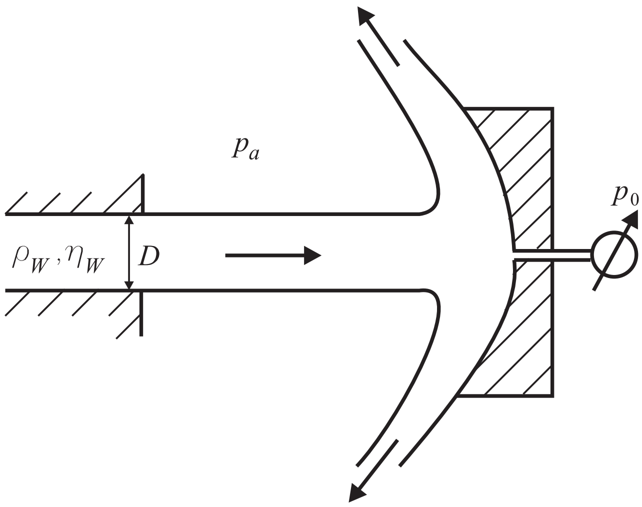
$$\rho u \frac{\partial}{\partial x}\left(c_p T + \frac{u^2}{2}\right) + \rho v \frac{\partial}{\partial y}\left(c_p T + \frac{u^2}{2}\right) = u\eta \frac{\partial^2 u}{\partial y^2} + \eta \left(\frac{\partial u}{\partial y}\right)^2 + \lambda \frac{\partial^2 T}{\partial y^2}$$

Determine with the method of differential equations

a) the dimensionsless form of the differential equation,
b) the dimensionless parameters of the problem.
c) Determine the isentropic coefficient γ , if the equation is independent of the Mach-number $M_\infty = \dfrac{u_\infty}{c_\infty}$.

Hint: $c = \sqrt{\gamma R T}$ $\qquad c_p = \dfrac{\gamma R}{\gamma - 1}$

11.14 The steady flow of a water jet colliding with a deflector blade shall be analyzed experimentally.



a) Determine with the method of differential equations from the x-momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\eta}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

the dimensionless parameters of the problem. Use only the given quantities as reference quantities and define the reference velocity by using the pressure difference $p_0 - p_a$.

b) Interpret the solutions of a).
Given: $D,\ p_a,\ p_0,\ \eta_W,\ \rho_W$

11.15 The laminar boundary layer flow on a flat plate, neglecting the viscous heat, can be described with the continuity, the momentum, and the energy equation in the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \eta\frac{\partial^2 u}{\partial y^2}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \lambda\frac{\partial^2 T}{\partial y^2}$$

a) Determine the dimensionless parameters of the problem.

b) Reformulate the resulting parameters by using well known parameters of fluid mechanics.

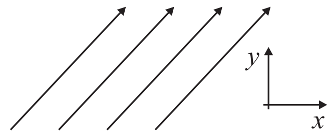
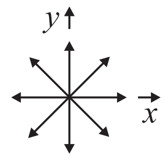
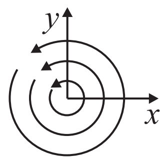
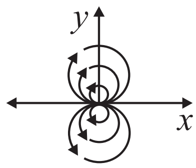
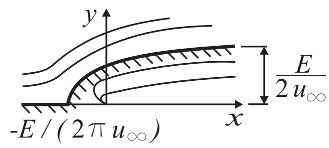
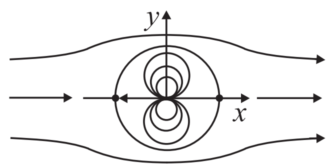
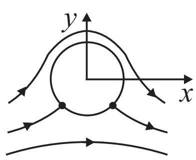
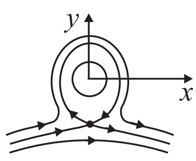
Assuming constant material quantities the flow field is independent of the temperature field. Both distributions can be computed separately.

c) Specify the assumptions to determine the temperature distribution in the boundary layer directly from the velocity distribution.

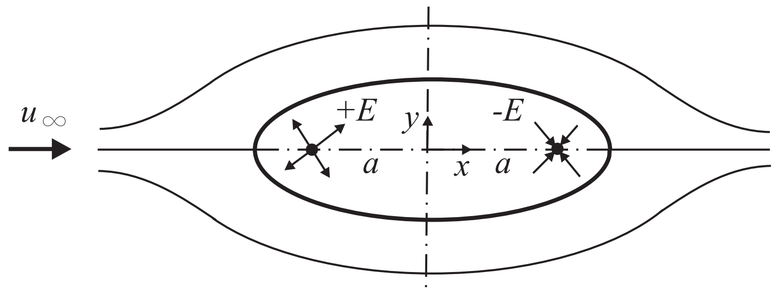
Hint for c): Compare the differential equations and assume that the velocity distribution $\vec{v}(x,\ y)$ is already known.

14 Potential flows

Complex Potential F(z)	**Potential** $\phi(x,y)$	**Streamfunction** $\psi(x,y)$
$(u_\infty - iv_\infty)z$ Parallel flow	$u_\infty x + v_\infty y$	$u_\infty y - v_\infty x$
$\frac{E}{2\pi}\ln z$ Source E $>$ 0, Sink E $<$ 0	$\frac{E}{2\pi}\ln r = \frac{E}{2\pi}\ln\sqrt{x^2+y^2}$	$\frac{E}{2\pi}\varphi = \frac{E}{2\pi}\arctan\frac{y}{x}$
$-\frac{\Gamma}{2\pi}i\ln z$ Vortex, $\Gamma < 0$ clockwise $\Gamma > 0$ counterclockwise	$\frac{\Gamma}{2\pi}\arctan\frac{y}{x}$	$-\frac{\Gamma}{2\pi}\ln\sqrt{x^2+y^2}$
$\frac{m}{z}$ Dipole	$\frac{mx}{x^2+y^2}$	$-\frac{my}{x^2+y^2}$
$u_\infty z + \frac{E}{2\pi}\ln z$ Parallel Flow+Source/Sink	$u_\infty x + \frac{E}{2\pi}\ln r$	$u_\infty y + \frac{E}{2\pi}\varphi$
$u_\infty(z + \frac{R^2}{z})$ Parallel Flow + Dipole = Cylinder Flow	$u_\infty x(1 + \frac{R^2}{x^2+y^2})$	$u_\infty y(1 - \frac{R^2}{x^2+y^2})$
$u_\infty(z + \frac{R^2}{z}) - \frac{\Gamma}{2\pi}i\ln z$ Cylinder Flow + Vortex	$u_\infty x(1 + \frac{R^2}{x^2+y^2}) + \frac{\Gamma}{2\pi}\varphi$	$u_\infty y(1 - \frac{R^2}{x^2+y^2}) - \frac{\Gamma}{2\pi}\ln r$
Parallel Flow + Vortex	$u_\infty x + \frac{\Gamma}{2\pi}\varphi$	$u_\infty y - \frac{\Gamma}{2\pi}\ln r$

velocity u	**velocity** v	**velocity** $c = \sqrt{u^2+v^2}$	**Streamlines** $\psi = const$
u_∞	v_∞	$c_\infty = \sqrt{u_\infty^2 + v_\infty^2}$	
$\frac{E}{2\pi}\frac{x}{x^2+y^2}$	$\frac{E}{2\pi}\frac{y}{x^2+y^2}$	$\frac{E}{2\pi r}$	
$-\frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}$	$\frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}$	$-\frac{\Gamma}{2\pi r}$	
$m\frac{y^2-x^2}{(x^2+y^2)^2}$	$-m\frac{2xy}{(x^2+y^2)^2}$	$\frac{m}{r^2}$	
$u_\infty + \frac{E}{2\pi}\frac{x}{x^2+y^2}$	$\frac{E}{2\pi}\frac{y}{x^2+y^2}$		
$2u_\infty \sin^2\varphi$	on the cylinder: $-2u_\infty \sin\varphi\cos\varphi$	$2u_\infty \lvert\sin\varphi\rvert$	
$2u_\infty \sin^2\varphi$ $-\frac{\Gamma}{2\pi R}\sin\varphi$	on the cylinder: $+2u_\infty \sin\varphi\cos\varphi$ $+\frac{\Gamma}{2\pi R}\cos\varphi$	$2u_\infty \lvert\sin\varphi\rvert$ $-\frac{\Gamma}{2\pi R}$	
$u_\infty - \frac{\Gamma}{2\pi}\frac{y}{x^2+y^2}$	$+\frac{\Gamma}{2\pi}\frac{x}{x^2+y^2}$		

14.1 The twodimensional flow around the sketched body shall be described with the potential theory by superimposing a parallel flow, a source, and a s ink with a distance a from the center, respectively.



Determine

a) the positions x_s of the stagnation points.
b) What is the equation to determine the contour of the body?

14.2 Proof, if the stream function and the potential exist for the following velocity fields!

a) $u = x^2 y, \qquad v = y^2 x$

b) $u = x, \qquad v = y$

c) $u = y, \qquad v = -x$

d) $u = y, \qquad v = x$

Compute the stream, function and the potential

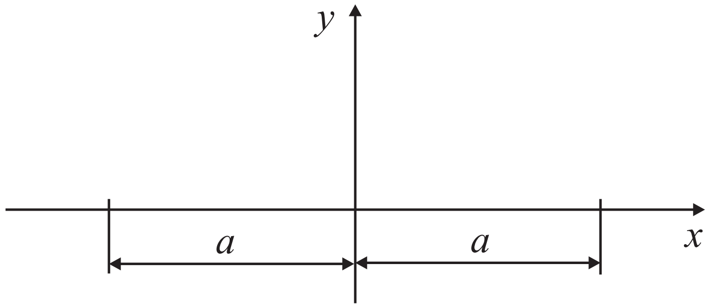
14.3 The stream function is given $\psi = \psi_1 + \psi_2$
with $\psi_1 = -\dfrac{\Gamma}{2\pi} \ln \sqrt{(x-a)^2 + y^2}$
$\psi_2 = -\dfrac{2\Gamma}{2\pi} \ln \sqrt{(x+a)^2 + y^2}$

Given: $a, \ \Gamma > 0$

Determine

a) the coordinates of the stagnation point,
b) the pressure coefficient on the x-axis $c_p(x, y = 0)$ such, that $c_p = 0$ in the origin of the

coordinate system.



14.4 The complex stream function is given

$$F(z) = \frac{2}{3}\frac{u_\infty}{\sqrt{L}}\, z^{\frac{3}{2}} + \frac{E}{2\pi}\ln(z)$$

Given: $L,\ u_\infty$

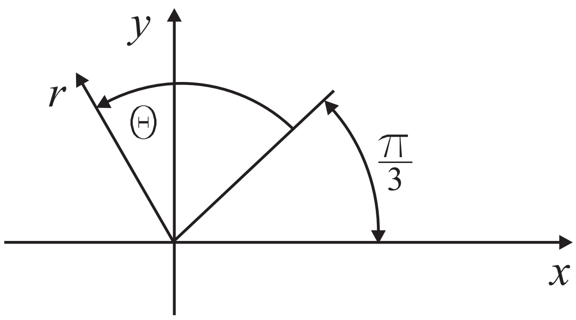
Determine

a) the potential $\phi(r, \theta)$ and the streamfunction $\psi(r, \theta)$.
b) the components of the velocity v_r, v_θ.
c) the constant E such that a stagnation point is at $(x = -L, y = 0)$,
d) the equation that describes the contour $r_k(\theta)$.

Hints:

$$z = x' + iy' = re^{i\theta}$$

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$



14.5 A planar flow is described by the stream function $\psi = (\frac{U}{L})xy$. The pressure in $x_{ref} = 0, y_{ref} = 1\ m$ is $p_{ref} = 10^5\ N/m^2$.

$$U = 2\ m/s \qquad L = 1\ m \qquad \rho = 10^3\ kg/m^3$$

a) Proof, if the flow has a potential!

Determine

b) the stagnation points, the pressure coefficient, and the lines of constant total velocity

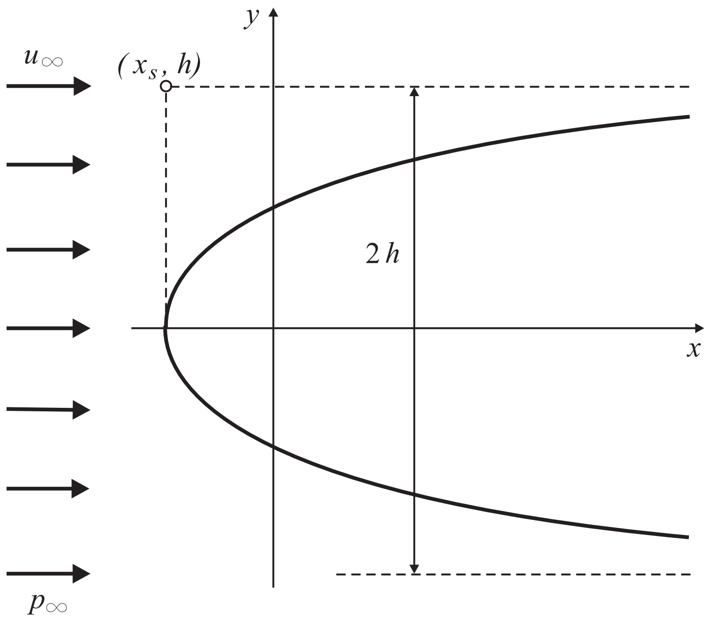
c) the velocity and the pressure at $x_1 = 2m, y_1 = 2m$,

d) the coordinates of a particle at $t = 0.5s$, if it passes at $t = -0$ the point x_1, y_1,

e) the pressure difference between these two points.

f) Sketch the stream lines.

14.6 A plane half-body with the width $2h$ is flown against with the velocity u_∞.



Gegeben: $u_\infty,\ p_\infty,\ h$

Determine:

a) the stagnation point and the velocity at $x = x_s,\ y = h$,

b) the contour of the half-body,

c) the pressure distribution on the contour,

d) the isobars,

e) the curve on that the pressure is $\dfrac{\rho}{4}u_\infty^2$ larger than the pressure p_∞ at infinity,

f) the lines of constant velocity,

g) the area in which the velocity v is larger than $\dfrac{u_\infty}{2}$,

h) the curve on which the streamlines have an inclination of 45^o,

i) the maximum deceleration for a particle on the x-axis between $x = -\infty$ and the stagnation point!

14.7 The following stream function is given

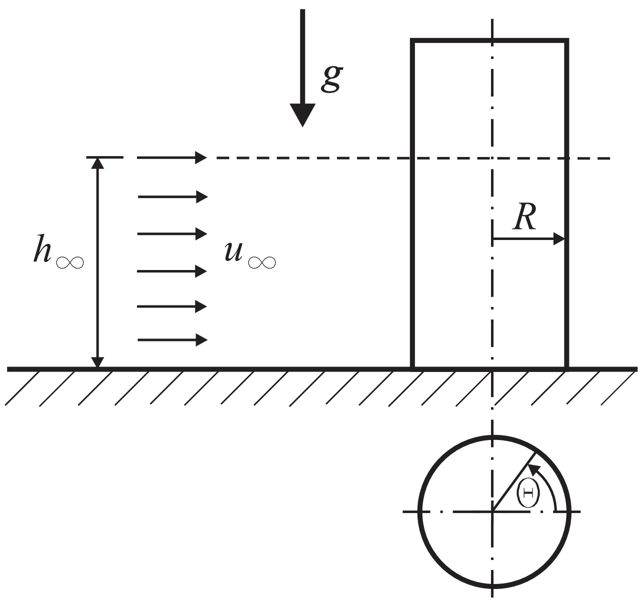
$$\psi = u_\infty y(1 - \frac{R^2}{x^2 + y^2}) \quad .$$

a) Sketch the stream lines for $x^2 + y^2 \geq R^2$.

Determine

b) the pressure distribution on the contour $\psi = 0$,
c) the time, a particle needs to come from point $x = -3R,\ y = 0$ to point $x = -2R,\ y = 0$.

14.8 A bridge pylon with a circular cross-section is flown against with the velocity u_∞. Far away from the pylon the water depth is h_∞.



$u_\infty = 1\ m/s \qquad h_\infty = 6\ m \qquad R = 2\ m \qquad \rho = 10^3\ kg/m^3 \qquad g = 10\ m/s^2$

Determine

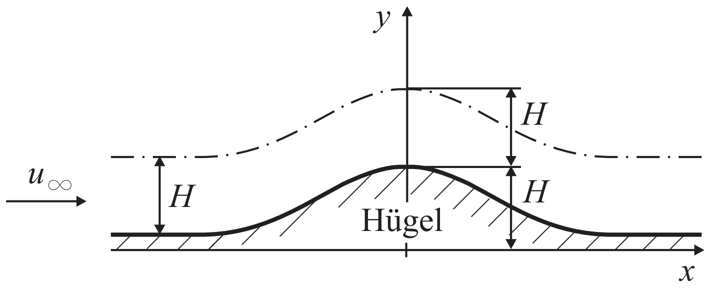
a) the water depth at the pylon wall as a function of θ,
b) the water depth in the stagnation points,
c) the smallest water depth over the ground.

Hint: Assume a twodimensional flow.

14.9 A wind energy facility shall be positioned on a hill. Assume, that the wind flow can be described with a potential flow to determine the energy. The flow over the hill is given by the stream function

$$F(z) = u_\infty z + 0.8\frac{u_\infty H^2}{z},$$

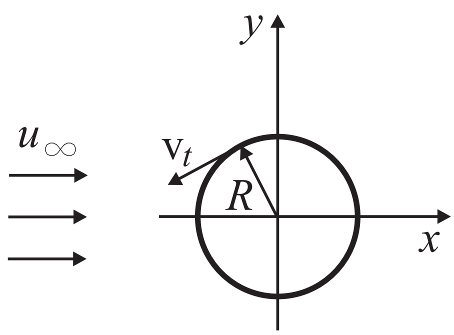
assuming that the countour of the hill is a streamline itself.



a) Compute the contour of the hill $f_k(x, y) = 0$ in cartesian coordinates,
b) Compute the power that can be extracted from the flow between the contour and the contour line in a distance H
I. far away from the hill ($x \to -\infty$) and
II. on top of the hill ($x = 0$).
c) Sketch the potential theoretical velocity profile on the hill ($x = 0$) as well as a realistic profile for viscous flow.

Given: $u_\infty,\ \rho,\ H$

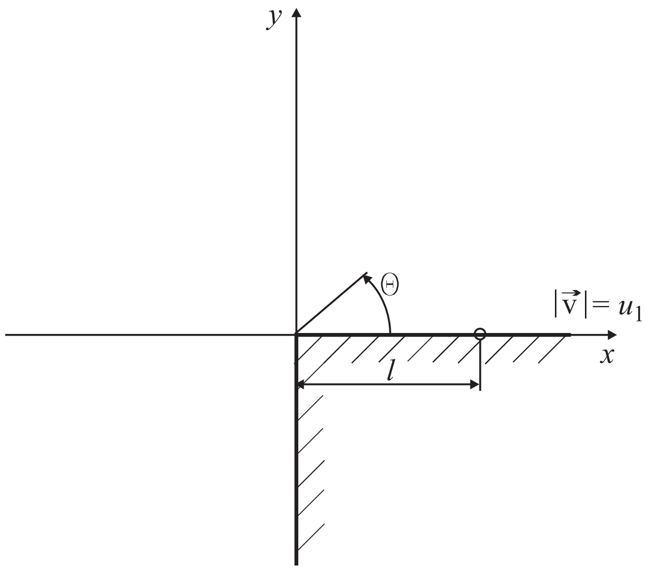
14.10 A rotating cylinder of length L is flown against with the velocityu_∞ normal to its axis. At the surface, the rotational part of the velocity is v_t.



a) Determine the circulation.
b) Discuss the flow field for $v_t = u_\infty$.
c) Compute the force on the cylinder.

14.11 The twodimensional flow at a 90^o-corner can be described with the complex stream function

$$F(z) = Az^{\frac{2}{3}}, A = konst.$$



Given: $|\vec{v}|_{(r=l,\ \theta=0)} = u_1$

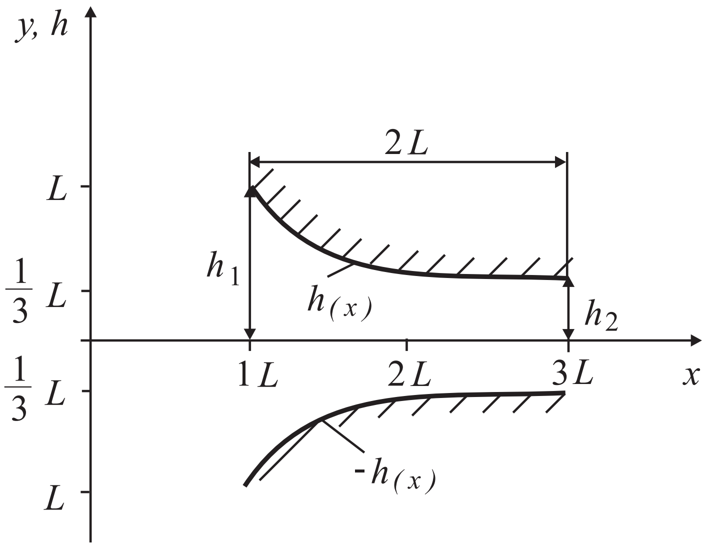
Determine:

a) the constant A
b) the pressure distribution c_p along the wall. Sketch it!
c) Sketch carefully the lines of constant pressure.
d) What is the equation $r = r(\theta)$ for the streamlines?
e) Sketch the flow field.

14.12 The stream function $\psi(x,y)$ for the flow of an incompressible fluid through the sketched plane nozzle is given.

$$\psi(x,y) = \frac{y}{h(x)} u_\infty L$$

Given: $u_\infty,\ L,\ B,\ h_1 = L,\ h_2 = \frac{1}{3}L$



a) Determine the upper and the lower contour $h(x)$ such that the flow can be described with the potential theory,
b) Compute the velocity distribution $u(x,y)$ and $v(x,y)$.
c) Determine the volume flux for a nozzle with the width B.

15 Laminar boundary layers

15.1 A flat plate is flown against parallel to the surface with water. Determine for a laminar boundary layer the momentum thickness as an integral of the wall shear stress

$$-\int_0^x \frac{\tau(x', y=0)}{\rho u_\infty^2} dx'!$$

15.2 A flat plate (length L, width B) is flown against parallel to the surface with air.

$$u_\infty = 10\ m/s \qquad L = 0.5\ m \qquad B = 1\ m \qquad \rho = 1.25\ kg/m^3 \qquad \nu = 1.5 \cdot 10^{-5}\ m^2/s$$

a) Sketch the velocity profiles $u(y)$ for different x!
b) Specify the bouindary conditions for the boundary equations!
c) Sketch the distribution of the shear stress $\tau(y)$ at a position $\tau(y)$!
d) Determine the boundary layer thickness at the end of the plate and the drag force!

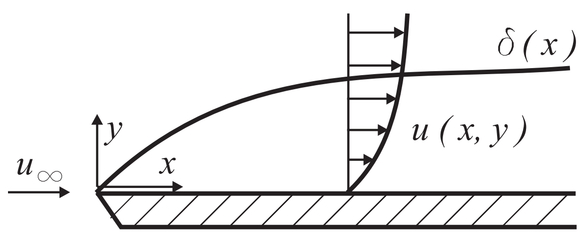
15.3 From exercise 15.2 the drag of a flate plate, wetted on both sides, of length x and width B can be determined with

$$W = \int_0^x \tau_w(x) B dx = \rho \int_0^{\delta(x)} u(u_\infty - u) B dy.$$

Using this equation and with the approximation of the velocity profile

$$\frac{u(x,y)}{u_\infty} = a_0 + a_1 \frac{y}{\delta} + a_2 \left(\frac{y}{\delta}\right)^2$$

the boundary layer thickness $\delta(x)$ is to be determined and compared with the Blasius solution $\delta(x) = 5.2\sqrt{\frac{\nu x}{u_\infty}}$.



Given: ν, u_∞

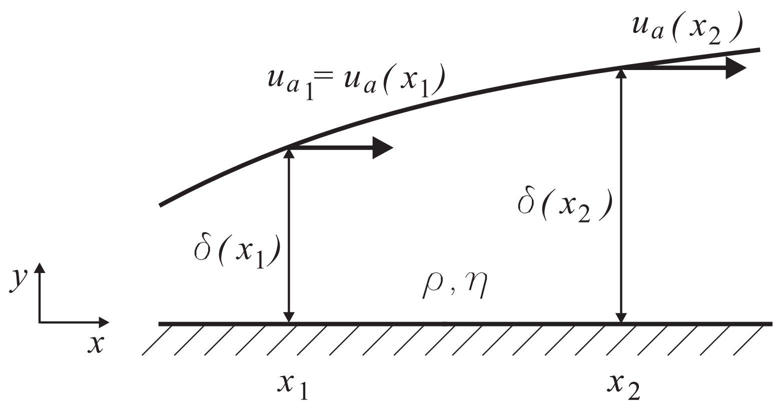
15.4 The velocity profile of a laminar incompressible boundar layer with constant viscosity η can be described with a polynomial:

$$\frac{u(x,y)}{u_a(x)} = a_0 + a_1(x)\left(\frac{y}{\delta}\right) + a_2(x)\left(\frac{y}{\delta}\right)^2 + a_3(x)\left(\frac{y}{\delta}\right)^3$$

The outer velocity $u_a(x)$ is given by the following approach:

$$u_a(x) = u_{a1} - C \cdot (x - x_1)^2.$$

u_{a1} is the outer velocity at x_1 and C is a positive constant. The boundary layer thickness at x_2 is $\delta(x_2)$.



Given: $\rho,\ \eta,\ x_1,\ u_{a1},\ \delta(x_2),\ C,\quad$ mit: $C > 0$

Determine:

a) the pressure gradient$\partial p/\partial x$ in the flow as a function of x.
b) the coefficient a_0 and the coefficients $a_1(x),\ a_2(x),\ a_3(x)$.

15.5 In the boundary layer of a flat plate the velocity profiles and the pressure distribution are measured. Thre pressure distribution on the surface is described with $\dfrac{p(x)}{p_0} = 1 - k\left(\dfrac{x}{l}\right)^2$, with $k = const < 1$ and the velocity profiles are presented with

$$\frac{u(x,y)}{u_a(x)} = \left(\frac{y}{\delta_0}\right)^{\frac{1}{2}}$$

with a constant boundary layer thickness δ_0.

Determine the wall shear stress τ_w using the Kármán integral equation

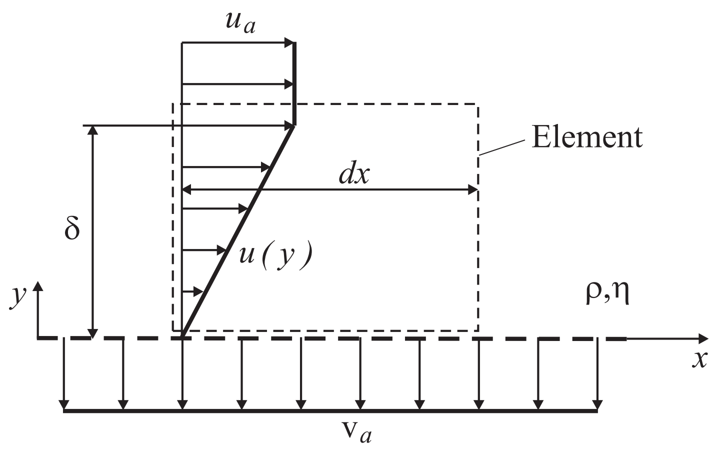
$$\frac{d\delta_2}{dx} + \frac{1}{u_a}\frac{du_a}{dx}(2\delta_2 + \delta_1) + \frac{\tau(y=0)}{\rho u_a^2} = 0$$

Given: $p_0,\ k,\ \delta_0,\ l$

Boundary layer equation (x-momentum):

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\eta}{\rho}\frac{\partial^2 u}{\partial y^2}$$

15.6 The fluid along a flat plate is sucked off through the porous wall with the velocity v_a.



Determine the suction velocity v_a by using a balance for the sketched element under the condition that the boundary layer thickness δ is independent of the length x. Assume that the tangential component of the velocity increases linearly.

Given : $\rho,\ \eta,\ \delta$

15.7 The velocity profile in the laminar boundary layer of a flat plate (length L) is described by a polynomial of fourth order

$$\frac{u}{u_a} = a_0 + a_1 \left(\frac{y}{\delta}\right) + a_2 \left(\frac{y}{\delta}\right)^2 + a_3 \left(\frac{y}{\delta}\right)^3 + a_4 \left(\frac{y}{\delta}\right)^4 .$$

a) Determine the coefficient of the polynomial!
b) Proof the following relationships:

$$\frac{\delta_1}{\delta} = 3/10$$

$$\frac{\delta_2}{\delta} = 37/315$$

$$\frac{\delta}{x} = 5.84/\sqrt{Re_x}$$

$$c_w = 1.371/\sqrt{Re_L}$$

15.8 In the stagnation point of a flat plate that is flown against normally to the outer flow $u_a(x)$ is accelerated in such a way that a constant boundary layer thickness δ_0 is generated. The velocity profile is assumed to be linear as a first approximation.

$$\frac{u(x,y)}{u_a(x)} = a_0 + a_1 \frac{y}{\delta_0}$$

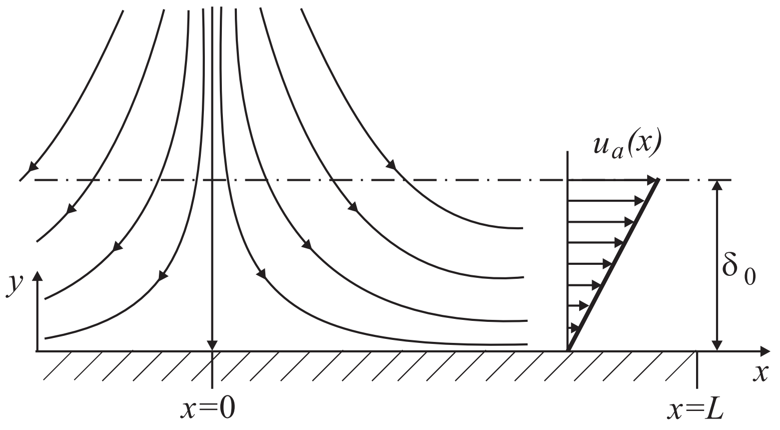
Determine:

a) the constants $a_0,\ a_1$
b) the distribution of the outer velocity $u_a(x)$ using the von Kármán integral equation.
c) the tangential force that is applied between $x = 0$ and $x = L$ on the plate with the width B.

Given: $\delta_0,\ L,\ \eta,\ \rho,\ B$

von Kármán integral equation

$$\frac{d\delta_2}{dx} + \frac{1}{u_a}\frac{du_a}{dx}(2\delta_2 + \delta_1) = \frac{\tau_w}{\rho u_a^2}$$



15.9 The velocity profile of a laminar boundary layer in a flow along a flat plate (length l) is approximated with

A) a polynomial of third order

$$\frac{u}{u_a} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

and

B) a sinusoidal approach

$$\frac{u}{u_a} = \sin\left(\frac{\pi}{2}\,\frac{y}{\delta}\right).$$

a) Determine $\delta_1,\ \delta_2,\ \delta$ and c_w !
b) Compute for

$$u_a = 1\ m/s \qquad L = 0.5\ m \qquad B = 1\ m \qquad \rho = 10^3\ kg/m^3 \qquad \nu = 10^{-6}\ m^2/s$$

the boundary layer thickness at the end of the plate and the drag force.

16 Turbulent boundary laters

16.1 A flat plate is flown against parallel to the surface with air.

$u_\infty = 45\ m/s \qquad \nu = 1,5 \cdot 10^{-5}\ m^2/s$

Determine

a) the transition point for $Re_{crit} = 5 \cdot 10^5$,
b) the velocity in the point $x = 0,1\ m,\ y = 2 \cdot 10^{-4}\ m$ using the Blasius solution. What is the coordinate y with the same velocity at $x = 0,15\ m$?

Sketch

c) the distribution of the boundary layer thickness $\delta(x)$ and a velocity profile for $x < x_{crit}$ and $x > x_{crit}$.
d) the wall shear stress as a function of x for $dp/dx < 0$, $dp/dx = 0$ and $dp/dx > 0$.

16.2 Representing the Reynolds averaged Navier-Stokes equations the averaged form of the x-momentum equation of a twodimensional, unsteady, incompressible flow is to be determined.

$$\rho\left(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y}\right) = f_x - \frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

a) Proof

$$\rho\left(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y}\right) = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right).$$

b) Determine the averaged form odf the x-momentum equation in the time intervall [0, T].

16.3 Two planar rectanguular plates have the same lengths L_1 and L_2. Plate no. 1 is flown against parallel to L_1 and die plate no. 2 is flown against parallel to L_2 with the velocity u_∞.

$L_1 = 1\ m \qquad L_2 = 0,5\ m \qquad \nu = 10^{-6}\ m^2/s$

a) Determine the ratio of viscous forces for $u_\infty = 0,4 m/s; 0,8 m/s; 1,6 m/s$.

b) What is the velocity for plate 2, assuming that the velocity for plate 1 is $u_\infty = 0,196\ m/s$ and the drag coefficients are the same?

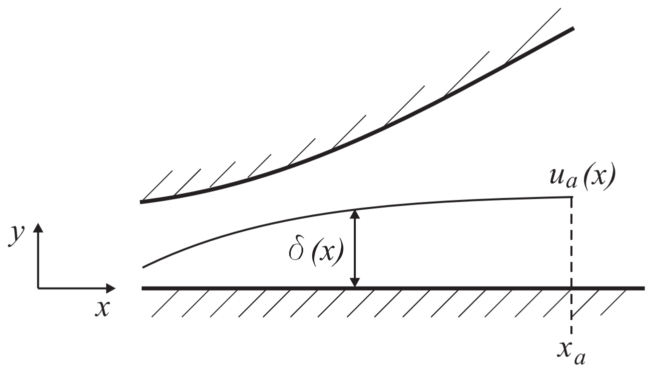
Hint: $c_w = \dfrac{0,074}{Re_L^{1/5}} - \dfrac{1700}{Re_L}$ for $5 \cdot 10^5 < Re_L < 10^7$

17 Boundary layer separation

17.1 The lower border of a divergent channel is formed by a flat plate. At $x = x_a$ the flow separates. The velocity profile is described with a polynomial of third order:

$$\frac{u(x,y)}{u_a(x)} = a_0(x) + a_1(x)\frac{y}{\delta(x)} + xa_2\left(\frac{y}{\delta(x)}\right)^2 + a_3(x)\left(\frac{y}{\delta(x)}\right)^3$$

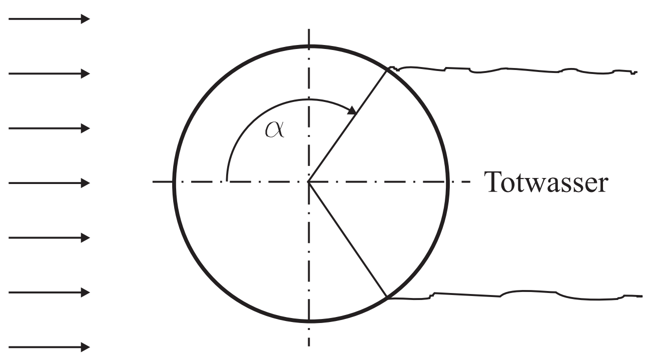
Given: x_a



Determine the velocity profile $\dfrac{u(x_a, y/\delta(x_a))}{u_a(x_a)}$ at the separation point.

Sketch 3 velocity profiles for $x < x_a$; $x = x_a$; $x > x_a$.

17.2 Assume that the flow on a circular cylinder separatesat $\alpha = 120^o$, the pressure up to the separation is determined from the potential flow, and the pressure in the separation region is constant. bestimmt und der Druck im Totwasser konstant ist.



Determine thedrag coefficient of the cylinder by neglecting the viscous drag.

17.3 A cylinder with radius R is flown against normal to its axis with the velocity u_∞. The velocity of the frictionless outer flow at the cylinder wall is

$$u_a(x) = 2u_\infty \sin\frac{x}{R}$$

Assume

$$\delta(x) = 5\sqrt{\frac{\eta x}{\rho u_a(x)}}$$

for the boundary layer thickness.

a) Determine the local velocity profiles in the boundary layer for the (x, y) coordinate system with the following approach

$$\frac{u}{u_a(x)} = a_0(x) + a_1(x)\left(\frac{y}{\delta}\right) + a_2(x)\left(\frac{y}{\delta}\right)^2 + a_3(x)\left(\frac{y}{\delta}\right)^3$$

b) Determine the angle where the separation occurs ($\tau_w = 0$)!

Hints:
- boundary layer equation

$$\rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \eta\frac{\partial^2 u}{\partial y^2}$$

- Approximation for $\varphi \approx \pi/2$:

$$\sin\varphi \approx 1$$
$$\cos\varphi \approx \pi/2 - \varphi$$

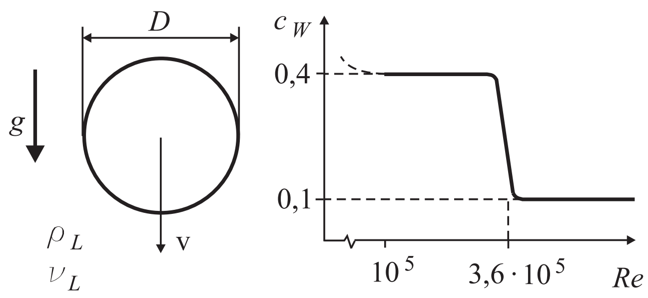
17.4 A sphere and a cylinder made from the same material are falling at constant velocity through air. The axis of the cylinder is normal to the falling direction. For $0 < Re \leq 0,5$ the drag coefficient for a sphere $c_w = 24/Re$ and the cylinnder $c_w = 8\pi/[Re(2 - lnRe)]$ are given.

$\rho = 800\ kg/m^3$ $\quad\rho_L = 1,25\ kg/m^3$ $\quad\nu_L = 15 \cdot 10^{-6}\ m^2/s$ $\quad g = 10\ m/s^2$

Determine

a) the maximum diameters for these laws being valid,
b) the corresponding sink velocity.

17.5 A sphere is falling steady with the constant velocity v_1 through non-moving air. A downward blast increases the velocity up to v_2.



$D = 0,35\ m \qquad G = 4,06\ N \qquad v_1 = 13\ m/s \qquad v_2 = 18\ m/s$
$\rho_L = 1,25\ kg/m^3 \qquad \nu_L = 15 \cdot 10^{-6}\ m^2/s$

a) What is the drag coefficient before the blast?
b) What is the final steady velocity of the sphere after the decay of the blast?

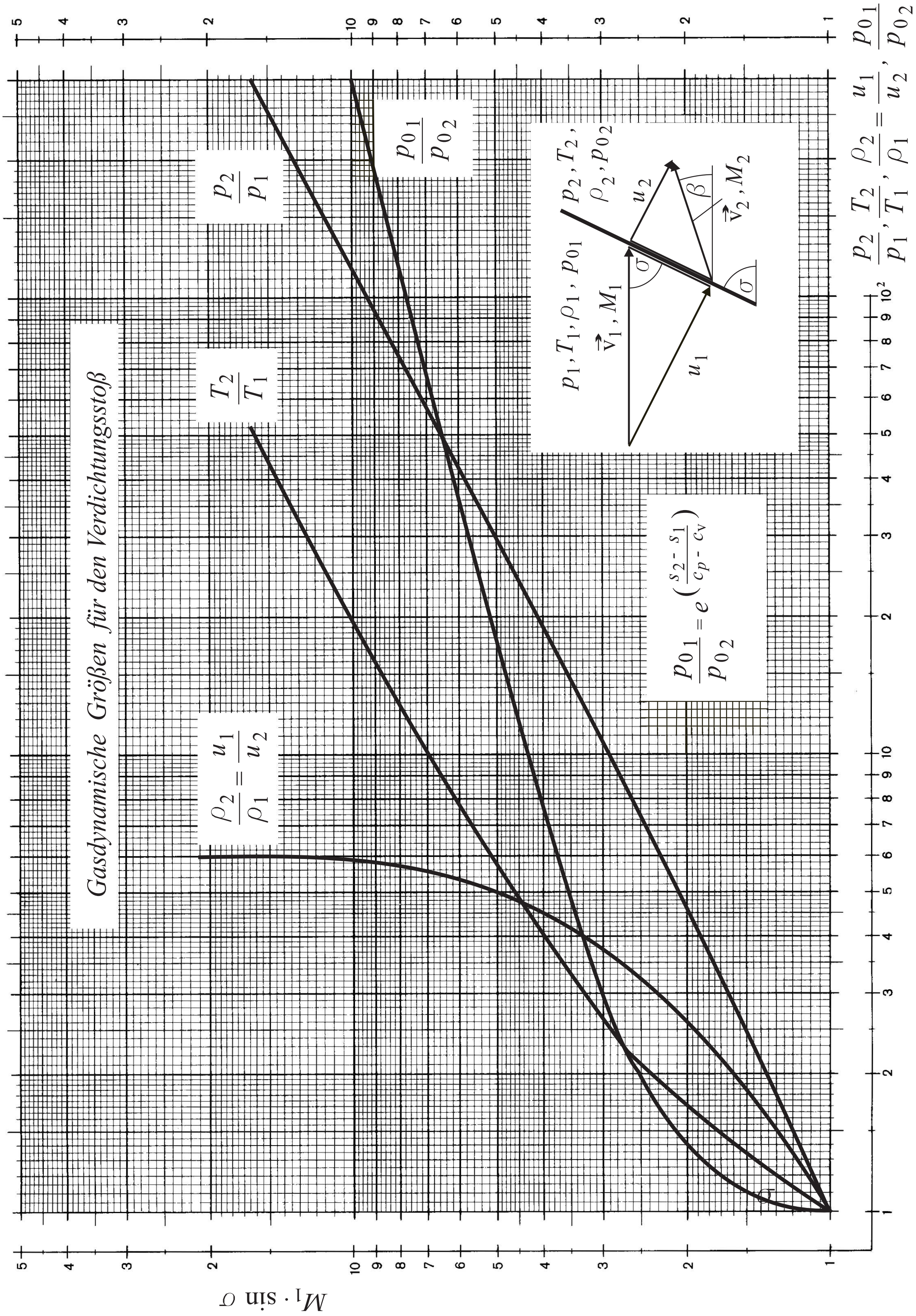
17.6 A sphere with the diameter D and the density ρ_K is shot vertically with the initial velocity v_0 upwards through non-moving air.

$D = 0,1\ m \qquad v_0 = 30\ m/s \qquad g = 10\ m/s^2 \qquad \rho_L = 1,25\ kg/m^3$
$\rho_H = 750\ kg/m^3 \qquad \rho_M = 7,5 \cdot 10^3\ kg/m^3$

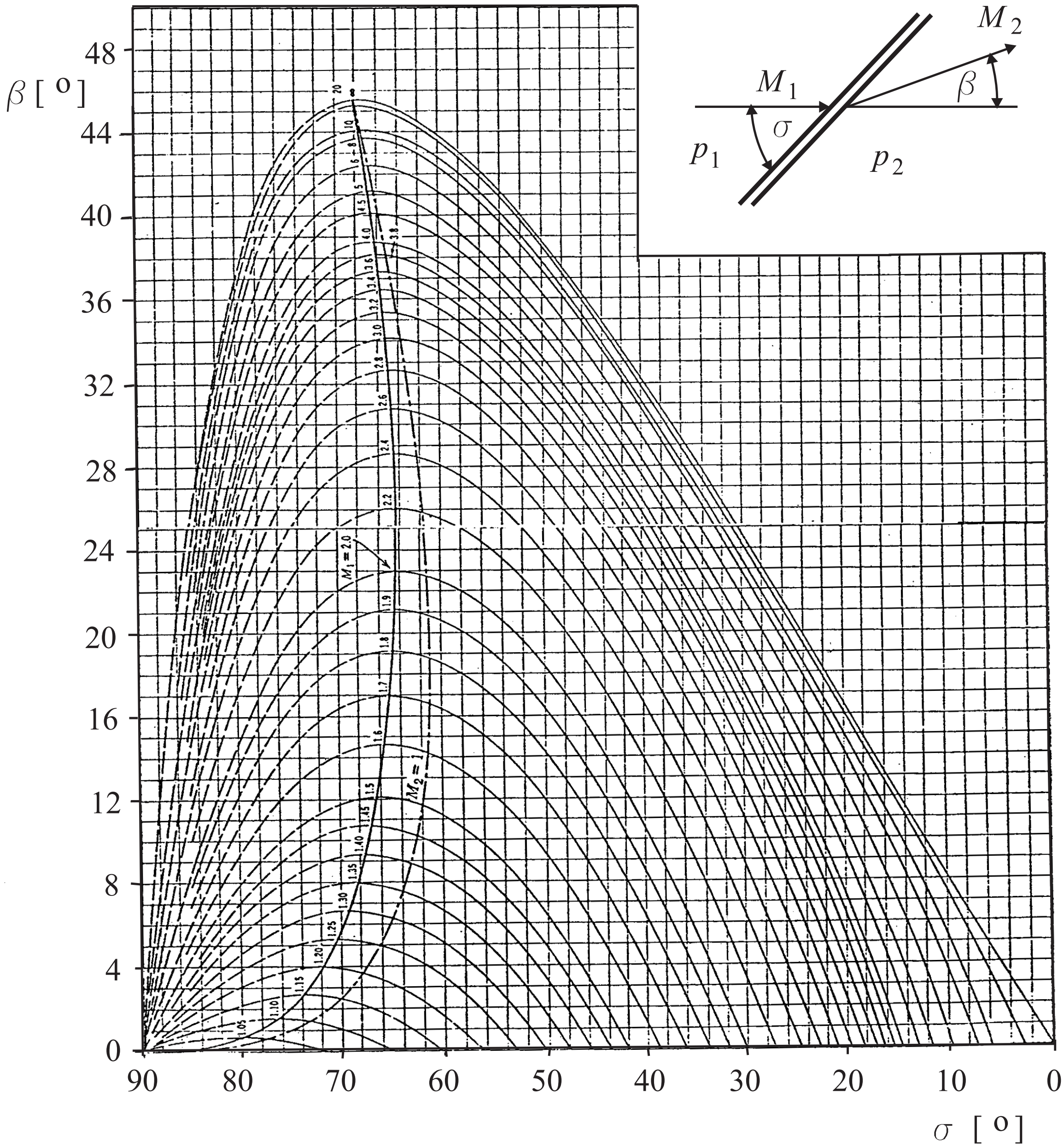
Determine generally with the assumption of a constant drag coefficient

a) the ceiling,
b) the ascend time,
c) the velocity when the sphere hits the ground,
d) the descend time.
e) Determine these values for a wooden sphere with the density ρ_H and for a metallic sphere with the density ρ_M, if $c_w = 0,4$ and $c_w = 0$.

19 compressible flow



Winkelbeziehung über den schiefen Stoß



19.1 A plane flies over an observer horizontally

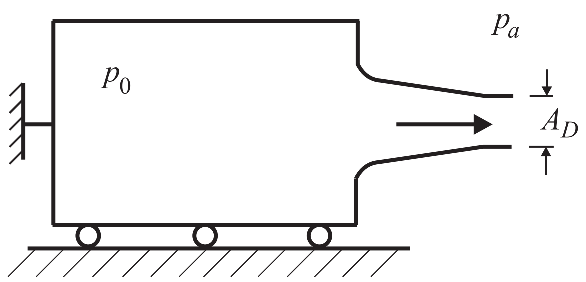
$H = 577\ m \quad v = 680\ m/s \quad T = 287\ K \quad R = 287\ Nm/kgK \quad \gamma = 1.4$

a) What is the Machnumber of the plane?
b) What is the distance covered by the plane before it can be heard by the observer?
c) When was the sound created?

19.2 One jet passes another at a distance b.
$v_A = 510\ m/s \quad v_B = 680\ m/s \quad b = 170\ m \quad T = 287\ K \quad R = 287\ Nm/(kgK) \quad \gamma = 1.4$

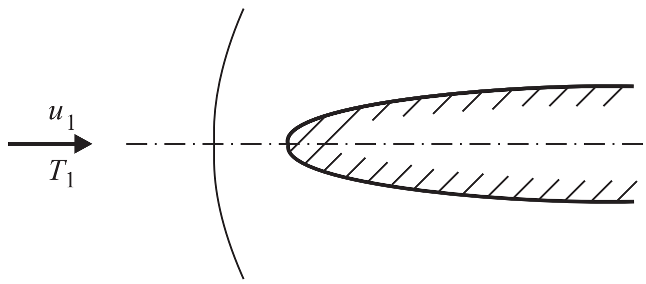
How mauch later the pilot of the passed jet can hear the sound of the faster plane?

19.3 Air is flowing isentropically ($\gamma = 1.4$) from a large and frictionless supported container through a well rounded nozzle into the open air.



a) Determine the dimensionless thrust $F_s/p_0 A_D$ for the pressure ratios $p_a/p_0 = 1;\ 0.6;\ 0.2;\ 0$!
b) What are these values for an incompressible fluid?

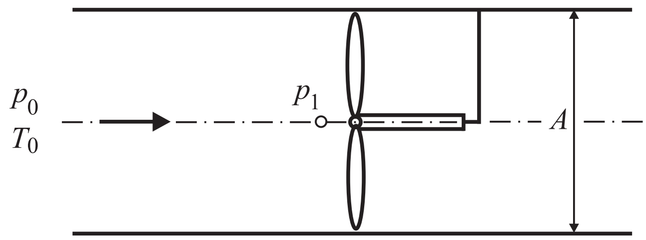
19.4 An airplane flies at supersonic speed. A shock is generated, that is normal in front of the airplane nose.



$u_1 = 680\ m/s \quad T_1 = 287\ K \quad R = 287\ Nm/(kgK) \quad \gamma = 1.4$

Determine the temperature change across the shock.

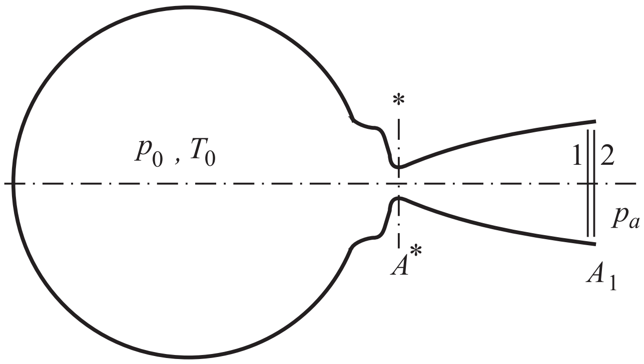
19.5 A turbine engine sucks air from the atmosphere. Immediately before the compressor the pressure is p_1.



$p_0 = 10^5\ N/m^2 \quad T_0 = 287\ K \quad p_1 = 0.74 \cdot 10^5\ N/m^2 \quad A = 9 \cdot 10^3\ mm^2$
$R = 287\ Nm/(kgK) \quad \gamma = 1.4$

Compute the mass flux flowing through the engine!

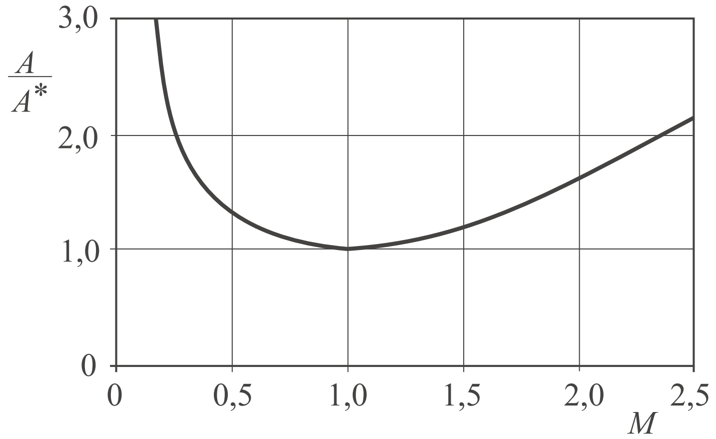
19.6 Air is flowing from a large reservoir through a well rounded nozzle into the open air. At the exit cross section A_1a normal shock develops.



$$A_1 = 0.018\ m^2 \quad T_0 = 287\ K \quad A^* = 0.01\ m^2 \quad p_a = 10^5\ \frac{N}{m^2} \quad R = 287\ \frac{J}{kg\ K}$$

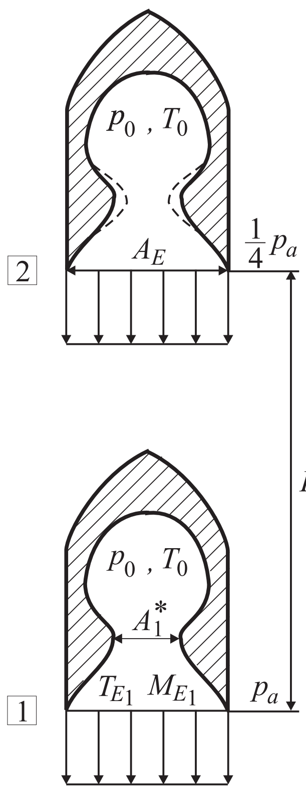
a) Compute the mass flux
b) Sketch the distribution of the static pressure along the nozzle axis.

Hints: $\dfrac{p_2}{p_1} = 1 + \dfrac{2\gamma}{\gamma+1}(M_1^2 - 1)$



19.7 A rocket is equipped with a Laval nozzle. At liftoff (1) the jet has the velocity $M_{E_1} = 2$ with a jet temperature T_{E_1}. The surrounding pressure is p_a. At the altitude H (2) the surrounding pressure is only $\frac{1}{4}p_a$. Under the condition that p_0, T_0 and A_E are constant during the flight, the smallest cross section A^* has to be adapted in such a way that no shocks develop in the jet. Determine the ratio between the smallest cross sections A_2^*/A_1^*.

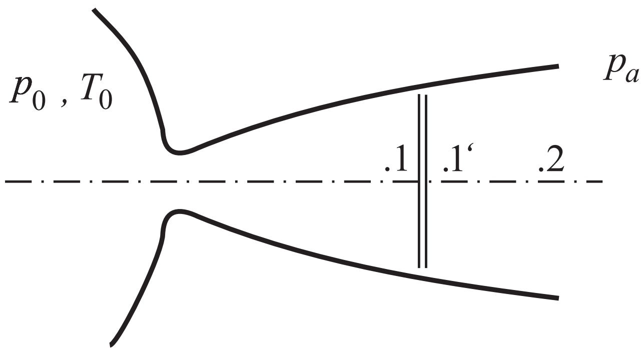
$\gamma = 1.4 \quad M_{E_1} = 2$



$$c_p = \frac{\gamma R}{\gamma - 1}; \qquad \frac{T}{T_0} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}};$$

$$\frac{A^*}{A} = \frac{M}{\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{1}{2}\frac{\gamma+1}{\gamma-1}}}$$

19.8 Air flows through a Laval nozzle. At positio '1' a normal shock is located.



$\dot{m} = 200\ kg/s \quad T_0 = 300\ K \quad p_0 = 2.2 \cdot 10^5\ N/m^2 \quad p_a = 10^5\ N/m^2$

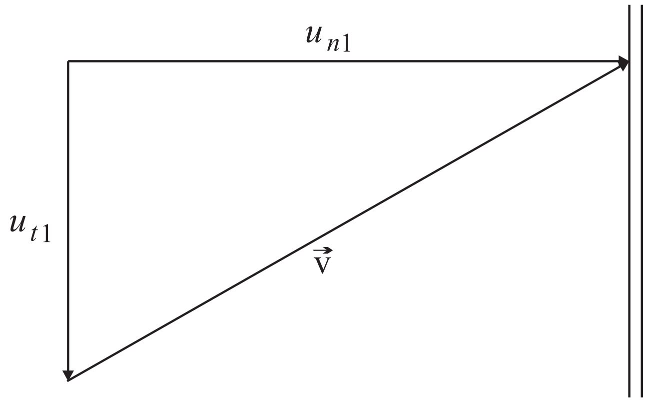
$\frac{p_{01}}{p_{01'}} = 1.39$ (ratio of total pressure across the shock) $\quad R = 287\ Nm/(kg\ K) \quad \gamma = 1.4$

Determine at the exit of the nozzle ('2')

a) the Mach number M_2,
b) the velocity u_2,
c) the stagnation density ρ_{02},
d) the exit cross section A_2.

Hint: $\quad \frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$

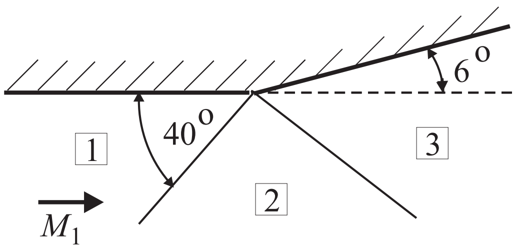
19.9 The velocity V_1 in front of a straight shock is given by the normal component $u_{n1} = 400 m/s$ and the tangential component $u_{t1} = 300 m/s$. The static temperature changes to $T_2 = 1.2 \cdot T_1$



Determine with $\gamma = 1.4$ and $R = 287\ Nm/(kgK)$:

a) the Mach number M_1 and the statictTemperature T_1 in front of the shock,
b) the Mach number M_2 and the deflection angle β behind the shock,
c) for $M_1 = const.$ the velocity components u_{n1} and u_{t1} in such a way that $M_2 = 1$. Compute the deflection angle β.

19.10 An oblique shock in a supersonic flow at $M_1 = 2.2$ hits a flat wall at an angle of 40^o. At the impinging point the wall is kinked outwars with 6^o.



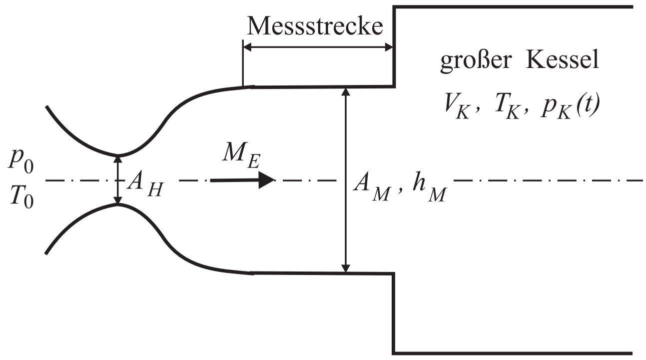
$\gamma = 1.4 \quad M_1 = 2.2 \quad R = 287\ Nm/(kgK)$

Determine: $M_2, \quad M_3, \quad \frac{p_2}{p_1}, \quad \frac{p_3}{p_2}, \quad \frac{T_2}{T_1}, \quad \frac{T_3}{T_1}$

19.11 Atmospheric air ($T = 280\ K,\ p = 1\ bar$) is sucked through a supersonic wind tunnel into an evacuated boiler (see sketch). The laval nozzle is designed for M_E in the test section.

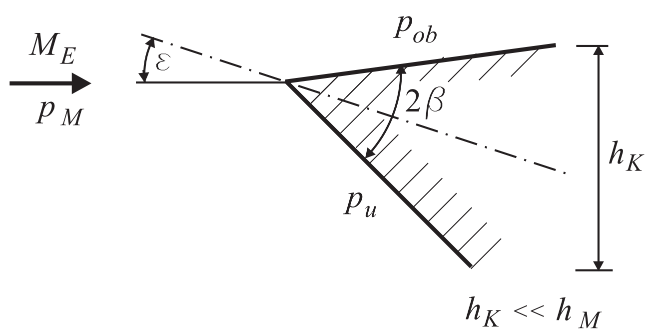
$p = 1\ bar \quad T = 280\ K \quad \gamma = 1.4 \quad R = 287\ Nm/(kgK)$
$M_E = 2.3 \quad A_H = 0.1\ m^2 \quad V_K = 1000\ m^3$

During the test the reservoir temperature in the boiler is $T_K = 280\ K = const.$. At the time



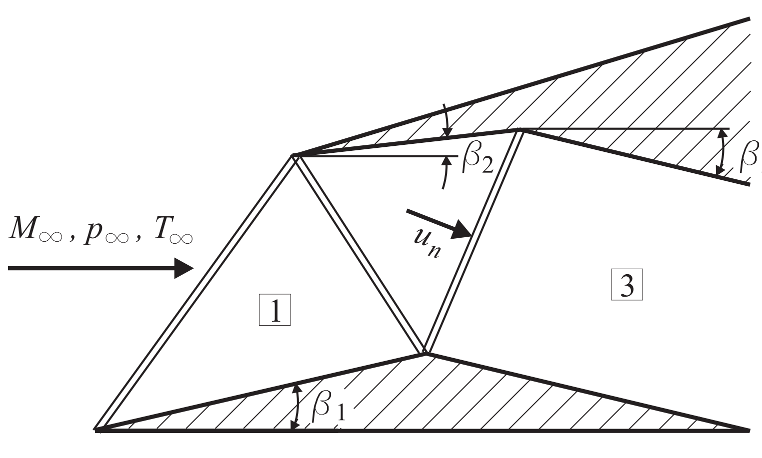
$t = 0$ the boiler pressure is $p_k(t = 0) = 0.08\ bar$.

a) Determine the available testing time Δt. (undisturbed flow in the test section, $M_E = 2.3$.)
b) Determine for the boiler pressure $p_K = 0.16\ bar$ the following quantities:
Shock angle σ, deflection angle β, $p_{02'}$, M_2, T_2, T_{02} and the velocity V_2 behind the shock.
c) A wedge $2 \cdot \beta_K = 40^o$ is mounted in the test section.



What is the maximum angleof attack ϵ without the development of a detached shock? What is the pressure difference $p_u - p_{ob}$ and the Mach numbers M_{ob} and M_u for this case?

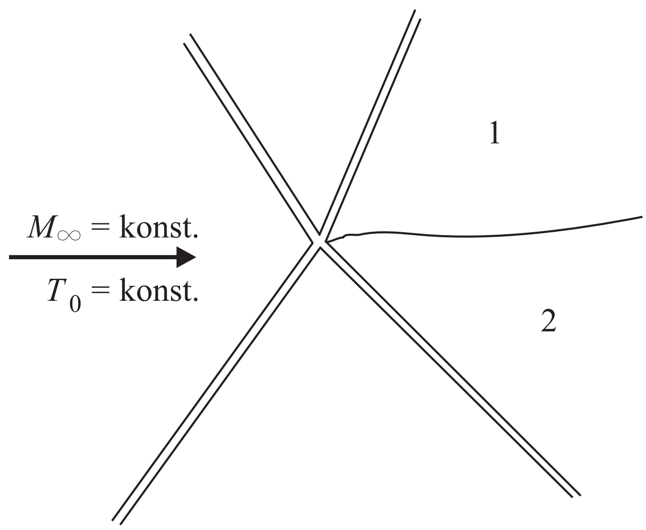
19.12 The sketched diffusor with three shocks decelerates the flow from M_∞ to M_3. The conditions of the incoming flow, and the normal velocity component u_n in front of the last shock are known.



$M_\infty = 3 \quad T_\infty = 270\ K \quad p_\infty = 1\ bar \quad \gamma = 1.4 \quad R = 287\ J/(kg\ K)$
$\beta_1 = 15^o \quad \beta_2 = 10^o \quad u_n = 649\ m/s$

a) Compute the Mach numbers M_1 and M_2 as well as the shock angles σ_1 and σ_2.
b) Determine for point '3' the Mach number M_3, the contour angle β_3, the shock angle σ_3 and the static pressure p_3.

19.13 The interaction of two oblique shocks of different intensities is sketched. The flow behind the shocks is splitted from a discontinuity line. The ratio of the total pressures in the areas '1' and '2' and the Mach number in '1' are known.

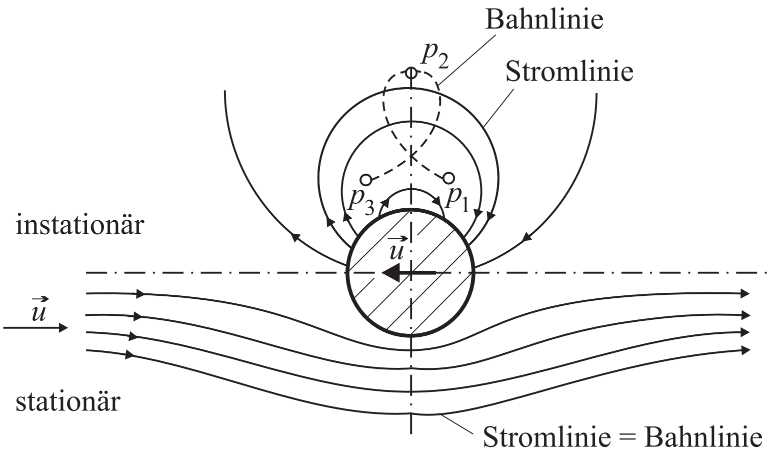


$\gamma = 1.4 \quad R = 287\ Jkg^{-1}K^{-1} \quad p_{01}/p_{02} = 1.2 \quad M_1 = 1.6$

a) Calculate the Mach number M_2 in area '2'.
b) Compute the ratio of the velocities $|\vec{V}_2|/|\vec{V}_1|$ and the density ratio ρ_2/ρ_1.

3 Fluid Kinematics

3.1



Unsteady flow for the resting observer, steady flow for the moving observer.

3.2

a)
$$\frac{dy}{dx} = \frac{v}{u} = -\frac{v_0}{u_0}\tan(\omega t)$$

Integration :
$$y = \left[-\frac{v_0}{u_0}\tan(\omega t)\right] x + c$$

Straight lines with slope 0, -1, -∞

b)
$$x(t) = \int u\ dt + c_1 = \frac{u_0}{\omega}\ \sin(\omega t) + c_1$$

$$y(t) = \int v\ dt + c_2 = \frac{v_0}{\omega}\ \cos(\omega t) + c_2$$

$$\Longrightarrow \left(\frac{\omega}{u_0}\right)^2 (x - c_1)^2 + \left(\frac{\omega}{v_0}\right)^2 (y - c_2)^2 = 1$$

Circles with radius 1 m

c) Circle around the origin

4 Basic equations for fluids 4.1

a)

1) $$\vec{\nabla} \times \vec{\nabla} p = \vec{\nabla} \times \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{pmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial y}\right) \quad , \quad \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) \quad , \quad \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) \right)^T$$

$$= \left(\frac{\partial^2 p}{\partial y\, \partial z} - \frac{\partial^2 p}{\partial z\, \partial y} \quad , \quad \frac{\partial^2 p}{\partial z\, \partial x} - \frac{\partial^2 p}{\partial x\, \partial z} \quad , \quad \frac{\partial^2 p}{\partial x\, \partial y} - \frac{\partial^2 p}{\partial y\, \partial x} \right)^T = 0 \qquad ,$$

If the second partial derivatives of p exist the orser of the differentiation is irrelevant.

2) analogous to 1)
3) analogous to 1)

b) $$\vec{\nabla} \cdot \vec{v} = 0$$

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}\left(\frac{\vec{v}^2}{2}\right) - \vec{v} \times \left(\vec{\nabla} \times \vec{v}\right) \right) = \rho \cdot \vec{g} - \vec{\nabla}\, p - \eta \vec{\nabla} \times \left(\vec{\nabla} \times \vec{v}\right)$$

4.2

a) in general (substantial acceleration): $\frac{dv}{dt} = \frac{\partial v}{\partial t} + v \cdot \frac{\partial v}{\partial x}$ (onedimensional)

From the continuity equation $A \cdot v = const$ follows $v \cdot \frac{\partial v}{\partial x} = 0$

$$\frac{\partial v}{\partial t} = \frac{\partial v_{\text{piston}}}{\partial t} \neq 0$$

$$\Longrightarrow \frac{dv}{dt} = \frac{\partial v_{\text{piston}}}{\partial t} \text{(local acceleration)}$$

b) in general (substantial acceleration): $\frac{dv}{dt} = \frac{\partial v}{\partial t} + v \cdot \frac{\partial v}{\partial x}$

Constant inflow velocity $v = v_0 \quad \Longrightarrow \frac{\partial v}{\partial t} = 0$

$$\text{Continuity}: \quad A(x) \cdot v(x) = konst$$

$$\Longrightarrow \frac{\partial A(x)}{\partial x} \cdot v(x) \;+\; \frac{\partial v(x)}{\partial x} \cdot A(x) \;=\; 0$$

$$\Longrightarrow \frac{\partial v(x)}{\partial x} \;=\; -\; \frac{\partial A(x)}{\partial x} \cdot \frac{v(x)}{A(x)} \;\neq\; 0$$

$$\Longrightarrow \frac{dv}{dt} \;=\; v(x) \cdot \frac{\partial v(x)}{\partial x} \;=\; -\; \frac{\partial A(x)}{\partial x} \cdot \frac{v(x)}{A(x)} \;\cdot v(x)$$

(only convective acceleration)

4.3

a)

$$\frac{\partial}{\partial \mathrm{x}} = \frac{\partial \mathrm{r}}{\partial \mathrm{x}} \cdot \frac{\partial}{\partial \mathrm{r}} + \frac{\partial \phi}{\partial \mathrm{x}} \frac{\partial}{\partial \phi} + \frac{\partial \mathrm{z}}{\partial \mathrm{x}} \cdot \frac{\partial}{\partial \mathrm{z}}$$

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{1}{2}\left(x^2 \;+\; y^2\right)^{-\frac{1}{2}} \cdot 2x \;=\; \frac{x}{\sqrt{x^2 \;+\; y^2}} \;=\; \frac{x}{r} \;=\; \cos\phi \\
\frac{\partial \phi}{\partial x} &= -\; \frac{y}{1 \;+\; \left(\dfrac{y}{x}\right)^2} \cdot x^{-2} \;=\; -\; \frac{y}{x^2 \;+\; y^2} \;=\; -\; \frac{y}{r^2} \;=\; -\; \frac{\sin\phi}{r} \\
\frac{\partial z}{\partial x} &= 0
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \frac{\partial}{\partial x} &= (\cos\phi) \cdot \frac{\partial}{\partial r} \;+\; \left(\frac{-\;\sin\phi}{r}\right) \frac{\partial}{\partial \phi} \\
\text{analogous} \quad \frac{\partial}{\partial y} &= (\sin\phi) \cdot \frac{\partial}{\partial r} \;+\; \left(\frac{\cos\phi}{r}\right) \frac{\partial}{\partial \phi} \\
\frac{\partial}{\partial z} &= \frac{\partial}{\partial z}
\end{aligned}$$

$$\begin{aligned}
v_x &= \frac{dx}{dt} = \frac{dr}{dt} \cdot \cos\phi \;-\; r \cdot \sin\phi \frac{d\phi}{dt} \;=\; v_r \cdot \cos\phi \;-\; \sin\phi \cdot v_\phi \quad \text{mit} \quad v_\phi \;=\; \frac{r \cdot d\phi}{dt} \\
v_y &= \frac{dy}{dt} = \frac{dr}{dt} \cdot \sin\phi \;+\; r \cdot \cos\phi \frac{d\phi}{dt} \;=\; v_r \cdot \sin\phi \;+\; \cos\phi \cdot v_\phi \\
v_z &= \frac{dz}{dt} = \frac{dz}{dt}
\end{aligned}$$

$$\Longrightarrow \frac{\partial u}{\partial x} \;=\; \frac{\partial v_x}{\partial x} \;=\; \frac{\partial v_x}{\partial r} \cdot \frac{\partial r}{\partial x} \;+\; \frac{\partial v_x}{\partial \phi} \frac{\partial \phi}{\partial x} \;+\; \frac{\partial v_x}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial r}{\partial x} = \cos\phi \qquad \frac{\partial \phi}{\partial x} = \frac{-\ \sin\phi}{r} \qquad \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial v_x}{\partial r} = \frac{\partial v_r}{\partial r}\cdot\cos\phi \ - \ \sin\phi\frac{\partial \phi}{\partial r}$$

$$\frac{\partial v_x}{\partial \phi} = -\ v_r\cdot\sin\phi \ - \ \cos\phi\cdot v_\phi \ - \ \sin\phi\frac{\partial v_\phi}{\partial \phi}$$

$$\frac{\partial v_x}{\partial z} = 0$$

$\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial w}{\partial z}$ are computed analogous.

Introducing in eq. 1) results in

$$\frac{\partial \rho}{\partial t} + \rho\cdot\left(\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\cdot v_r) + \frac{1}{r}\,\frac{\partial}{\partial \phi}\,v_\phi + \frac{\partial}{\partial z}\,v_z\right) = 0$$

b) analogous to a)

$v_x = v_r\cdot\sin\Theta\cdot\cos\phi \ - \ v_\phi\cdot\sin\phi \ + \ v_\Theta\cdot\cos\Theta\cdot\cos\phi$

$v_y = v_r\cdot\sin\Theta\cdot\sin\phi \ - \ v_\phi\cdot\cos\phi \ + \ v_\Theta\cdot\cos\Theta\cdot\sin\phi$

$v_z = v_r\cdot\cos\Theta \ - \ v_\Theta\cdot\sin\Theta$

$$\Longrightarrow \frac{\partial \rho}{\partial t} + \rho\cdot\left(\frac{1}{r^2}\,\frac{\partial}{\partial r}\,\left(r^2\cdot v_r\right) + \frac{1}{r\cdot\sin\Theta}\,\frac{\partial}{\partial \Theta}\,(\sin\Theta\ v_\Theta) + \frac{1}{r\cdot\sin\Theta}\,\frac{\partial v_\Theta}{\partial \phi}\right) = 0$$

4.4

incompressible, steady flow in r-direction:

$$\Longrightarrow \frac{\partial}{\partial t} = 0 \ , \ v_\Theta = 0 \ , \ v_z = 0 \ , \ \rho = const.$$

Symmetrical problem $\Longrightarrow \dfrac{\partial}{\partial \Theta} \;=\; 0$

$\Longrightarrow$ Conti: $\dfrac{\partial (r\; v_r)}{\partial r} \;=\; 0$

Momentum eq.: $\rho\; v_r\; \dfrac{\partial\; v_r}{\partial r} \;=\; -\dfrac{\partial p}{\partial r} \;+\; \eta \left[\dfrac{\partial}{\partial r} \left(\dfrac{1}{r} \underbrace{\dfrac{\partial (r\; v_r)}{\partial r}}_{=\;0} \right) \;+\; \dfrac{\partial^2\; v_r}{\partial z^2} \right]$

4.5
steady flow $\dfrac{\partial\; v_r}{\partial t} \;=\; \dfrac{\partial\; v_z}{\partial t} \;=\; 0$

fully developed $\dfrac{\partial\; v_r}{\partial z} \;=\; \dfrac{\partial\; v_z}{\partial z} \;=\; 0$

invompressible $\rho \;=\; konst$

Conti.: $\dfrac{\partial (r\; v_r)}{\partial r} \;=\; 0 \;\Longrightarrow v_r \;=\; 0$

momentum eq.: $\dfrac{\partial p}{\partial r} \;=\; 0$

$$-\frac{\partial p}{\partial z} \;+\; \frac{\eta}{r}\, \frac{\partial}{\partial r} \left(r\, \frac{\partial v_z}{\partial r} \right) \;=\; 0$$

energy eq.: $\rho\; c_p \left(\dfrac{\partial T}{\partial t} \;+\; v_z\; \dfrac{\partial T}{\partial z} \right) \;=\; v_z\; \dfrac{\partial p}{\partial z} \;+\; \lambda \left(\dfrac{1}{r}\, \dfrac{\partial}{\partial r} \left(r\; \dfrac{\partial T}{\partial r} \right) \;+\; \dfrac{\partial^2\; T}{\partial z^2} \right) \;+\; \eta \left(\dfrac{\partial\; v_z}{\partial r} \right)^2$

4.6
steady: $\dfrac{\partial}{\partial t} \;=\; 0,$ frictionless: $\eta\; \nabla^2\; \vec{v} \;=\; 0$

$\Longrightarrow$ Navier-Stokes eqns for incompressible, steady, frictionless flows:

$$\frac{\partial u}{\partial x} \;+\; \frac{\partial v}{\partial y} \;=\; 0$$

$$\rho \left(u\; \frac{\partial u}{\partial x} \;+\; v\; \frac{\partial u}{\partial y} \right) \;=\; -\frac{\partial p}{\partial x} \;+\; \rho\; g_x$$

$$\rho \left(u\; \frac{\partial v}{\partial x} \;+\; v\; \frac{\partial v}{\partial y} \right) \;=\; -\frac{\partial p}{\partial y} \;+\; \rho\; g_y$$

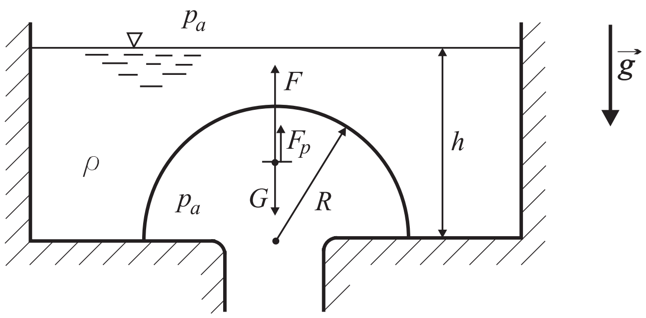
5 Hydrostatics

5.1

$$\text{Lift} = \text{Weight}$$
$$F_A = G$$

$$F_A = \rho_1\, h\, a^2\, g + \rho_2\, (a - h)\, a^2\, g$$
$$G = \rho_K\, a^3\, g$$
$$\Longrightarrow h = \frac{\rho_2 - \rho_K}{\rho_2 - \rho_1}\, a = 6,67 \cdot 10^{-2}\, m$$

5.2



$$\sum F = 0$$
$$F - G + F_p = 0$$
$$F = G - F_p$$

F_p is the resulting pressure force. F_p can be determined with the principle of Archimedes taking into account that the body is not fully wet.

$$F_p = V_{HK} \cdot \rho\, g - \rho\, g\, h \cdot A_{HK}$$

$V_{HK} \hat{=}$ Volume of the half sphere $\qquad A_{HK} \hat{=}$ Base area of the halph sphere

$$F_p = \frac{1}{2}\,\frac{4}{3}\,\pi\, R^3\, \rho\, g - \rho\, g\, h\, \pi\, R^2$$
$$F_p = \rho\, g\, \pi\, R^2 \left(\frac{2}{3}\, R - h\right)$$
$$\Longrightarrow F = G - \rho\, g\, \pi\, R^2 \left(\frac{2}{3}\, R - h\right)$$

5.3

Force equilibrium(τ: volume of the buoy,τ_H: encosed volume of air)

$$\left.\begin{array}{lll} \text{1) at the surface :} & g\, m_B = \frac{1}{3}\,\tau\, g\, \rho_w \\ \text{2) indepth H :} & g\, m_B = \tau_H\, g\, \rho_w \end{array}\right\} \Longrightarrow \tau_H = \frac{1}{3}\,\tau$$

enclosed air mass is constant:

$$\Longrightarrow \frac{2}{3}\,\tau\,\rho_{L_0} \;=\; \tau_H\cdot\rho_{LH} \;=\; \frac{1}{3}\,\tau\cdot\rho_{LH} \Longrightarrow 2\,\rho_{L_0} \;=\; \rho_{LH}$$

$$\Longrightarrow 2\,\frac{p_a \;+\; \frac{1}{3}\,\rho_w\;g\;h}{R_L\;T_L} \;=\; \frac{p_a \;+\; \rho_w\;g\left(H \;+\; \frac{1}{3}\;h\right)}{R_L\;T_L}$$

$$\Longrightarrow H \;=\; \frac{p_a}{\rho_w\;g} \;+\; \frac{1}{3}\;h$$

5.4

a) $$\tau_0 \;=\; \frac{G_0}{\rho_M\;g} \;=\; 1,07\cdot 10^5\;m^3$$

b) $$\Delta G \;=\; \rho_M\;A(h_0 \;-\; h_1)\;g$$

$\Longrightarrow A \;=\; 1,95\cdot 10^4\;m^2$

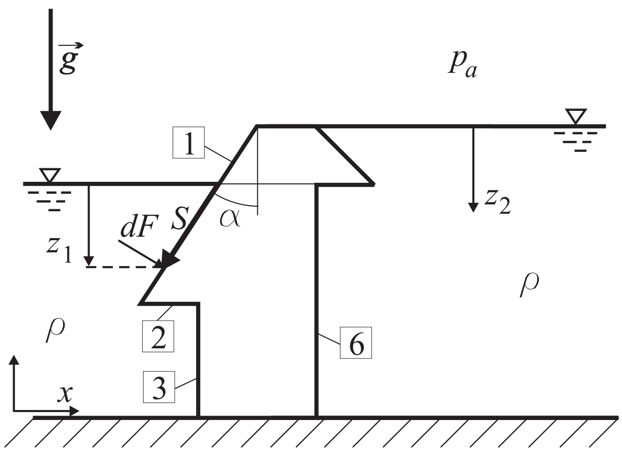
c) $$G_0 \;-\; \Delta G \;=\; \rho_M\;\tau_1\;g \;=\; \rho_F\;\tau_2\;g$$

$$\Longrightarrow \tau_2 \;-\; \tau_1 \;=\; \frac{G_0 \;-\; \Delta G}{\rho_M\cdot g}\left(\frac{\rho_M}{\rho_F} \;-\; 1\right) \;=\; 2,44\cdot 10^3\;m^3$$

d) $$\tau_2 \;-\; \tau_1 \;=\; (h_2 \;-\; h_1)\;A$$

$\Longrightarrow h_2 \;=\; 10,625\;m$

5.5



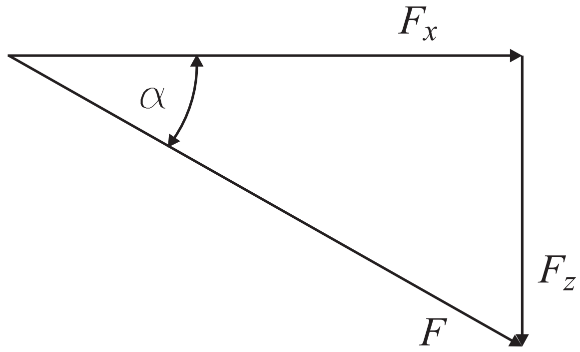
$\boxed{1}$ $$F_1 \;=\; \int d\,F_1 \;=\; \int p\,(z_1){\cdot}L{\cdot}ds$$

$$\text{coordinate transformation: with } S \;=\; \frac{z_1}{\cos\alpha} \quad;\quad ds \;=\; \frac{d\;z_1}{\cos\alpha}$$

$$F_{1x} \;=\; F_1\cdot\cos\alpha$$

$$F_{1z} \;=\; -\,F_1\cdot\sin\alpha$$

$$\Longrightarrow F_{1x} \;=\; \int\limits_0^{2a}\cos\alpha\;p(z_1)\cdot L\;\frac{d\;z_1}{\cos\alpha} \;=\; \int\limits_0^{2a}\rho\;g\;z_1\;L\;d\;z_1 \;=\; 2\;\rho\;g\;a^2\;L$$



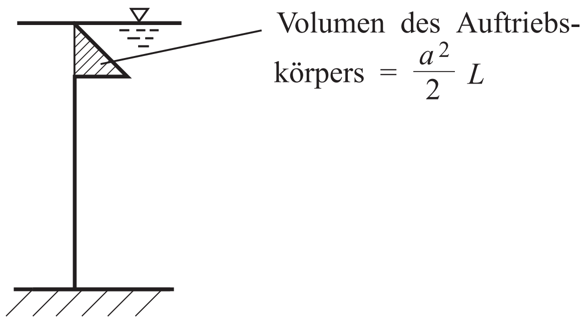
$$F_{1z} \;=\; -\int_0^{2a} \sin\alpha\; p(z_1)\cdot L\;\frac{d\;z_1}{\cos\alpha} \;=\; -\int_0^{2a} \tan\alpha\cdot\rho\;g\;z_1\;L\;d\;z_1 \;=\; -\frac{4}{3}\;\rho\;g\;a^2\;L$$

$$\text{with } \tan\alpha = \frac{2}{3}$$

$\boxed{2}$ $$F_{2x} \;=\; 0 \quad ; \quad F_{2z} \;=\; 2\rho\,g\,a^2\,L$$

$\boxed{3}$ $$F_{3x} \;=\; +\int_{2a}^{4a} p(z_1)\cdot L\,dz_1 \;=\; +\int_{2a}^{4a} \rho\,g\,z_1\,L\,dz_1 \;=\; 6\,\rho\,g\,a^2\,L \quad ; \quad F_{3z} \;=\; 0$$

$\boxed{4}$, $\boxed{5}$, $\boxed{6}$



$$\Longrightarrow F_{45z} \;=\; \frac{1}{2}\rho\cdot g\cdot a^2\cdot L \quad ; \quad F_{6z} \;=\; 0$$

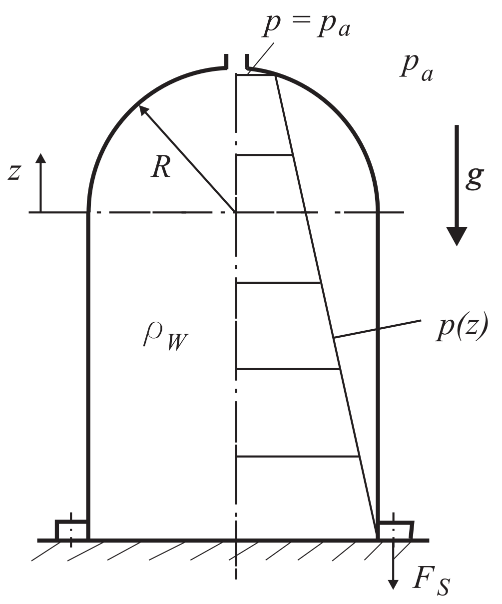
$$F_{456x} \;=\; -\frac{\rho\;g\;5a}{2}\cdot 5a\cdot L \;=\; -\frac{25}{2}\;\rho\;g\;a^2\cdot L$$

$$F_x \;=\; \sum_i F_{ix} \;=\; -\frac{9}{2}\;\rho\;g\;a^2\;L$$

$$F_z \;=\; \sum_i F_{iz} \;=\; +\frac{7}{6}\;\rho\;g\;a^2\;L$$

$$F_{ges} \;=\; \sqrt{F_x^2 \;+\; F_z^2} \;=\; 4.65\;\rho\;g\;a^2\;L$$

5.6



Screw force $F_S \; = \; ?$
The origin of the force is the different pressure distributions $p_i(z)$, $p_a(z)$. The force F_S depends on the shape of the boiler.
With $G_{\text{Gefäß}} \; = \; 0 \Longrightarrow F_{p \text{ res}} \; - \; F_S \; = \; 0$ the resulting pressure force $F_{p \text{ res}}$ can be determined.

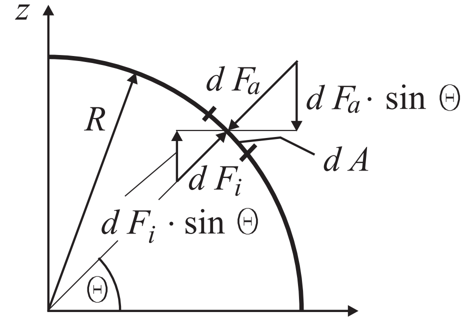
pressure distributions
EHE (inside) $\quad p_i(z) \; = \; p_a \; + \; \rho_W \cdot g(R \; - \; z)$

outside: $\quad \dfrac{d\,p_L}{dz} \; = \; \rho_L \; g \; << \; \rho_W \; g \; = \; - \dfrac{d\,p_i}{dz}$

due to $\rho_L << \rho_W$ pressure gradient can be neglected outside.

$$\Longrightarrow \quad p_a(z) \; = \; p_a$$

Resulting pressure force from the integration of all forces:

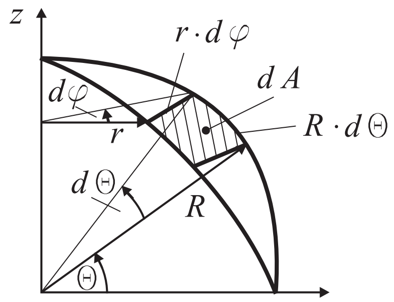


$$dF_i \; = \; p_i \cdot dA$$

$$dF_a = p_a \cdot dA$$

<u>z - component:</u> $dF_{p_z} = (dF_i - dF_a) \cdot \sin\Theta = (p_i(z) - p_a)\, dA \sin\Theta$

<u>surface element dA</u>



$$dA = R \cdot d\Theta \cdot r \cdot d\varphi \quad \text{mit} \quad r = R \cdot \cos\Theta$$

$$\Longrightarrow \quad dA = R^2 \cdot \cos\Theta d\Theta\; d\varphi$$

<u>EHE:</u> $p_i - p_a = \rho_W \cdot g(R - z) = \rho_W \cdot g(R - R \cdot \sin\Theta)$

$$F_{pz} = \int dF_{pz} = \int_0^{2\pi}\int_0^{\pi/2} \rho_W \cdot gR(1 - \sin\Theta)\; R^2 \cos\Theta \cdot \sin\Theta\; d\Theta\; d\varphi$$

<u>substitution:</u> $\sin\Theta = \eta$

$$F_{p_z} = 2\pi\rho_W \cdot g\; R^3 \int_0^1 (\eta - \eta^2)\; d\eta$$

$$\Longrightarrow \quad F_{p_z} = \frac{\pi}{3}\rho_W \cdot g\; R^3 = F_S$$

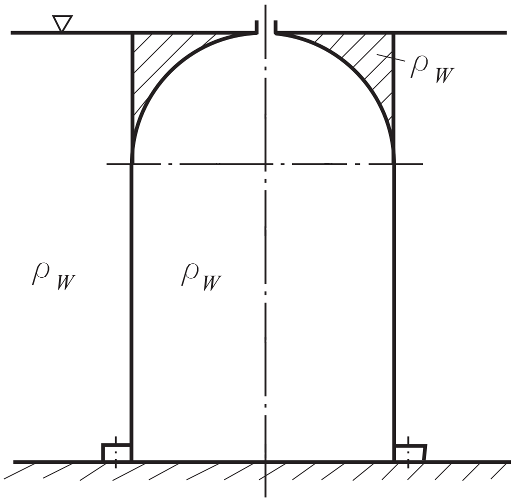
<u>alternative solution</u>

Assume a completely submerged body

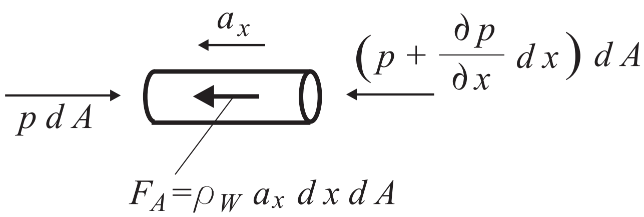
$$F_S = 0, \quad p_i(z) \quad \text{unchanged}$$

the difference is the weight of the hatched water

$$\begin{aligned} \Longrightarrow \quad F_S = G_W &= \rho_W \cdot g\; (V_{\text{cylinder}} - V_{\text{halfsphere}}) \qquad V \mathrel{\hat{=}} \text{volume} \\ &= \rho_W \cdot g\; (\pi\; R^3 - \frac{1}{2} \cdot \frac{4}{3}\; R^3\pi) \\ &= \rho_W \cdot g \cdot R^3 \cdot \frac{\pi}{3} \end{aligned}$$



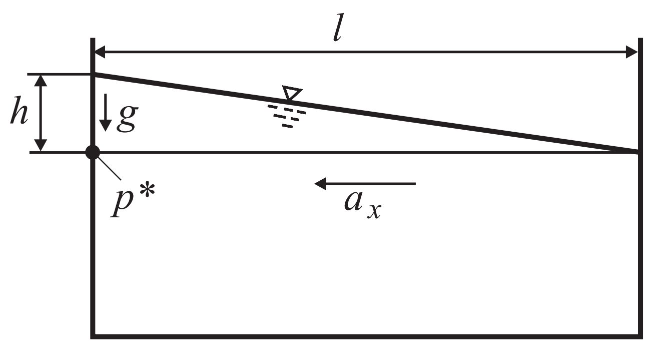
5.7



1.

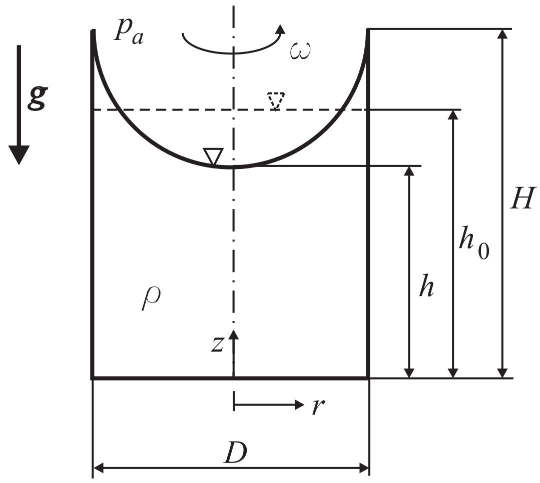
$$\sum F = 0 = p\; dA - \left(p + \frac{\partial p}{\partial x}\; dx\right)\; dA - \rho_W\; a_x\; dx\; dA \Longrightarrow \frac{\partial p}{\partial x} = -\rho_W\; a_x$$

2. $\Longrightarrow p = -\rho_W\; a_x \cdot x + c_1$ and $p = -\rho_W\; g \cdot z + c_2$
$\Longrightarrow p^* = p_a + \rho g h = p_a + \rho a_x l$
$\Longrightarrow l = \dfrac{g}{a_x}\, h$



5.8

boundary condition: $r = 0, \quad z = h: \quad p = p_a$



$$p - p_a = \rho\, g(h - z) + \frac{1}{2}\,\rho\omega^2\, r^2$$

a) Surface: $p = p_a$; $z_0(r)$ is the surface coordinate

$$\Longrightarrow \quad z_0(r) = h + \frac{\omega^2\, r^2}{2g}$$

$$r = R: \quad z_0 = H$$

$$\Longrightarrow \quad \omega^2 = 2g\frac{H-h}{R^2}$$

$$\Longrightarrow \quad z_0(r) = h + (H-h)\frac{r^2}{R^2}$$

Volume of water:

$$\pi\, R^2\, h_0 = \int\limits_0^R z_0(r)\, 2\pi\, r\, dr = \pi\, R^2\, h + \frac{1}{2}\,\pi\, R^2\,(H-h)$$

$$\Longrightarrow \quad h = 2h_0 - H = 0,4\ m$$

$$\Longrightarrow \quad \omega = \sqrt{\frac{4g}{R^2}\,(H-h_0)} = 13,9\ s^{-1}$$

b)

$$r = R: \quad p = p_a + \rho\, g\,(H-z)$$

$$z = 0: \quad p = p_a + \rho\, g\, h + \frac{\rho}{2}\,\omega^2\, r^2$$

from hint: $dp = \rho\,\omega^2\, r\, dr - \rho\, g\, dz$

$$\Longrightarrow \int\limits_{p_a}^{p} dp = \int\limits_{r=0}^{r} \rho\,\omega^2\, r\, dr - \int\limits_{z=h}^{z} \rho\, g\, dz$$

$$\Longrightarrow p \; - \; p_a \; = \; \frac{\rho\,\omega^2\,r^2}{2} \; - \; \rho\,g\,(z-h)$$

$$\Longrightarrow p \; - \; p_a \; = \; \rho\,g\,(h-z) \; + \; \frac{1}{2}\,\rho\,\omega^2\,r^2$$

5.9

$z \; = \; H: \quad 0 \; = \; F_A \; - \; F_G \; - \; F_N \;;\; F_A \; = \; \rho_{L\;(H)}\; g\tau_B \;,\; F_G \; = \; \rho_{G\;(H)}\; g\tau_B$

$z \; = \; h: \quad 0 \; = \; F_A \; - \; F_G \; - \; F_N \;;\; F_A \; = \; \rho_{L\;(h)}\; g\;\frac{\tau_B}{2} \;,\; F_G \; = \; \rho_{G\;(h)}\; g\;\frac{\tau_B}{2}$

$$\Longrightarrow \quad F_N \; = \; \left(\rho_{L\;(H)} \; - \; \rho_{G\;(H)}\right)\; g\tau_B \; = \; \left(\rho_{L\;(h)} \; - \; \rho_{G\;(h)}\right)\; g\frac{1}{2}\tau_B$$

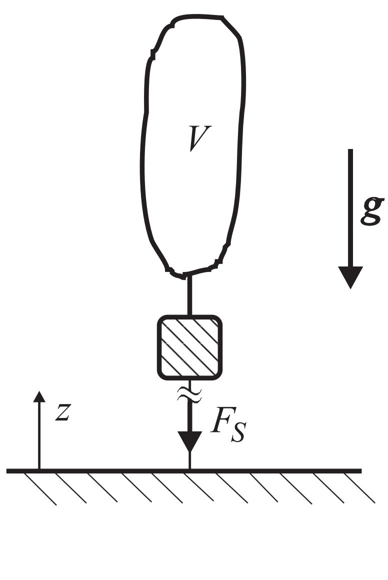
$$\rho_{L\;(H)} \; = \; \rho_0\;\exp\left(-\frac{g\;H}{R_L\;T_0}\right), \quad \rho_{L\;(h)} \; = \; \rho_0\;\exp\left(-\frac{g\;h}{R_L\;T_0}\right) \quad (T \; = \; const)$$

open balloon: $p_L \; = \; p_G \quad \Longrightarrow \rho_L\;R_L\;T_0 \; = \; \rho_G\;R_G\;T_0 \quad \Longrightarrow \rho_G \; = \; \rho_L\;\frac{R_L}{R_G}$

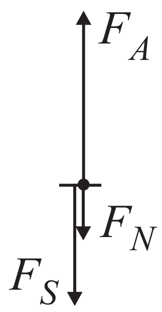
$$\rho_{L\;(H)}\left(1 \; - \; \frac{R_L}{R_G}\right) \; = \; \rho_{L\;(h)}\left(1 \; - \; \frac{R_L}{R_G}\right)\frac{1}{2} \quad \Longrightarrow \quad \rho_0\;\exp\left(-\;\frac{g\;H}{R_L\;T_0}\right) \; = \; \frac{1}{2}\,\rho_0\;\exp\left(-\frac{g\;h}{R_L\;T_0}\right)$$

$$h \; = \; H \; - \; \frac{R_L\;T_0}{g}\;\ln 2 \; = \; 4290\;m$$

5.10



a)



$$F_A \;=\; F_S \;+\; F_N \quad \text{and} \quad F_S \;=\; F_A \;-\; F_N$$

$$\Longrightarrow F_S \;=\; \rho_0 \cdot V_0 \cdot g \;-\; m \cdot g \;=\; (\rho_0 \cdot V_0 \;-\; m) \cdot g \qquad (\text{mit } m = m_{Gas} + m_{structure})$$

b)

General equation für $V(z)$:

$$V \;=\; \frac{m_g}{\rho_g(z)} \qquad \text{with perfect gas assumption} \qquad \rho_g(z) \;=\; \frac{p_g\ (z)}{R_g \cdot T_g}$$

$$\Longrightarrow \quad V(z) \;=\; \frac{m_g \cdot R_g \cdot T_g}{p_g(z)}$$

pressure equalization: $\quad p_g \;=\; p_a(z) \;=\; \rho_a(z) \cdot R_a \cdot T_a$

$$\Longrightarrow \quad V(z) \;=\; \frac{m_g \cdot R_g \cdot T_g}{\rho_a(z) \cdot R_a \cdot T_a} \quad \text{temperatureequalization} \quad \Longrightarrow \quad T_g \;=\; T_a$$

$\rho_a(z) \;=\; ?$

,,Scale Height Relation": $\quad p_a(z) \;=\; p_0 \cdot e^{\frac{-g}{R_a \cdot T_0} \cdot z}$

$$\rho_a(z) \;=\; \frac{p_a(z)}{R_a \cdot T_0}$$

$$\Longrightarrow \quad \rho_a(z) \cdot R_a \cdot T_0 \;=\; \rho_0 \cdot R_a \cdot T_0 \cdot e^{\frac{-g}{p_0} \cdot \rho_0 \cdot z}$$

$$\Longrightarrow \quad V \;=\; \frac{m_g \cdot R_g}{\rho_0 \cdot R_a} \cdot e^{\frac{g \cdot \rho_0}{p_0} \cdot z}$$

For $\;z \;=\; 0 \qquad V \;=\; V_0$

$$\Longrightarrow \quad V_0 \;=\; \frac{m_g \cdot R_g}{\rho_0 \cdot R_a} \quad \Longrightarrow \quad V(z) \;=\; V_0 \cdot e^{\frac{g \cdot \rho_0}{p_0} \cdot z}$$

for $z = z_1$ follows:

$$V_1 = V_0 \cdot e^{\dfrac{g \cdot \rho_0}{p_0} \cdot z_1}$$

$$\Longrightarrow \quad z_1 = \frac{p_0}{g \cdot \rho_0} \cdot \ln \frac{V_1}{V_0}$$

c)

$\longrightarrow$ For $z \geq z_1$ the balloon is rigid.

$\longrightarrow$ gas mass and volume is contant $\Longrightarrow$ no pressure equalization.

Ceiling is reached when $F_a(z_2) = G$

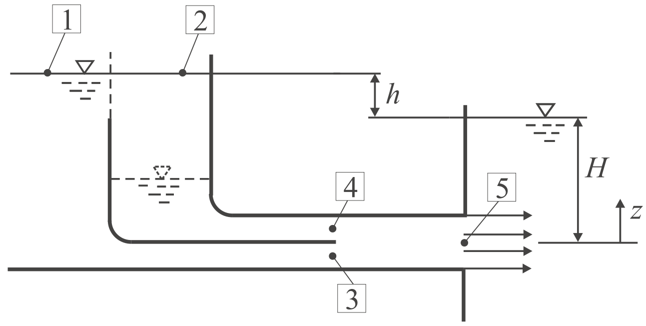
$$F_a(z_2) = \rho_a(z_2) \cdot V_1 \cdot g = \rho_0 \cdot V_1 \cdot g \cdot e^{\dfrac{-g \cdot \rho_0}{p_0} \cdot z} = m \cdot g$$

$$\Longrightarrow \quad z_2 = \ln \frac{\rho_0 \cdot V_1}{m} \cdot \frac{p_0}{g \cdot \rho_0}$$

6 Continuity and Bernoulli's equation

6.1

a) Volume flux $\dot{V} = v_5 \cdot A$



Bernoulli $\boxed{1} \longrightarrow \boxed{3}$: $p_a + \rho\, g\, (H+h) = p_3 + \dfrac{\rho}{2}\, v_3^2$

Bernoulli $\boxed{2} \longrightarrow \boxed{4}$: $p_a + \rho\, g\, (H+h) = p_4 + \dfrac{\rho}{2}\, v_4^2$

with $p_3 = p_4 \Longrightarrow v_3 = v_4$

$\Longrightarrow$ no mixing losses:

Bernoulli $\boxed{1} \longrightarrow \boxed{5}$: $p_a + \rho\, g\, (H+h) = p_5 + \dfrac{\rho}{2}\, v_5^2$

with $p_5 = p_a + \rho\, g\, H \Longrightarrow v_5 = \sqrt{2\, g\, h} \Longrightarrow \dot{V} = \sqrt{2\, g\, h}\; A$

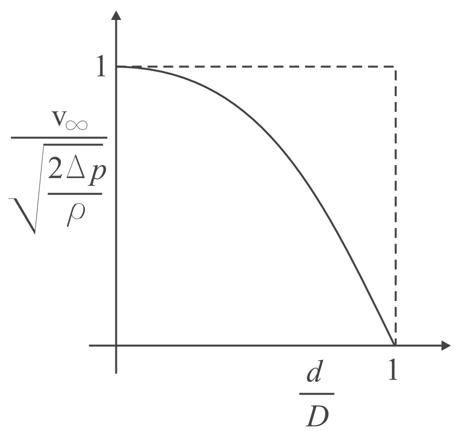
6.2

Bernoulli : $$p_0 = p_\infty + \frac{\rho}{2}\, v_\infty^2 = p_2 + \frac{\rho}{2}\, v_2^2$$

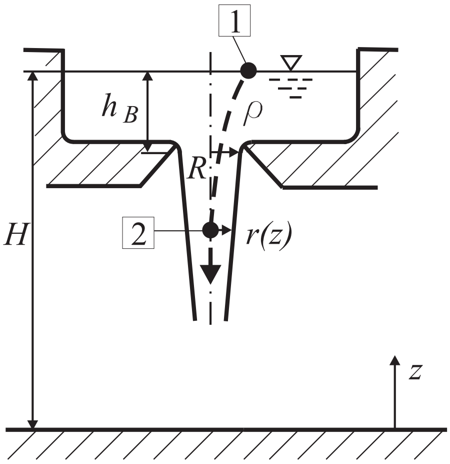
$$\Longrightarrow \Delta p = p_0 - p_2 = \frac{\rho}{2}\, v_2^2$$

Conti : $$v_\infty\, \pi\, D^2 = v_2\, \pi\, (D^2 - d^2)$$

$$\Longrightarrow v_2 = \sqrt{\frac{2\, \Delta p}{\rho}} = v_\infty \frac{D^2}{D^2 - d^2} \qquad \Longrightarrow \qquad \frac{v_\infty}{\sqrt{\dfrac{2\Delta p}{\rho}}} = 1 - \left(\frac{d}{D}\right)^2$$



6.3



Bernoulli $\boxed{1} \longrightarrow \boxed{2}$: $p_a \; + \; \rho \; g \; H \; = \; p_a \; + \; \rho \; g \; z \; + \; \frac{\rho}{2} \; v^2(z)$

$$\Longrightarrow v(z) \; = \; \sqrt{2 \; g \; H \; - \; 2 \; g \; z} \; = \; \sqrt{2 \; g(H - z)}$$

conti :

$$\pi \, R^2 \, v_R \; = \; \pi \, r^2(z) \, v(z)$$

$$\Longrightarrow r(z) \; = \; R \sqrt{\frac{v_R}{v(z)}}$$

$$r(z) \; = \; R \sqrt{\frac{v(z = H - h_B)}{v(z)}}$$

$$r(z) \; = \; R \sqrt[4]{\frac{2g(H - H + h_B)}{2g(H - z)}}$$

$$r(z) \; = \; R \sqrt[4]{\frac{h_B}{H - z}}$$

6.4

a) Bernoulli : $p_1 + \frac{\rho}{2} \, v_1^2 \; = \; p_2 + \frac{\rho}{2} \, v_2^2$

conti : $v_1 \, A_1 \; = \; v_2 \, A_2 \; = \; v_3 \, A_3$

$$\Longrightarrow v_2 \; = \; \sqrt{\frac{2 \; \Delta p}{\rho \; [1 - (A_2/A_1)^2]}} \; = \; 12 \; m/s$$

$$\Longrightarrow v_1 \; = \; 4 \; m/s \qquad v_3 \; = \; 6 \; m/s$$

b) Bernoulli : $p_2 + \frac{\rho}{2} \, v_2^2 \; = \; p_3 + \frac{\rho}{2} \, v_3^2$

$$p_3 = p_a = 10^5\ N/m^2 \text{(Outflow to the surrounding)}$$

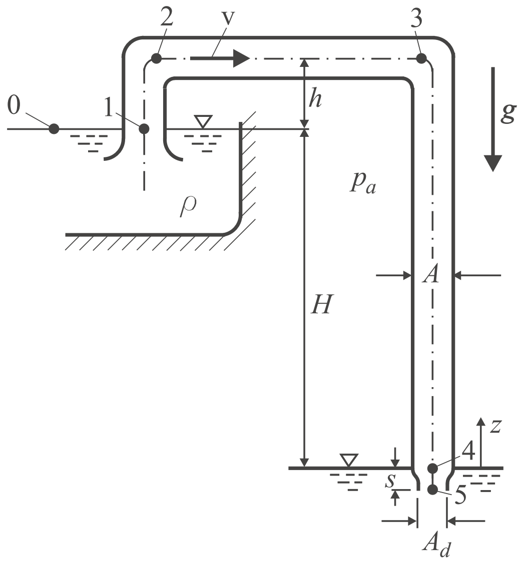
$$\Longrightarrow p_2 = 0,46 \cdot 10^5\ N/m^2$$

$$\Longrightarrow p_1 = 1,1 \cdot 10^5\ N/m^2$$

$$\text{Bernoulli :} \quad p + \rho\, g\, h = p_a + \frac{\rho}{2}\, v_3^2$$

$$p = 1,08 \cdot 10^5\ N/m^2$$

6.5



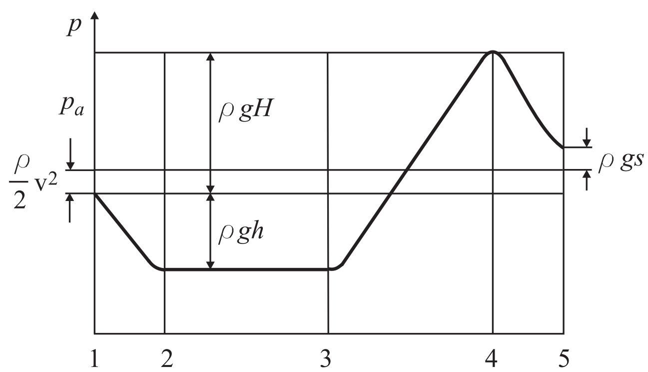
a) Bernoulli $\boxed{0} \longrightarrow \boxed{5}$:

$$p_a + \rho\, g\, H = p_5 - \rho\, g\, s + \frac{\rho}{2}\, v_5^2$$

$$p_5 = p_a + \rho\, g\, s$$

$$\Longrightarrow \dot{V} = A_d \sqrt{2\, g\, H} = 4\ m^3/s$$

b)



c) bubbles occur, if $p_2 = p_3 = p_D$.

conti : $$v_5^* \, A_d^* = v^* \, A$$

Bernoulli : $$p_a = p_D + \rho \, g \, h + \frac{\rho}{2} \, v^{*2}$$

$$\Longrightarrow A_d^* = A \sqrt{\frac{p_a - p_D}{\rho \, g \, H} - \frac{h}{H}} = 0,244 \; m^2$$

6.6 Bernoulli $\boxed{0} \longrightarrow \boxed{3}$:

$$p_a + \rho \, g \, H = p_3 + \frac{\rho}{2} \, v_3^2 + \int\limits_0^3 \rho \, \frac{\partial v}{\partial t} \, ds \qquad , \qquad p_3 = p_a \, , \qquad v_2 = v_3$$

$$\int\limits_0^3 = \int\limits_0^1 + \int\limits_1^2 + \int\limits_2^3$$

$$I_{01} = \int\limits_0^1 \rho \, \frac{\partial v}{\partial t} \, ds \simeq 0 \quad (h_1 << L)$$

$$I_{12} = \int\limits_1^2 \rho \, \frac{\partial v}{\partial t} \, ds \quad , \quad v = v_2 \frac{h_2}{h} \quad , \quad h = h_1 + \frac{h_2 - h_1}{L} \, x$$

$$I_{12} = \rho \, \frac{dv_2}{dt} \int\limits_1^2 \frac{h_2}{h_1 + \dfrac{h_2 - h_1}{L} \, x} \, dx = \rho \, \frac{dv_2}{dt} \, \frac{h_2 \, L}{h_2 - h_1} \, \ln \frac{h_2}{h_1} = \rho \, \frac{dv_2}{dt} \, \overline{L}$$

$$I_{23} = \rho \int\limits_2^3 \frac{\partial v}{\partial t} \, ds = \rho \, L \, \frac{dv_2}{dt}$$

introduce in Bernoulli:

$$p_a + \rho \, g \, H = p_a + \frac{\rho}{2} \, v_3^2 + \rho \, \frac{dv_3}{dt} \, (\overline{L} + L)$$

$$\frac{dv_3}{dt} = \frac{1}{\overline{L} + L} \left(g \, H - \frac{v_3^2}{2} \right)$$

$$t \rightarrow \infty : \quad g \, H - \frac{1}{2} \, v_{3e}^2 = 0 \Longrightarrow v_{3e} = \sqrt{2 \, g \, H}$$

local acceleration:

$$b_l = \frac{\partial v}{\partial t} = \frac{dv_3}{dt} \, \frac{h_2}{h} \quad , \quad b_{l \; (v_3 = \frac{1}{2} \; v_{3e}, \; x = \frac{L}{2})} = \frac{1}{\overline{L} + L} \, g \, H \, \frac{3}{4} \, \frac{2 \, h_2}{h_1 + h_2}$$

convective acceleration:

$$b_k = v \, \frac{\partial v}{\partial x} = - \, v_3^2 \, \frac{h_2^2}{h^3} \, \frac{dh}{dx} \qquad , \qquad \frac{dh}{dx} = \frac{h_2 - h_1}{L}$$

$$b_{k(v_3=\frac{1}{2}\ v_{3e},\ x=\frac{L}{2})} \;=\; 4\ \frac{g\ H}{L}\ \frac{h_2^2\ (h_1-h_2)}{(h_1+h_2)^3}$$

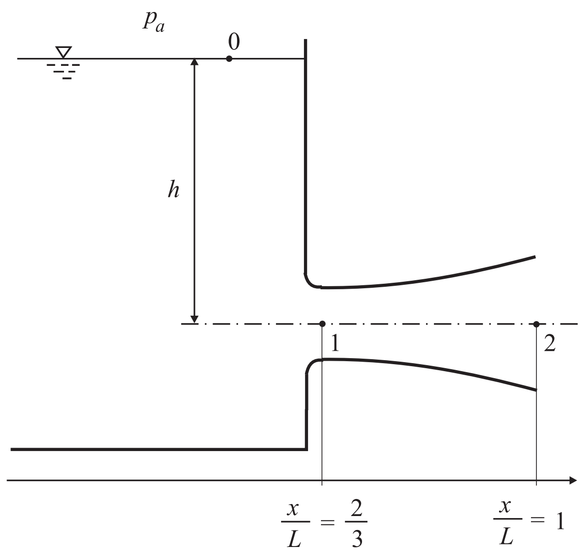
substantial acceleration:

$$b_s \;=\; \frac{dv}{dt} \;=\; \frac{\partial v}{\partial t} \;+\; v\ \frac{\partial v}{\partial x} \;=\; b_l \;+\; b_k$$

$$b_s \;=\; \frac{3}{2}\ \frac{g\ H}{\overline{L}+L}\ \frac{h_2}{h_1+h_2} \;+\; 4\ \frac{g\ H}{L}\ \frac{h_2^2\ (h_1-h_2)}{(h_1+h_2)^3}$$

6.7

a) Bernoulli $\boxed{0} \longrightarrow \boxed{2}$:



$$(1) \qquad p_a \;+\; \rho\ g\ h \;=\; p_2 \;+\; \frac{\rho}{2}\ v_2^2 \;+\; \rho \int\limits_{2/3\ L}^{L} \frac{\partial v}{\partial t}\ dx$$

conti: $\qquad v \cdot A \;=\; v_2 \cdot A_2$

$$\Longrightarrow \quad \rho \int\limits_{2/3\ L}^{L} \frac{\partial v}{\partial t}\ dx \;=\; \frac{\partial v_2}{\partial t}\rho \int\limits_{2/3\ L}^{L} \frac{A_2}{A}\ dx \;=\; \frac{\partial v_2}{\partial t}\rho \int\limits_{2/3\ L}^{L} \frac{L^2}{x^2}\ dx \;=\; \frac{\partial v_2}{\partial t}\ \frac{L}{2}\ \rho$$

Introduce in (1):

$$\left.\frac{\partial v_2}{\partial t}\right|_{t=0} \;=\; \left.\frac{dv_2}{dt}\right|_{t=0} \;=\; \frac{2\ g\ h - v_2^2}{L} \;=\; \frac{2\ g\ h}{L}$$

b) Bernoulli $\boxed{\text{x}} \longrightarrow \boxed{2}$:

$$p \;+\; \frac{\rho}{2}\ v^2 \;=\; p_2 \;+\; \frac{\rho}{2}\ v_2^2 \;+\; \rho \int\limits_{x}^{L} \frac{\partial v}{\partial t}\ dx$$

conti: $$v = \frac{A_2}{A} \cdot v_2$$

$$\Longrightarrow \quad p = p_2 + \frac{\rho}{2} v_2^2 \left(1 - \left(\frac{A_2}{A}\right)^2\right) + \frac{\rho}{L}\left(2\, g\, h - v_2^2\right) \int_x^L \frac{L^2}{x^2}\, dx$$

$$\Longrightarrow p = p_2 + \frac{\rho}{2}\, v_2^2 \left(1 - \frac{L^4}{x^4}\right) + \rho\left(2\, g\, h - v_2^2\right)\left(-1 + \frac{L}{x}\right)$$

Extremum for $p(x)$ from $\dfrac{dp}{dx} = 0$ bei $v_2 = \dfrac{1}{2}\sqrt{2\, g\, h}$:

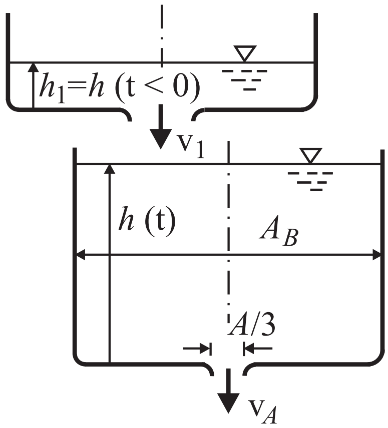
$$\Longrightarrow 0 = \frac{\rho}{2} \cdot \frac{1}{4}\, 2gh \left(\frac{4\, L^4}{x^5}\right) + \rho\left(2\, g\, h - \frac{1}{4}\, 2\, g\, h\right) \cdot \left(-\frac{L}{x^2}\right)$$

$$\Longrightarrow 0 = \frac{L^4}{x^5} - \frac{3}{2}\, \frac{L}{x^2} \quad \Longrightarrow \quad x = \sqrt[3]{\frac{2}{3}}\, L \approx 0,87\, L$$

$$\left.\frac{d^2\, p}{d\, x^2}\right|_{x=x_m} = \frac{\rho\, g\, h}{L^2}\left(\frac{-5}{(2/3)^2} + \frac{3}{2} \cdot 2 \cdot \frac{1}{2/3}\right) = \frac{\rho\, g\, h}{L^2}\left(-\frac{27}{4}\right) < 0$$

$$\Longrightarrow \text{ Maximum bei } x = \sqrt[3]{\frac{2}{3}} \cdot L$$

6.8



conti : $$A_B\, \frac{dh}{dt} = v_1\, A - v_A \cdot \frac{A}{3}$$

Torricelli : $$v_1 = \sqrt{2\, g\, h_1} \quad , \quad v_A = \sqrt{2\, g\, h(t)} \left(\frac{dh}{dt} << v_A\right)$$

introducing $$\frac{3}{\sqrt{2\, g}}\, \frac{A_B}{A}\, \frac{dh}{dt} = 3\sqrt{h_1} - \sqrt{h(t)}$$

$$\Longrightarrow T = \frac{3}{\sqrt{2\, g}}\, \frac{A_B}{A} \int_{h_1}^{4h_1} \frac{dh}{3\sqrt{h_1} - \sqrt{h(t)}}$$

$$= \frac{3}{\sqrt{2\, g}}\, \frac{A_B}{A}\, 2\left[\left(3\sqrt{h_1} - \sqrt{h(t)}\right) - 3\sqrt{h_1}\, \ln\left(3\sqrt{h_1} - \sqrt{h(t)}\right)\right]_{h_1}^{4h_1} = 108\, s$$

6.9

a) steady Bernoulli $\boxed{0} \longrightarrow \boxed{1}$ for $t \longrightarrow \infty$

$$p_a \; + \; \rho \; g \; h \; = \; p_a \; + \; \rho \; g \; h \; + \; \frac{\rho}{2} \; v_{max}^2 \; - \; \Delta p$$

$$\Longrightarrow v_{max} \; = \; \sqrt{\frac{2\Delta p}{\rho}}$$

unsteady Bernoulli $\boxed{0} \longrightarrow \boxed{1}$

$$p_a \; + \; \rho \; g \; h \; = \; p_a \; + \; \rho \; g \; h \; + \; \frac{\rho}{2} \; v_{(t)}^2 \; - \; \Delta p \; + \; \int\limits_L \rho \; \frac{\partial v}{\partial t} \; ds$$

$$\Longrightarrow \Delta p \; = \; \frac{\rho}{2} \; v_{(t)}^2 \; + \; \rho \; L \; \frac{dv}{dt}$$

$$\Longrightarrow dt \; = \; \frac{\rho \; L}{\Delta p - \frac{\rho}{2} \; v^2} \; dv \; = \; \frac{2 \; L}{\frac{2 \; \Delta p}{\rho} \; - \; v^2} \; dv \; = \; \frac{2 \; L}{v_{max}^2 \; - \; v_{(t)}^2} \; dv$$

$$\text{Integration :} \quad T \; = \; \frac{L}{v_{max}} \; \ln \; \frac{v_{max} \; + \; v}{v_{max} \; - \; v} \Bigg|_{v=0}^{v=.999 \; v_{max}} \; = \; \frac{L}{v_{max}} \; \ln \left(\frac{1 + .999}{1 - .999} \right)$$

$$v_{max} \; = \; \frac{L}{T} \cdot 7.6 \; = \; \frac{10 \; m}{7.6 \; s} \cdot 7.6 \; = \; 10 \; m/s$$

Bernoulli $\boxed{1} \longrightarrow \boxed{2}$ for $t \longrightarrow \infty$

$$\frac{\rho}{2} \; v_{max}^2 \; = \; \rho \; g \; H \quad \Longrightarrow \quad H \; = \; \frac{v_{max}^2}{2 \; g} \; = \; 5 \; m$$

b) $$v_{H/2} \; = \; \frac{v_{max}}{\sqrt{2}} \Longrightarrow t_{H/2} \; = \; \frac{L}{v_{max}} \; \ln \; \frac{v_{max} \left(1 + \frac{1}{\sqrt{2}} \right)}{v_{max} \left(1 - \frac{1}{\sqrt{2}} \right)} \; = \; 1,763 \; sec$$

6.10

a)

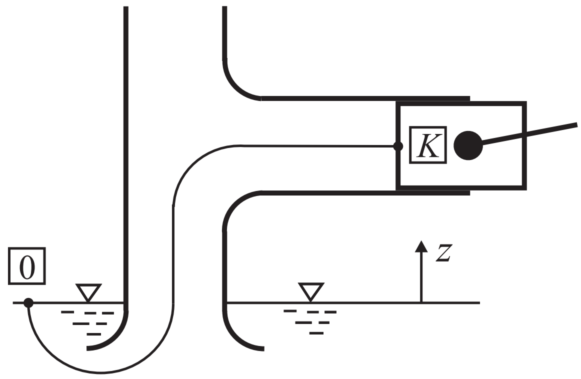
Bernoulli $\boxed{0} \longrightarrow \boxed{K}$:

$$p_a \; = \; p_K \; + \; \frac{\rho}{2} \; v_K^2 \; + \; \rho \; g \; h \; + \; \rho \int\limits_0^{L+\xi} \frac{\partial v}{\partial t} \; ds$$

conti :

$$v \cdot A \; = \; v_K \cdot A$$

$$\Longrightarrow \rho \int\limits_0^{L+\xi} \frac{\partial v}{\partial t} \; ds \; = \; \rho \; \frac{dv_K}{dt} \; (L + \xi)$$



$$v_K = \frac{d\xi}{dt} = -\xi_0\,\omega\,\sin\,(\omega t); \quad v_K^2 = \xi_0^2\,\omega^2\,\sin^2\,(\omega t)$$

$$\frac{dv_K}{dt} = \frac{d^2\,\xi}{dt^2} = -\xi_0\,\omega^2\,\cos\,(\omega t)$$

$$\Longrightarrow p_K = p_a - \frac{\rho}{2}\,\xi_0^2\,\omega^2\,\sin^2\,(\omega t) - \rho\,g\,h + \rho\,\xi_0\,\omega^2\,\cos\,(\omega t)\,(L + \xi_0\,\cos\,(\omega t))$$

$$\Longrightarrow \frac{p_K}{\rho\,L^2\,\omega^2} = \frac{p_a - \rho\,g\,h}{\rho\,L^2\,\omega^2} - \frac{1}{2}\,\frac{\xi_0^2}{L^2}\,\sin^2\,(\omega t) + \frac{\xi_0}{L}\,\cos\,(\omega t) + \frac{\xi_0^2}{L^2}\,\cos^2\,(\omega t)$$

$$\Longrightarrow \frac{p_K}{\rho\,L^2\,\omega^2} = \frac{p_a - \rho\,g\,h}{\rho\,L^2\,\omega^2} + \frac{\xi_0}{L}\left[\cos\,(\omega t) - \frac{1}{2}\,\frac{\xi_0}{L}\,(1 - 3\cos^2\,(\omega t))\right]$$

b) Minimum of $\frac{\xi_0}{L}\left[\cos\,(\omega t) - \frac{1}{2}\,\frac{\xi_0}{L}\,(1 - 3\cos^2\,(\omega t))\right]$ at $t = \frac{\pi}{\omega}$ für $\xi_0 << L$!

$$\Longrightarrow p_D = p_a - \rho\,g\,h - \rho\,\xi_0\,(L - \xi_0)\,\omega_K^2$$

$$\Longrightarrow \omega_K = \sqrt{\frac{p_a - \rho g h - p_D}{\rho\,\xi_0\,(L - \xi_0)}}$$

c)

$$\Delta T = \frac{2\pi}{\omega} \quad \text{Period of piston movement}$$

$$\dot{V} = \frac{1}{\Delta T}\int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} v_K(t)\,A\,dt = \frac{\omega}{2\pi}\,A\int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} v_K\,dt$$

$$\dot{V} = \frac{\omega}{2\pi}\,A\int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} (-\xi_0\,\omega\,\sin\,(\omega t))\,dt = \frac{\omega}{2\pi}\,A\,\xi_0\,\omega \left.\frac{\cos\,\omega t}{\omega}\right|_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}}$$

$$\Longrightarrow \dot{V} = \frac{\omega}{\pi}\,A\,\xi_0$$

6.11

a) Bernoulli from surface $\longrightarrow$ pipe end :

$$\begin{aligned}
1)\ \omega &= 0:\ p_a + \rho\, g\, (h+l) \ = \ p_a + \frac{\rho}{2}\, v^2 \\
&\Longrightarrow v\ (t<0) \ = \ \sqrt{2\ g\ (h+l)} \\
2)\ \omega &= \omega_r:\ p_a + \rho\, g\, (h+l) \ = \ p_a + \frac{\rho}{2}\, v^2 - \frac{\rho}{2}\, \omega_r^2\, R^2 \\
&\Longrightarrow v\ (t\to\infty) \ = \ \sqrt{2\ g\ (h+l) + \omega^2\ R^2}
\end{aligned}$$

unsteady Bernoulli:

$$p_a + \rho\, g\, (h+l) \ = \ p_a + \frac{\rho}{2}\, v^2 - \frac{\rho}{2}\, \omega_r^2\, R^2 + \rho(l+R)\, \frac{dv}{dt}$$

$$\Longrightarrow \frac{dv}{dt} \ = \ \frac{2g(h+l) + \omega_r^2\, R^2 - v^2}{2(l+R)} \ = \ \frac{a^2 - v^2}{2(l+R)}$$

Integration :

$$\Delta T \ = \int\limits_{v(t<0)}^{\frac{1}{2}v(t\to\infty)} \frac{2(l+R)\ dv}{a^2 - v^2} \ = \ \frac{2(l+R)}{2a}\ \ln\ \frac{a+v}{a-v}\Bigg|_{v(t<0)}^{\frac{1}{2}v(t\to\infty)}$$

$$\text{with}\ \ v(t\to\infty) \ = \ a$$

$$\Longrightarrow \Delta T \ = \ \frac{l+R}{a}\ \ln\ \frac{3\ \omega_r^2\ R^2}{(a + v(t<0))^2}$$

b)

$$\frac{\partial p}{\partial r} \ = \ \rho\, \omega^2\, r \ \Longrightarrow \ p(r) \ = \ \frac{1}{2}\, \rho\, \omega_r^2\, r^2 + c$$

boundary condition :

$$r \ = \ R \Longrightarrow p \ = \ p_a \Longrightarrow c \ = \ p_a - \frac{1}{2}\, \rho\, \omega_r^2\, R^2$$

$$\Longrightarrow \ p \ = \ p_a - \frac{\rho\, \omega_r^2}{2}\, (R^2 - r^2)$$

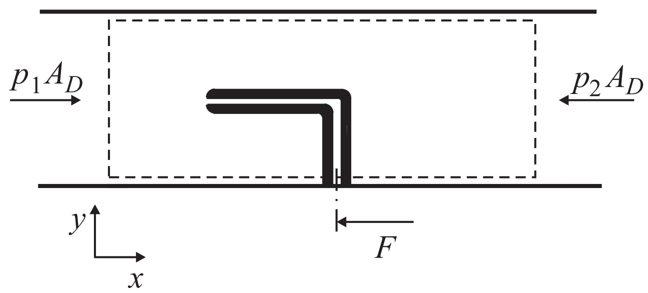
$$\text{point } \boxed{1} \ :\ r \ = \ 0 \ \Longrightarrow \ p_1 \ = \ p_a - \frac{\rho\, \omega_r^2}{2}\, R^2$$

7 Momentum and momentum of momentum equation

7.1

Force equilibrium in the u-tubes:

$$
\begin{aligned}
p_1 + \rho_{Hg} g \Delta h_1 &= p_{01} = p_2 + \rho_{Hg} g \Delta h_2 \\
\Longrightarrow p_1 - p_2 &= \rho_{Hg} g (\Delta h_2 - \Delta h_1)
\end{aligned}
$$

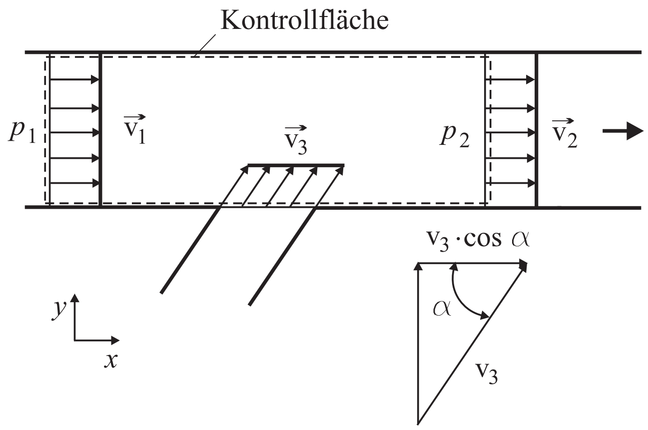


Momentum equation in x-direction: $\quad 0 = (p_1 - p_2)A_D - F \quad \Longrightarrow \quad F = (p_1 - p_2)A_D$

$$\Longrightarrow \quad F = \rho_{Hg} g (\Delta h_2 - \Delta h_1) \frac{\pi D^2}{4}$$

7.2

Momentum: $\quad \dfrac{dI_x}{dt} = -\rho v_1^2 A + \rho v_2^2 A - v_3 \cos\alpha \rho v_3 A_3 = (p_1 - p_2)A$



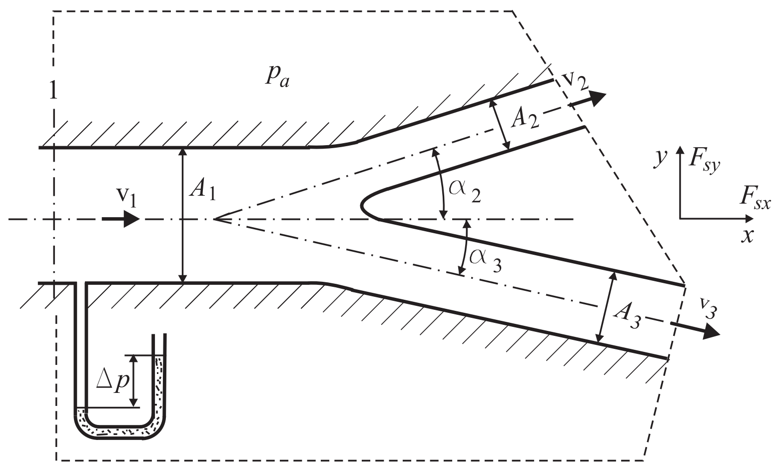
conti: $\quad v_1 A + v_3 A_3 = v_2 A \quad$ mit $\quad A_3 = \dfrac{1}{4} A$

$$\Longrightarrow \quad v_1 + \frac{1}{4} v_3 = v_2$$

$$\Longrightarrow \quad v_3 = 4(v_2 - v_1)$$

$$\Longrightarrow \quad p_2 - p_1 = \rho\,(v_1^2 - v_2^2 + 4(v_2 - v_1)^2 \cos\alpha)$$

7.3



a)

$$\begin{aligned} p_1 + \frac{\rho}{2}v_1^2 &= p_a + \frac{\rho}{2}v_2^2 = p_a + \frac{\rho}{2}v_3^2 \\ \Delta p &= p_1 - p_a \\ v_1 A_1 &= v_2 A_2 + v_3 A_3 \end{aligned}$$

$$\begin{aligned} v_1 &= \sqrt{\frac{2\Delta p}{\rho}\frac{1}{\left(\dfrac{A_1}{A_2+A_3}\right)^2 - 1}} = 2,58\ m/s \\ v_2 &= v_3 = \frac{A_1}{A_2+A_3}v_1 = 5,16\ m/s \end{aligned}$$

b) Momentum in x-direction:

$$\begin{aligned} -\rho v_1^2 A_1 + \rho v_2^2 A_2 \cos\alpha_2 + \rho v_3^2 A_3 \cos\alpha_3 &= (p_1 - p_a)A_1 + F_{sx} \\ \Longrightarrow \quad F_{sx} &= -866,4\ N \end{aligned}$$

Momentum in y-direction:

$$\begin{aligned} \rho v_2^2 A_2 \sin\alpha_2 - \rho v_3^2 A_3 \sin\alpha_3 &= F_{sy} \\ \Longrightarrow \quad F_{sy} &= -238,4\ N \end{aligned}$$

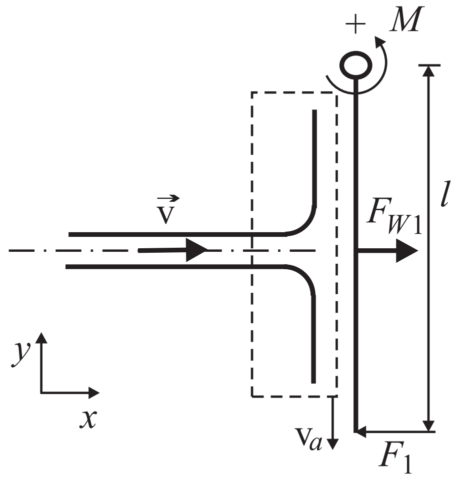
c)

$$A_2 \sin\alpha_2 - A_3 \sin\alpha_3^* = 0$$

$$\Longrightarrow \quad \alpha_3^* = 12,37^\circ$$

7.4

a)



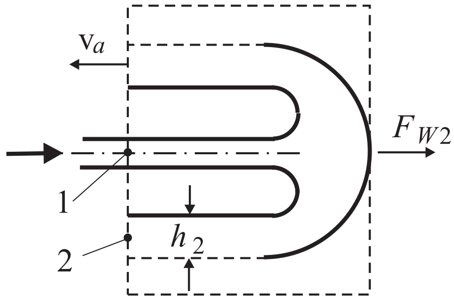
$$\sum \vec{M} = 0 \quad \Longrightarrow aF_{w1} - lF_1 = 0 \quad \Longrightarrow F_1 = \frac{a}{l}F_{w1}$$

Momentum: $\int \rho\vec{v}(\vec{v}\cdot\vec{n})dA = \Sigma F$

x-direction: $\vec{v}_a \perp \vec{v}; \quad \sum F = -F_{w1}; \quad p = p_a$

$$\Longrightarrow -\rho v^2 hB = -F_{w1} \Longrightarrow F_1 = \rho\frac{a}{l}v^2 hB$$

b)



Momentum in x-direction: $-\rho v^2 hB - 2\rho v_2^2 h_2 B = -F_{w2}$

Bernoulli: $1 \longrightarrow 2 \quad p_1 + \frac{\rho}{2}v_1^2 = p_2 + \frac{\rho}{2}v_2^2$

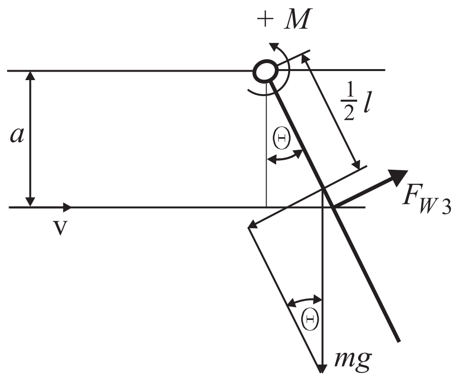
$$p_1 = p_2 = p_a; \quad v_1 = v \quad \Longrightarrow v_2 = v$$

conti: $vh = 2vh_2 \quad \Longrightarrow h_2 = \frac{1}{2}h$

$$\Longrightarrow F_{w2} = 2\rho v^2 hB = 2F_{w1}$$

$$\sum \vec{M} = 0 \Longrightarrow F_2 = \frac{a}{l}F_{w2} \Longrightarrow F_2 = 2\rho\frac{a}{l}v^2 hB = 2F_1$$

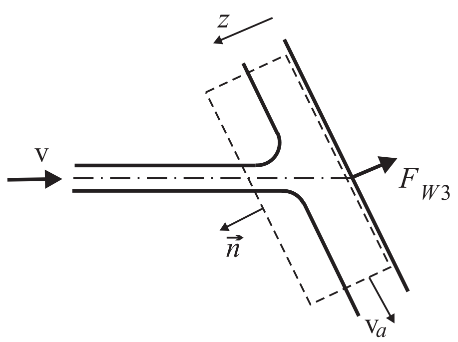
c)



$$\sum \vec{M} = 0 \Longrightarrow -\frac{l}{2}mg\sin\theta + \frac{a}{\cos\theta}F_{w3} = 0 \Longrightarrow F_{w3} = \frac{1}{2}\frac{l}{a}mg\sin\theta\cos\theta$$

Momentum in z-direction: $\rho v\cos\theta vhB = F_{w3}$

$$\Longrightarrow \rho v^2\cos\theta hB = \frac{1}{2}\frac{l}{a}mg\sin\theta\cos\theta$$
$$\Longrightarrow \sin\theta = \rho\frac{2a}{l}\frac{hB}{mg}v^2$$
$$\Longrightarrow \theta = \arcsin\left(\frac{2a}{l}\frac{\rho hBv^2}{mg}\right)$$

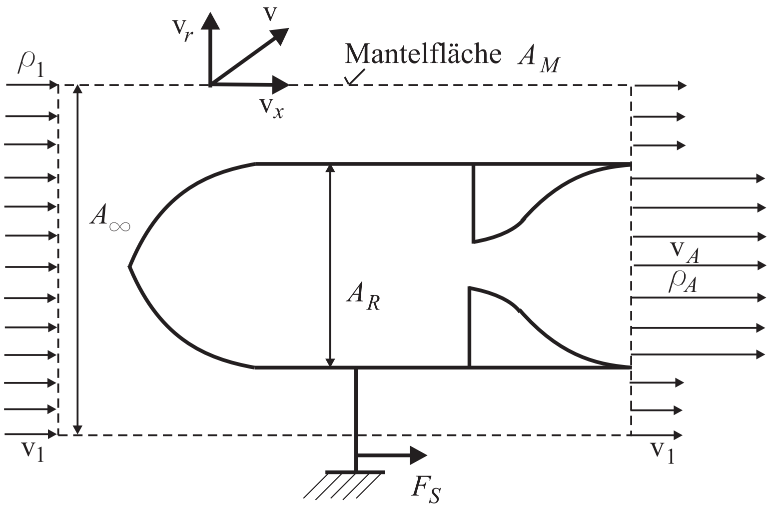


7.5

a) Conti:

$$\rho_1 v_1 A_\infty = \rho_1 v_1 (A_\infty - A_R) + \Delta\dot{m}$$

$$\Longrightarrow \quad \Delta\dot{m} = \rho_1 v_1 A_R$$



b) Momentum in x-direction

$$-\rho_1 v_1^2 A_\infty + \rho_1 v_1^2 (A_\infty - A_R) + \rho_A v_A^2 A_R + \int_{A_M} \rho_1 v_x v_r dA = F_s$$

Für $A_\infty / A_R \gg 1 : v_x = v_1$

$$\int_{A_M} \rho_1 v_x v_r dA = v_1 \int_{A_M} \rho_1 v_r dA = v_1 \Delta\dot{m}$$

$$\Longrightarrow \quad F_s = \rho_A v_A^2 A_R$$

$$\Longrightarrow \quad P = F_s v_1 = \rho_A v_A^2 v_1 A_R$$

7.6

a)

Bernoulli1 $\longrightarrow 1'$:

$$p_a + \frac{\rho}{2} v_1^2 = p_{1'} + \frac{\rho}{2} v'^2$$

Bernoulli2$'$ $\longrightarrow 2$:

$$p_{2'} + \frac{\rho}{2} v'^2 = p_a + \frac{\rho}{2} v_2^2$$

Konti :

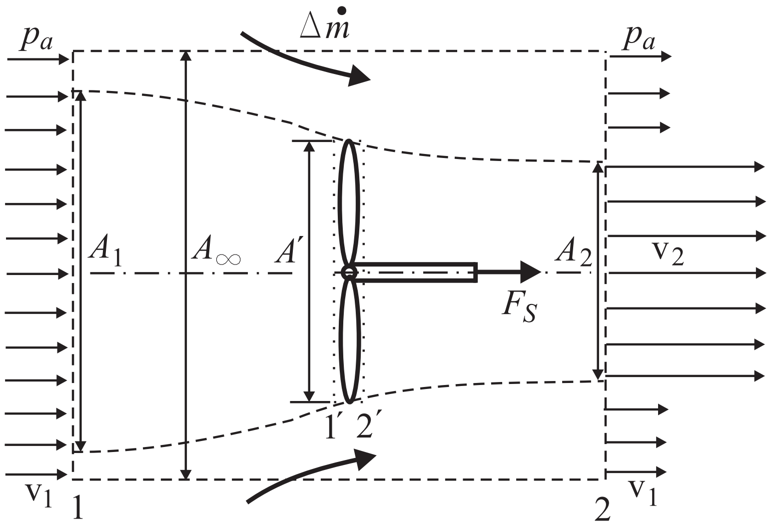
$$v_1 A_1 = v' A' = v_2 A_2$$

Momentum in $x-$ direction for the small surface :

$$0 = (p_{1'} - p_{2'}) A' + F_s$$

Momentum in $x-$ direction for the large control surface

$$-\rho v_1^2 A_\infty + \rho v_2^2 A_2 + \rho v_1^2 (A_\infty - A_2) - \Delta\dot{m} v_1 = F_s \quad \text{(see Exercise 7.5)}$$



conti: $\quad \rho v_1 A_\infty + \Delta\dot{m} = \rho v_2 A_2 + \rho v_1 (A_\infty - A_2)$

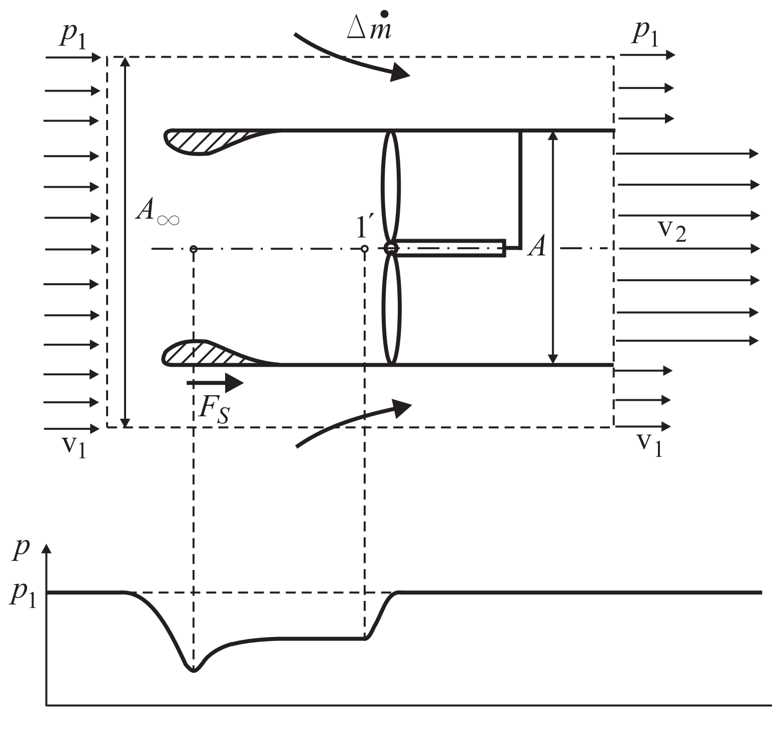
$$\Longrightarrow \quad v' = \frac{v_1 + v_2}{2} = 6,5\ m/s$$

b)

$$\eta = \frac{F_s v_1}{F_s v'} = \frac{v_1}{v'} = 0,769$$

7.7

a)



b) Bernoulli 1 $\longrightarrow$ 1':

$$\begin{aligned} p_1 + \frac{\rho}{2} v_1^2 &= p_{1'} + \frac{\rho}{2} v_{1'}^2 \\ \Longrightarrow \quad v_{1'} = v_2 &= \sqrt{\frac{2}{\rho}(p_1 - p_{1'}) + v_1^2} \\ \dot{m} = \rho A v_{1'} &= 13 \cdot 10^3 \ kg/s \end{aligned}$$

c) Momentum in x-direction:

$$-\rho v_1^2 A_\infty - \Delta\dot{m} v_1 + \rho v_2^2 A + \rho v_1^2 (A_\infty - A) = F_s \qquad \text{(see Exercise 7.5)}$$

Konti: $$\rho v_1 A_\infty + \Delta\dot{m} = \rho v_1 (A_\infty - A) + \rho v_2 A$$

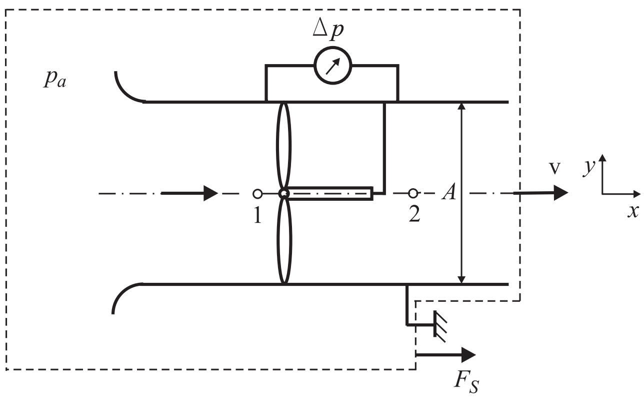
$$\Longrightarrow \quad F_s = \rho v_2 (v_2 - v_1) A = 0,39 \cdot 10^5 \ N$$

d)

$$P = \dot{V}(p_{02} - p_{01'}) = \dot{V}(p_1 - p_{1'}) = 448,5 \ kW$$

7.8
well rounded inlet

a) Bernoulli $\infty \longrightarrow 1$:



$$\begin{aligned} p_a &= p_1 + \frac{\rho}{2} v_1^2 \\ \Delta p &= p_2 - p_1 = p_a - p_1 \\ \Longrightarrow \dot{V} &= vA = \sqrt{\frac{2\Delta p}{\rho}} A \end{aligned}$$

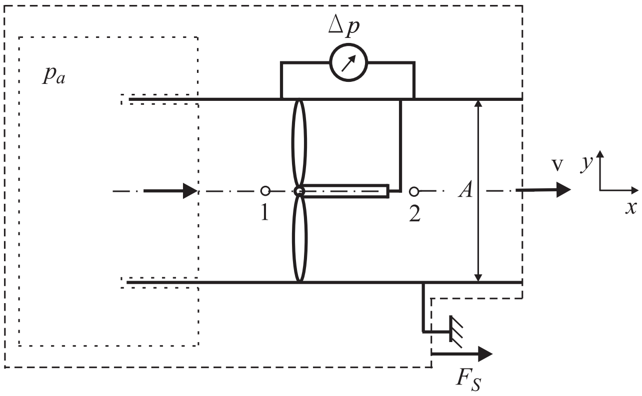
b)

$$P = \dot{V}(p_{02} - p_{01}) = \sqrt{\frac{2\Delta p}{\rho}}\Delta p A$$

c) Momentum in x-direction:

$$\rho v^2 A = F_s = 2\Delta p A$$

sharp edged inlet



a) Momentum in x-direction for the dotted control surface:

$$\begin{aligned} \rho v^2 A &= (p_a - p_1)A \\ \Delta p &= p_2 - p_1 = p_a - p_1 \\ \Longrightarrow \quad \dot{V} &= vA = \sqrt{\frac{\Delta p}{\rho}}A \end{aligned}$$

b)

$$P = \dot{V}(p_{02} - p_{01}) = \sqrt{\frac{\Delta p}{\rho}}\Delta p A$$

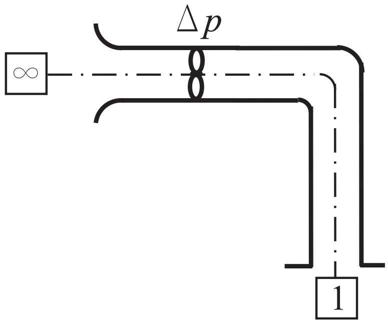
c) Momentum in x-direction for the dashed line

$$\rho v^2 A = F_s = \Delta p A$$

7.9

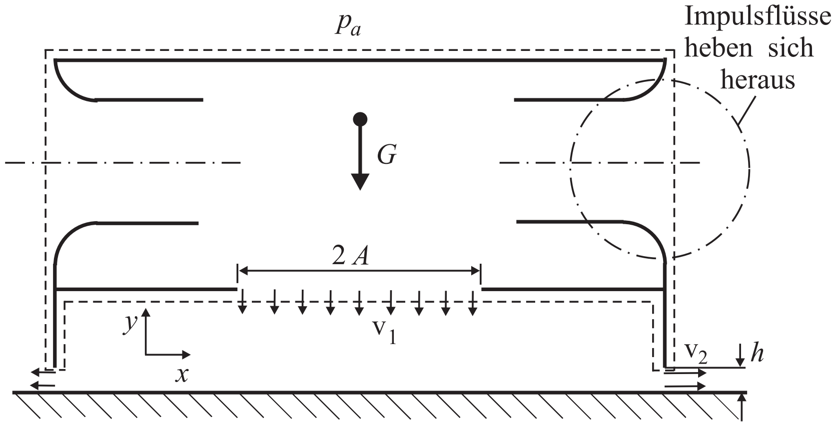
a)

Bernoulli $\infty \Longrightarrow 1$: $\qquad p_a = \frac{\rho}{2}v_1^2 + p_1 - \Delta p$



Momentum in y-direction: $$\frac{dI_y}{dt} = -\rho v_1^2 2A = (p_1 - p_a)BL - G$$

$$\Longrightarrow \rho v_1^2 = \frac{1}{2A}\left(G - (p_1 - p_a)BL\right)$$



$$\Longrightarrow p_a = \frac{1}{4A}\left(G - (p_1 - p_a)BL\right) + p_1 - \Delta p$$
$$\Longrightarrow p_a\left(1 - \frac{BL}{4A}\right) = p_1\left(1 - \frac{BL}{4A}\right) - \Delta p + \frac{G}{4A}$$
$$\Longrightarrow p_1 = p_a + \frac{\Delta p - \frac{G}{4A}}{1 - \frac{BL}{4A}} = p_a + \frac{1}{4}\left(\frac{G}{4A} - \Delta p\right)$$
$$\Longrightarrow \frac{\rho}{2}v_1^2 = p_a - p_1 + \Delta p \quad \Longrightarrow v_1 = \sqrt{\frac{1}{2\rho}\left(5\Delta p - \frac{1}{4}\frac{G}{A}\right)}$$
$$\dot V = v_1 2A = 2A\sqrt{\frac{1}{2\rho}\left(5\Delta p - \frac{1}{4}\frac{G}{A}\right)}$$

b)

Power: $P = \Delta p \dot V$

pressure loss: $\Delta p_v = p_{o1} - p_{o2} = p_1 + \frac{\rho}{2}v_1^2 - p_a - \frac{\rho}{2}v_2^2$

conti: $$2Av_1 = 2h(B+L)v_2 \quad \Longrightarrow v_1 = v_2$$
$$\Longrightarrow \Delta p_v = p_1 - p_a = \frac{1}{4}\left(\frac{G}{4A} - \Delta p\right)$$

$$P_v = \Delta p_v \dot{V} = \frac{\dot{V}}{4}\left(\frac{G}{4A} - \Delta p\right)$$

7.10

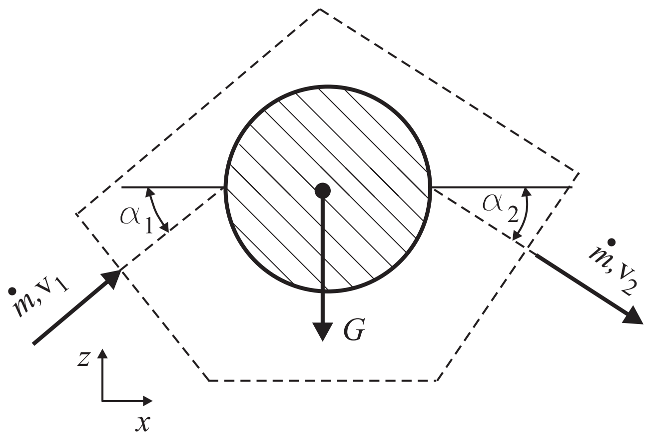
In frictionless flow the Bernoulli equation is valid. If the distance from the ball is large enough a constant flow velocity is assumed. In a constant velocity field the pressure gradient across the flow is zero.

a) Bernoulli $\infty \longrightarrow 1 \longrightarrow 2 \longrightarrow \infty$:

$$p_a + \rho\frac{v_1^2}{2} = p_1 + \rho\frac{v_1^2}{2} = p_2 + \rho\frac{v_2^2}{2} = p_a + \rho\frac{v_2^2}{2}$$

$$\Longrightarrow v_1 = v_2$$

b)



Momentum in x-direction:

$$-\dot{m}v_1\cos\alpha_1 + \dot{m}v_2\cos\alpha_2 = \sum F_{p_x} + F_{st_x} = 0$$

$$p = p_a$$

$$\Longrightarrow v_1 = v_2 \qquad \Longrightarrow \alpha_1 = \alpha_2,$$

with the absolute values $\alpha_{1,2}$.

c)

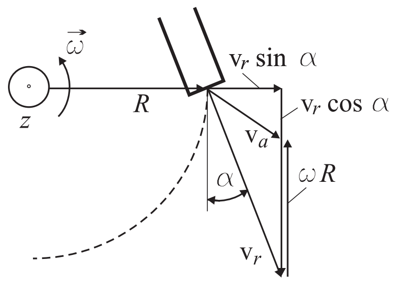
Momentum in y-direction:

$$-\dot{m}v_1\sin\alpha_1 + (\dot{m}v_2(-\sin\alpha_1)) = -G$$

$$\Longrightarrow \quad \dot{m} = \frac{G}{2v_1 \sin \alpha_1}$$

7.11

a)

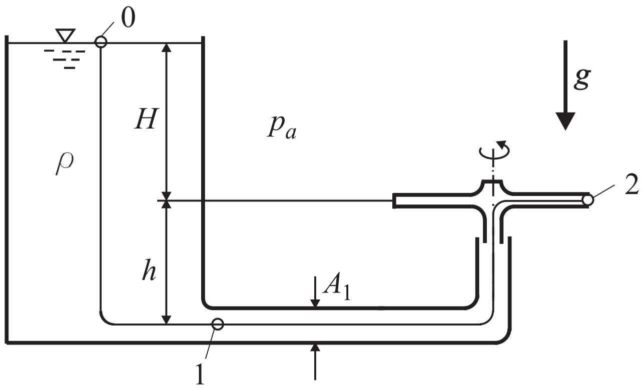


$\vec{v}_a = v_{absolut} \qquad \vec{v}_r = v_{relativ} \qquad \vec{\omega} \cdot R = v_{\text{Fahrzeug}} \qquad \vec{v}_a = \vec{v}_r + \vec{\omega} R$

v_r Bernoulli with acceleration term

$$0 \quad \rightarrow 2: \quad p_a = p_a + \frac{\rho}{2} v_r^2 - \int_{s0}^{s2} \rho(\vec{b} \cdot d\vec{s}) = p_a + \frac{\rho}{2} v_r^2 - \rho \left(gH + \frac{w^2 R^2}{2} \right)$$

$$\Longrightarrow \quad v_r = \sqrt{2gH + w^2 R^2} = 16 \frac{m}{s}$$



b) Steady flow in a moving coordinate system.

$$\Longrightarrow \int_{KF} (\vec{r} \times \vec{v}) \rho \vec{v} \cdot \vec{n} dA = \Sigma \vec{M} = \Sigma \left(\vec{r} \times \vec{F} \right)_{KV}$$

$$\Longrightarrow \vec{M} = \int_{KF} (\vec{r} \times \vec{v}_a) \, \rho \vec{v}_r \cdot \vec{n} dA$$

$$|\vec{r} \times \vec{v}_a| = |R(\omega R - v_r \cos \alpha)|$$

torsional moment for 3 arms:

$$\Longrightarrow M = 3 \rho v_r A R \left(\omega R - v_r \cos \alpha \right) = -6.8 \ Nm$$

The torsional moment $\vec{M}$ operates in the direction of $\vec{\omega}$. $\vec{F}$ and $\vec{v}$ are in opposite direction.

$$\dot{V} = 3v_r A = 2.4 \cdot 10^{-3}\ \frac{m^3}{s}$$

c) Bernoulli 0 $\longrightarrow$ 1:

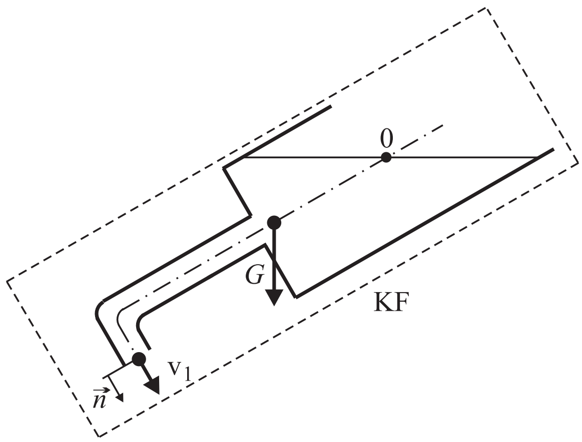
$$\begin{aligned}
p_a + \rho g(h+H) &= p_1 + \frac{\rho}{2}v_1^2 \\
v_1 &= \frac{\dot{V}}{A_1} \\
\Longrightarrow p_1 = p_a + \rho g(h+H) - \frac{\rho}{2}\frac{\dot{V}^2}{A_1^2} &= 0.82 \cdot 10^5\ \frac{N}{m^2}
\end{aligned}$$

d)

$$\frac{dM}{d\omega} = 0 \quad \Longrightarrow \text{maximum for } \omega = 0 \quad \Longrightarrow \quad v_r = \sqrt{2gH}$$

$$\Longrightarrow M = 3\rho\sqrt{2gH}AR\left(-\sqrt{2gH}\cos\alpha\right) = -6\rho gHAR\cos\alpha = 13\ Nm$$

7.12



Momentum of momentum: $\int_{KF} \rho(\vec{r} \times \vec{v})\vec{v} \cdot \vec{n} dA = \Sigma\vec{M}$

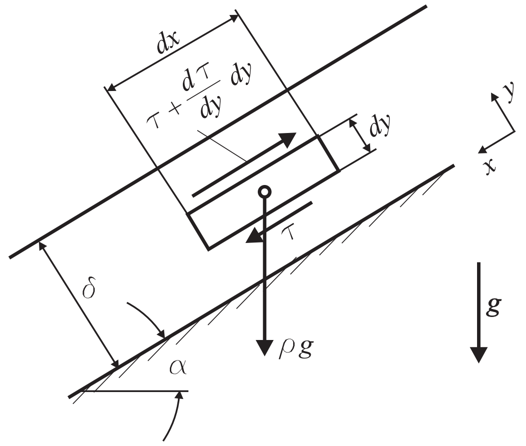
$$\Longrightarrow \rho l v_1^2 A = Ghg\sin\alpha$$

Bernoulli 0 $\longrightarrow$ 1 : $\rho g l \cos\alpha = \frac{\rho}{2}v_1^2$

$$
\begin{aligned}
&\Longrightarrow \quad v_1 = \sqrt{2gl\cos\alpha} \\
&\Longrightarrow \quad A\rho l 2gl\cos\alpha = Ghg\sin\alpha \\
\\
&\Longrightarrow \quad Gh\tan\alpha = 2\ \rho\ l^2\ A \\
&\Longrightarrow \quad \alpha = \arctan\left(2\frac{A\rho l^2}{Gh}\right)
\end{aligned}
$$

9 Laminar viscous flow

9.1



Volume flux : $\dot{V} = B\int_0^\delta u(y)dy$

Momentum in $x-$direction : $\frac{d\tau}{dy} = \rho g \sin\alpha$

$$\tau = -\eta\frac{du}{dy}$$

boundary conditions : $y = 0: \quad u = 0$

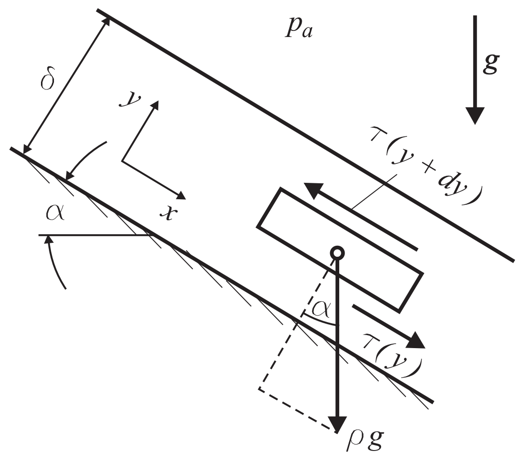
$$y = \delta: \quad \tau = 0$$

$$\Longrightarrow \quad u(y) = \frac{\rho g \sin\alpha}{\eta}\left(\delta y - \frac{y^2}{2}\right)$$

$$\Longrightarrow \quad \dot{V} = \frac{\rho g B \sin\alpha}{3\eta}\delta^3 = 1,2\cdot 10^{-3} \quad m^3/s$$

9.2

a)



Momentum in x-direction: $-\frac{\partial\tau}{\partial y} + \rho g \sin\alpha = 0$

with $\rho = \rho_0 e^{-\frac{T}{T_0}}$ and $\dfrac{T}{T_0} = \dfrac{y}{\delta}\dfrac{T_\delta - T_0}{T_0} + 1$

and the constant $k = -\dfrac{T_\delta - T_0}{T_0}$ follows

$$\frac{\partial \tau}{\partial y} = \rho_0 \frac{e^{\frac{y}{\delta}k}}{e} g \sin\alpha$$

Integration: $\tau = \dfrac{\rho_0 g \delta \sin\alpha}{ke} e^{\frac{y}{\delta}k} + C(x)$

Boundary condition: $\tau(y = \delta) = 0 \implies C = \dfrac{-\rho_0 \delta g \sin\alpha}{ke} \quad e^k$

$$\implies \tau(y) = \frac{\rho_0 \delta g \sin\alpha}{ke}\left(e^{\frac{y}{\delta}k} - e^k\right)$$

b)

$$\tau = -\eta \frac{\partial u}{\partial y} = -\rho\nu \frac{\partial u}{\partial y} = \frac{\rho_0 g \delta \sin\alpha}{ke}\left(e^{\frac{y}{\delta}k} - e^k\right)$$

$$\frac{\partial u}{\partial y} = \frac{-g\delta \sin\alpha}{k\nu}\left(1 - e^{k(1-\frac{y}{\delta})}\right)$$

Integration : $u = \dfrac{-g\delta\sin\alpha}{k\nu}\left(y + \dfrac{\delta}{k}e^{k(1-\frac{y}{\delta})}\right) + C(x)$

Boundary condition: $u(y = 0) = 0 \implies C = \dfrac{g\delta^2 \sin\alpha}{k^2\nu} e^k$

$$\implies u(y) = -\frac{g\delta\sin\alpha}{k\nu}\left(y + \frac{\delta}{k}(e^{k(1-\frac{y}{\delta})} - e^k)\right)$$

c)

Momentum in y-direction: $-\dfrac{\partial p}{\partial y} = \rho g \cos\alpha$

Integration: $p = f(y) + C(x)$

Boundary condition: $p(y = \delta) = p_a \implies C = p_a + f(y) \neq f(x)$

$$\implies \frac{\partial p}{\partial x} = 0$$

9.3

a)

PDE : $$\frac{dp}{dx} = \eta \frac{d^2u}{dy^2}$$

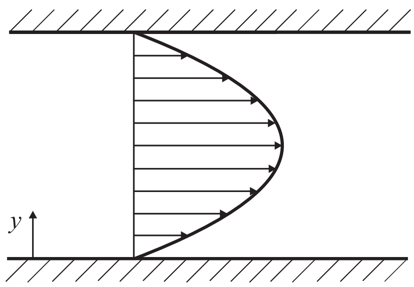
1. Integration : $$\frac{du}{dy} = \frac{1}{\eta}\frac{dp}{dx}y + C_1$$

2. Integration : $$u = \frac{1}{2\eta}\frac{dp}{dx}y^2 + C_1 y + C_2$$

1) non-moving bend

Boundary conditions

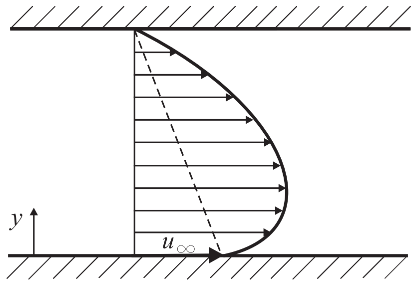
$$\begin{aligned} y=0: \quad u &= 0 \quad \longrightarrow \quad C_2 = 0 \\ y=h: \quad u &= 0 \quad \longrightarrow \quad C_1 = -\frac{h}{2\eta} \quad \frac{dp}{dx} \\ \Longrightarrow \quad u(y) &= \frac{h^2}{2\eta} \quad \frac{dp}{dx}\left((\frac{y}{h})^2 - \frac{y}{h}\right) \end{aligned}$$



2) moving bend

Boundary conditions:

$$\begin{aligned} y=0: \quad u &= u_\infty \quad \longrightarrow \quad C_2 = u_\infty \\ y=h: \quad u &= 0 \quad \longrightarrow \quad C_1 = -\left(\frac{h}{2\eta} \quad \frac{dp}{dx} + \frac{u_\infty}{h}\right) \\ \Longrightarrow \quad u(y) &= \frac{h^2}{2\eta} \quad \frac{dp}{dx}\left((\frac{y}{h})^2 - \frac{y}{h}\right) + u_\infty(1 - \frac{y}{h}) \end{aligned}$$



b)

Friction per unit width at the bottom: $\frac{F}{B} = l\eta \left.\frac{du}{dy}\right|_{y=h}$

1) non-moving bend

$$\frac{F}{B} = \eta l \left\{ \frac{1}{\eta} \quad \frac{dp}{dx} h - \frac{h}{2\eta} \quad \frac{dp}{dx} \right\}$$

$$\Longrightarrow \quad \frac{F}{B} = \frac{lh}{2}\frac{dp}{dx}$$

2) moving bend

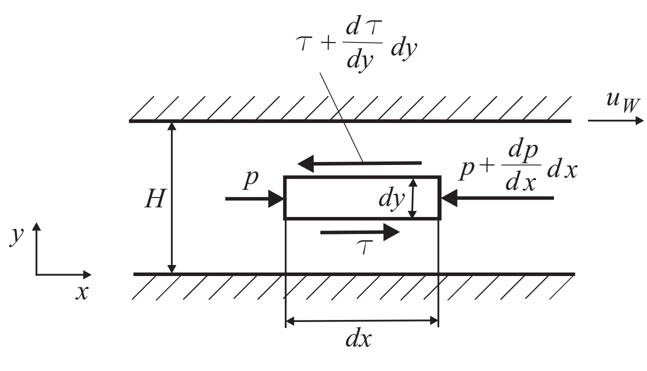
$$\frac{F}{B} = \eta l \left\{ \frac{1}{\eta}\frac{dp}{dx} h - \frac{h}{2\eta}\frac{dp}{dx} - \frac{u_\infty}{h} \right\}$$

$$\Longrightarrow \quad \frac{F}{B} = \frac{lh}{2}\frac{dp}{dx} - \eta \frac{l}{h} u_\infty$$

$$\Longrightarrow \quad |F|_{2)} \quad > \quad |F|_{1)}$$

9.4

a)



Momentum in $x-$direction :

$$\frac{dp}{dx} + \frac{d\tau}{dy} = 0$$

$$\tau = -\eta \frac{du}{dy}$$

$$\Longrightarrow \quad \frac{d^2u}{dy^2} = \frac{1}{\eta}\,\frac{dp}{dx}$$

Boundarycondition :

$$y = 0 : \quad u = 0$$

$$y = H : \quad u = u_w$$

Integration :

$$u(y) = \frac{1}{2\eta}\frac{dp}{dx}H^2\left[\left(\frac{y}{H}\right)^2 - \frac{y}{H}\right] + u_w\frac{y}{H}$$

b)

$$\frac{\tau(y=H)}{\tau(y=0)} = \frac{u_w + \dfrac{1}{2\eta}\dfrac{dp}{dx}H^2}{u_w - \dfrac{1}{2\eta}\dfrac{dp}{dx}H^2}$$

c)

$$\dot{V} = u_m BH = B\int_0^H u(y)dy = \left(\frac{u_w}{2} - \frac{dp}{dx}\frac{H^2}{12\eta}\right)BH$$

d)

$$u_{max} = -\frac{dp}{dx}\frac{H^2}{8\eta}$$

e)

$$\frac{dI_x}{dt} = B\int_0^H \rho u(y)^2 dy = \frac{6}{5}\rho u_m^2 BH$$

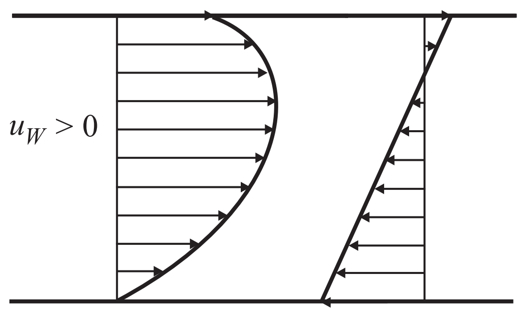
f)

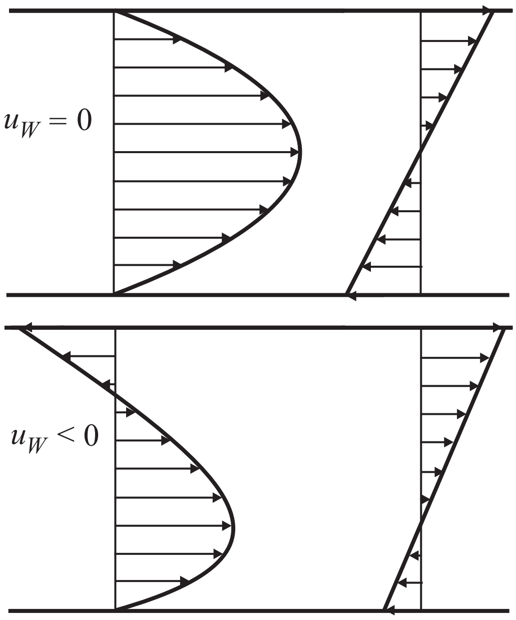
$$\tau_w = \eta\,\frac{\partial u}{\partial y} \qquad \text{using u from a)}$$

$$u_m = \frac{2}{3}\,u_{max} \qquad \text{see script p. 140}$$

$$\text{and dimensionless:} \quad \frac{\tau_w}{\frac{\rho}{2}u_m^2} = \frac{12}{Re}$$

g)





9.5

a) Computation of $\tau(r)$

$$\text{fully developed :} \qquad \frac{dp}{dx} = const. = \frac{\Delta p}{L} \Longrightarrow \frac{d(r\tau)}{dr} = r\frac{\Delta p}{L}$$

$$\Longrightarrow \quad d(r\tau) = r\frac{\Delta p}{L}dr$$

$$\text{Integration :} \qquad r\tau = c + \frac{r^2 \Delta p}{2L}$$

$$\Longrightarrow \quad \tau = \frac{c}{r} + \frac{r\Delta p}{2L}$$

boundary condition: $\quad r \longrightarrow 0 \Longrightarrow \tau \longrightarrow 0 \Longrightarrow c = 0$

$$\Longrightarrow \tau = \frac{r}{2}\frac{\Delta p}{L}$$

Computation of $u(y)$

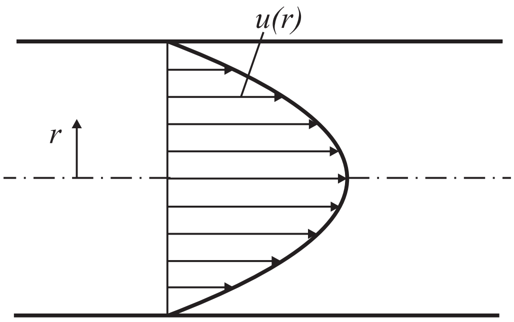
$$\tau = -\eta_{OdW}\left|\frac{du}{dr}\right|\frac{du}{dr} = \frac{r\Delta p}{2L}$$

$$\Longrightarrow \quad \left|\frac{du}{dr}\right|\frac{du}{dr} = -\frac{r\Delta p}{2\eta L}$$

for a given r is $\dfrac{du}{dr} < 0$

$$\Longrightarrow \frac{du}{dr} = -\sqrt{\frac{r\Delta p}{2\eta_{OdW}L}} \Longrightarrow du = -\sqrt{\frac{\Delta p}{2\eta_{OdW}L}}\sqrt{r}dr$$

$$\Longrightarrow \quad u = c - \sqrt{\frac{\Delta p}{2\eta_{OdW}L}}\frac{2}{3}r^{\frac{3}{2}}$$



boundary condition: $u(r=R)=0 \quad \Longrightarrow \quad c=\frac{2}{3}\sqrt{\frac{\Delta p}{2\eta_{OdW}L}}R^{\frac{3}{2}}$

$$\Longrightarrow \quad u(r)=\frac{1}{3}\sqrt{\frac{2\Delta p}{\eta_{OdW}L}}\left(R^{\frac{3}{2}}-r^{\frac{3}{2}}\right)$$

Computation of $\dot{V}$

$$\dot{V}=\int_0^R u(r)2\pi r dr=\frac{2\pi}{3}\sqrt{\frac{2\Delta p}{\eta_{OdW}l}}\int_0^R\left(R^{\frac{3}{2}}r-r^{\frac{5}{2}}\right)dr=\frac{2\pi}{3}\sqrt{\frac{2\Delta p}{\eta_{OdW}l}}\left(\frac{R^{\frac{7}{2}}}{2}-\frac{2R^{\frac{7}{2}}}{7}\right)$$

$$\Longrightarrow \quad \dot{V} \;=\; \frac{\pi}{7}\sqrt{\frac{2\Delta p}{\eta_{OdW}L}}R^{\frac{7}{2}}$$

$$\Longrightarrow \quad \Delta p \;=\; \left(\frac{7\dot{V}}{\pi R^{\frac{7}{2}}}\right)^2\frac{\eta_{OdW}L}{2}$$

9.6

a)

$$r \;=\; R-y$$

$$\dot{I}=\frac{dI_x}{dt}=\int_o^R \rho u^2 2\pi r dr=2\rho u_m^2\pi R^2\int_0^1\left(\frac{u}{u_m}\right)^2\frac{r}{R}d\left(\frac{r}{R}\right)$$

$$\delta \;=\; 0: \quad \frac{u}{u_m}=1$$

$$\Longrightarrow \quad \dot{I} \;=\; \rho u_m^2\pi R^2$$

$$\delta \;=\; R: \quad \frac{u}{u_m}=2\left[1-\left(\frac{r}{R}\right)^2\right]$$

$$\Longrightarrow \quad \dot{I} \quad = \quad 1,33\rho u_m^2 \pi R^2$$

$$\delta \quad = \quad \frac{R}{2}: \quad \frac{u}{u_m} = \begin{cases} \dfrac{96}{17}\dfrac{r}{R}\left(1-\dfrac{r}{R}\right) & \dfrac{R}{2} \leq r \leq R \\ \dfrac{24}{17} & 0 \leq r \leq \dfrac{R}{2} \end{cases}$$

$$\Longrightarrow \dot{I} = 2\rho u_m^2 \pi R^2 \left\{ \int_0^{0,5} \left(\frac{24}{17}\right)^2 \frac{r}{R} d\left(\frac{r}{R}\right) + \int_{0,5}^{1} \left[\frac{96}{17}\frac{r}{R}\left(1-\frac{r}{R}\right)\right]^2 \frac{r}{R} d\left(\frac{r}{R}\right) \right\} = 1,196\rho u_m^2 \pi R^2$$

b)

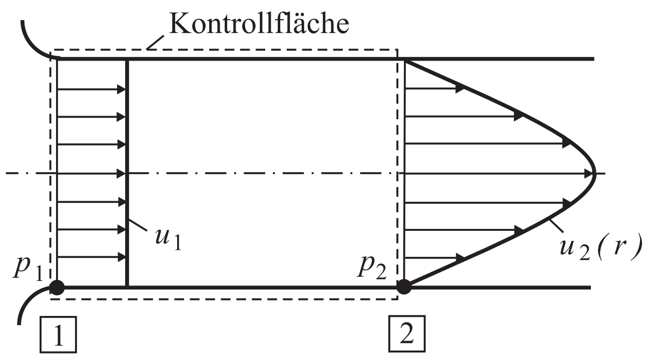
$$\tau_w = \eta u_m \frac{2/\delta}{1 - \dfrac{2}{3}\dfrac{\delta}{R} + \dfrac{1}{6}\left(\dfrac{\delta}{R}\right)^2}$$

$$\delta \longrightarrow 0: \quad \tau_w \longrightarrow \infty$$

$$\delta = R: \quad \tau_w = 4\frac{\eta \cdot u_m}{R}$$

$$\delta = \frac{R}{2}: \quad \tau_w = 5,65\frac{\eta \cdot u_m}{R}$$

9.7



Momentum in x-direction:

$$-\rho u_1^2 \pi R^2 \quad + \quad \int_0^R \rho u_2^2(r) 2\pi r dr = (p_1 - p_2)\pi R^2$$

$$\Longrightarrow \quad -\rho u_1^2 \pi R^2 \quad + \quad \rho 2\pi u_{2max}^2 \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right]^2 r dr = (p_1 - p_2)\pi R^2$$

$$\Longrightarrow \quad -\rho u_1^2 \pi R^2 \quad + \quad \rho \pi u_{2max}^2 \frac{1}{3} R^2 = (p_1 - p_2)\pi R^2$$

$$\Longrightarrow \quad p_1 - p_2 \quad = \quad \frac{1}{3} u_{2max}^2 \rho - \rho u_1^2$$

Continuity:

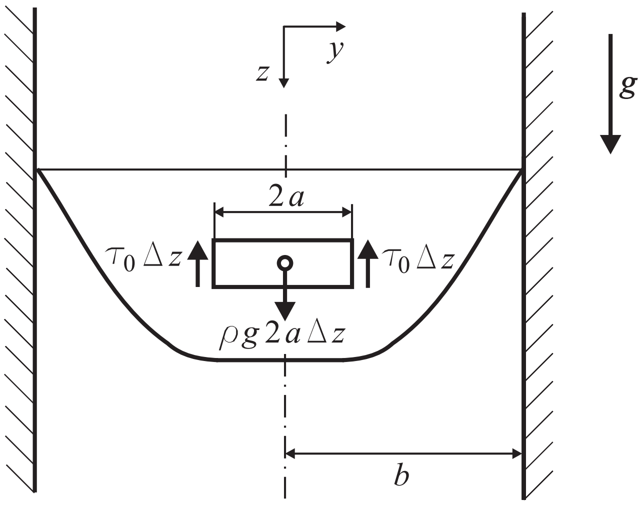
$$\begin{aligned} \bar{u}\pi R^2 &= u_1 \pi R^2 = \int_0^R u_2(r) 2\pi r dr = \frac{u_{2max}}{2}\pi R^2 \\ \Longrightarrow \quad u_1 &= \bar{u} = \frac{1}{2} u_{2max} \\ \Longrightarrow \quad p_1 - p_2 &= \frac{1}{3}\rho 4\bar{u}^2 - \rho\bar{u}^2 = \frac{1}{3}\rho\bar{u}^2 \end{aligned}$$

b)

$$\zeta_E = \frac{p_1 - p_2}{\frac{\rho}{2} \cdot \bar{u}^2} = \frac{2}{3}$$

9.8

a)



For $y \leq a$ the fluid behaves like a rigid body

equilibrium of forces :

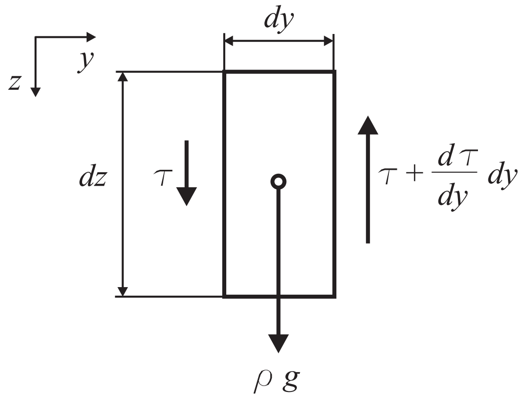
$$\begin{aligned} \rho g a \Delta z &= \tau_0 \Delta z \\ \Longrightarrow \quad a &= \frac{\tau_0}{\rho g} \end{aligned}$$

b)

$$a \;\leq\; y \;\leq\; b :$$

Momentum in $z-$direction :

$$\begin{aligned} \frac{d\tau}{dy} &= \rho g \\ \tau &= -\eta \frac{dw}{dy} + \tau_0 \end{aligned}$$



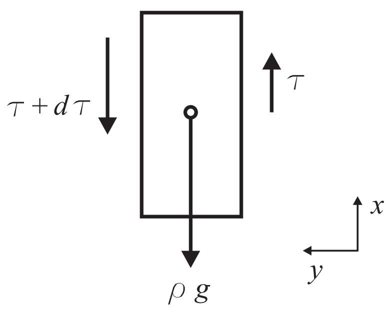
Boundary conditions : $\quad y = a : \quad \tau = \tau_0$

$\qquad\qquad\qquad\qquad\quad\; y = b : \quad w = 0$

$$\begin{aligned} \text{Integration :}\quad w(y) &= \frac{\rho g}{2\eta}[(b-a)^2 - (y-a)^2] \\ 0 \le y \le a :\quad w(y) &= \frac{\rho g}{2\eta}(b-a)^2 \end{aligned}$$

9.9

a)



Force equilibrium:

$$\tau dxB - \rho g dxdyB - (\tau + \frac{d\tau}{dy}\, dy)\, dxB = 0$$

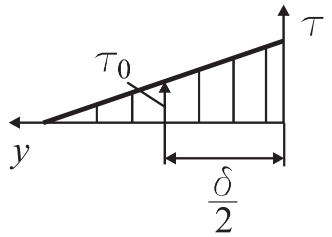
$$\Longrightarrow \quad \frac{d\tau}{dy} = -\rho g$$

Integration: $\qquad \tau = -\rho g y + C_1$

Boundary condition: $\quad y = \delta : \quad \tau = 0 \quad \Longrightarrow \quad \tau = \rho g(\delta - y)$

b)

$$\frac{du}{dy} < 0 \qquad \Longrightarrow \qquad \tau = \tau_0 - \eta \frac{du}{dy}$$

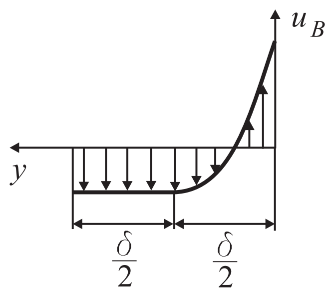


$$\begin{aligned}
\tau_0 - \eta\frac{du}{dy} &= \rho g(\delta - y) \\
\Longrightarrow \quad \frac{du}{dy} &= \frac{\tau_0 - \rho g(\delta - y)}{\eta} \quad \text{for} \quad |\tau| > |\tau_0| \\
\Longrightarrow \quad du &= \frac{\tau_0 - \rho g(\delta - y)}{\eta} dy
\end{aligned}$$

Integration : $$u(y) = \frac{\tau_0}{\eta}y - \frac{\rho g}{\eta}(\delta y - \frac{y^2}{2}) + C_2$$

Boundary condition: $y = 0 \quad : \quad u = u_B$

$$\Longrightarrow \quad u(y) = u_B + \frac{\rho g}{2\eta}(y^2 - \delta y) \quad \text{for} \quad |\tau| > |\tau_0|$$



Remark: $\frac{\delta}{2}$ from problem formulation

c)

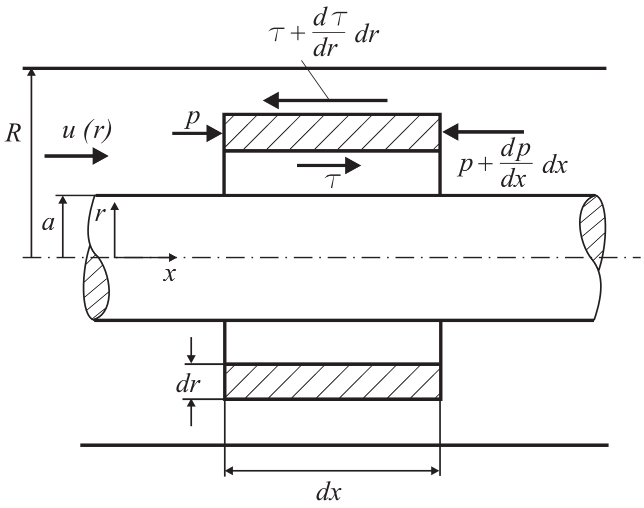
Assumption: $\dot{V} = 0 \quad \left(\text{with } [\dot{V}] = \frac{m^2}{s}\right) \quad$ oder: $\frac{\dot{V}}{B} = 0$

$$\dot{V} = \int_0^{\delta} u(y)dy = \int_0^{\frac{\delta}{2}} u(y)dy + \int_{\frac{\delta}{2}}^{\delta} u\left(\frac{\delta}{2}\right) dy = 0$$

$$\begin{aligned}
\text{with} \quad u\left(\frac{\delta}{2}\right) &= u_B - \frac{\rho g}{8\eta}\delta^2 \\
\int_0^{\frac{\delta}{2}} u(y)dy &= \int_0^{\frac{\delta}{2}} \left(u_B + \frac{\rho g}{2\eta}(y^2 - \delta y)\right) dy = u_B\frac{\delta}{2} + \frac{\rho g}{2\eta}\delta^3(\frac{1}{24} - \frac{1}{8}) \\
\int_{\frac{\delta}{2}}^{\delta} u(\frac{\delta}{2})dy &= \int_{\frac{\delta}{2}}^{\delta} \left(u_B - \frac{\rho g}{8\eta}\delta^2\right) dy = u_B\frac{\delta}{2} - \frac{\rho g}{\eta}\delta^3\frac{1}{16} \\
\Longrightarrow \quad \dot{V} &= u_B\delta - \frac{\rho g}{\eta}\delta^3\frac{5}{48} = 0 \quad \Longrightarrow u_{B,min} = \frac{\rho g}{\eta}\delta^2\frac{5}{48}
\end{aligned}$$

9.10

a)



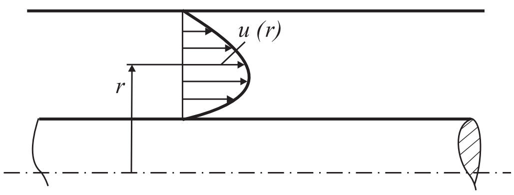
Momentum in $x-$direction :
$$\frac{dp}{dx} + \frac{1}{r}\frac{d(\tau r)}{dr} = 0$$

$$\tau = -\eta\frac{du}{dr}$$

$$\Longrightarrow \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) - \frac{1}{\eta}\frac{dp}{dx} = 0$$

Boundary conditions :
$$r = a \quad : \quad u = 0$$
$$r = R \quad : \quad u = 0$$

$$\Longrightarrow \quad u(r) = -\frac{1}{4\eta}\frac{dp}{dx}(R^2 - a^2)\left[\frac{R^2 - r^2}{R^2 - a^2} - \frac{ln(r/R)}{ln(a/R)}\right]$$



b)

$$\frac{\tau(r = a)}{\tau(r = R)} = \frac{R}{a}\frac{2a^2 \ln\dfrac{a}{R} + R^2 - a^2}{2R^2 \ln\dfrac{a}{R} + R^2 - a^2}$$

c)

$$
\begin{aligned}
u_m &= \frac{\dot{V}}{\pi(R^2-a^2)} = \frac{1}{\pi(R^2-a^2)} \int_a^R u(r) 2\pi r dr = \\
&= -\frac{1}{8\eta}\frac{dp}{dx}R^2\left[1+\left(\frac{a}{R}\right)^2 + \frac{1-(a/R)^2}{\ln(a/R)}\right]
\end{aligned}
$$

9.11

a)

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r\cdot v)\right] = 0$$

1. Integration: $\frac{1}{r}\frac{d}{dr}(rv) = c_1$

$$\Longrightarrow \quad d(rv) = c_1 r dr$$

2. Integration: $rv = \frac{1}{2}c_1 r^2 + c_2$

Boundary conditions:

1) $r = R_i: \quad v = 0$

2) $r = R_a: \quad v = \omega R_a$

1. BC: $0 = \frac{1}{2}c_1 R_i^2 + c_2 \quad \Longrightarrow \quad c_2 = -\frac{c_1}{2}R_i^2$

2. BC: $\omega R_a^2 = \frac{1}{2}c_1 R_a^2 + c_2$

$$
\begin{aligned}
\Longrightarrow \quad \omega R_a^2 &= \frac{1}{2}c_1 R_a^2 - \frac{c_1}{2}R_i^2 \\
\Longrightarrow \quad c_1 &= \frac{2\omega R_a^2}{R_a^2 - R_i^2} \\
\Longrightarrow \quad c_2 &= -\frac{\omega R_a^2 R_i^2}{R_a^2 - R_i^2}
\end{aligned}
$$

$$\begin{aligned}
\Longrightarrow \quad rv &= \frac{1}{2}r^2\frac{2\omega R_a^2}{R_a^2 - R_i^2} - \frac{\omega R_a^2 R_i^2}{R_a^2 - R_i^2} \\
\Longrightarrow \quad rv &= \frac{\omega(r^2 - R_i^2)R_a^2}{R_a^2 - R_i^2} \\
\Longrightarrow \quad v &= \frac{\omega R_a^2}{r}\frac{r^2 - R_i^2}{R_a^2 - R_i^2}
\end{aligned}$$

b)

$$\begin{aligned}
\tau &= -\eta r\frac{d}{dr}\left(\frac{v}{r}\right) \\
\Longrightarrow \quad \eta &= \frac{\tau}{-r\frac{d}{dr}\left(\frac{v}{r}\right)}
\end{aligned}$$

$$\begin{aligned}
M_z &= -\tau(r = R_i)2\pi R_i^2 L \\
\Longrightarrow \quad \eta &= \frac{M_z}{4\pi\omega R_i^2 L} \quad \left[1 - \left(\frac{R_i}{R_a}\right)^2\right] = 10^{-2}\frac{Ns}{m^2}
\end{aligned}$$

10 Turbulent pipe flows

10.1

The first is trivial and no proof is necessary. The time average of a quantity m in the time intervall $[0,\ T]$ is defined as

$$\bar{m} \ = \ \frac{1}{T} \cdot \int\limits_0^T \ m \cdot dt \quad .$$

Using this for the averaging of $f + g$ results in:

b) $$\overline{f \ + \ g} \ = \ \frac{1}{T} \cdot \int\limits_0^T \ (f + g) \cdot dt \ = \ \frac{1}{T} \cdot \int\limits_0^T \ f \cdot dt \ + \ \frac{1}{T} \cdot \int\limits_0^T \ g \cdot dt \ = \ \bar{f} \ + \ \bar{g} \quad .$$

c) $$\overline{\bar{f} \cdot g} \ = \ \frac{1}{T} \cdot \int\limits_0^T \ \bar{f} \cdot g \cdot dt \ = \ \bar{f} \cdot \frac{1}{T} \cdot \int\limits_0^T \ g \cdot dt \ = \ \bar{f} \cdot \bar{g} \quad .$$

d) $$\overline{\frac{\partial f}{\partial s}} \ = \ \frac{1}{T} \cdot \int\limits_0^T \ \frac{\partial f}{\partial s} \cdot dt \ = \ \frac{\partial}{\partial s} \left(\frac{1}{T} \cdot \int\limits_0^T \ f \cdot dt \right) \ = \ \frac{\partial \bar{f}}{\partial s} \quad .$$

e) $$\overline{\int\limits_s \ f \cdot ds} \ = \ \frac{1}{T} \cdot \int\limits_0^T \left(\int\limits_s \ f \cdot ds \right) \cdot dt \ = \ \frac{1}{T} \cdot \int\limits_s \left(\int\limits_0^T \ f \cdot dt \right) \cdot ds \ =$$

$$\int\limits_s \left(\frac{1}{T} \cdot \int\limits_0^T \ f \cdot dt \right) \cdot \ ds \ = \ \int\limits_s \ \bar{f} \cdot ds$$

10.2

The ratio between the average and the maximum velocity is

$$\frac{\bar{v}_m}{\bar{v}_{max}} \ = \ 2 \int\limits_0^1 \ \xi (1 \ - \ \xi)^{\frac{1}{n}} \ d\xi \ = \ \frac{2n^2}{(n+1)(2n+1)} \quad \text{mit} \quad \xi \ = \ \frac{r}{R} \quad .$$

The integral is solved using partial integration. The average velocity is at a distance

$$\frac{r_m}{R} = 1 - \left(\frac{\bar{v}_m}{\bar{v}_{max}}\right)^n$$

see table.

Re	n	$\bar{v}_m/\bar{v}_{max}$	r_m/R
$1 \cdot 10^5$	7	0.8166	0.7577
$6 \cdot 10^5$	8	0.8366	0.76
$1.2 \cdot 10^6$	9	0.8526	0.762
$2 \cdot 10^6$	10	0.8658	0.7633

Measuring $\bar{v}(r)$ at a distance $R - r_m$ from the wall, and with the known $\bar{v}_{max}$ the average velocity can be determined, and the volume flux $\dot{V} = v_m \ \pi \ R^2$ can be computed.

10.3

a)
$$r = R - y$$

$$\frac{\bar{u}_m}{\bar{u}_{max}} = \frac{\dot{V}}{\pi \ R^2 \ \bar{u}_{max}} = 2 \int_0^1 \left(1 - \frac{r}{R}\right)^{1/7} \frac{r}{R} \ d\left(\frac{r}{R}\right) = \frac{49}{60}$$

b)
$$\frac{\dot{I}}{\rho \ \bar{u}_m^2 \ \pi \ R^2} = \frac{\int_0^R \rho \ \bar{u}^2 \ 2\pi \ r \ dr}{\rho \ \bar{u}_m^2 \ \pi \ R^2} = 2\left(\frac{\bar{u}_{max}}{\bar{u}}\right)^2 \int_0^1 \left(1 - \frac{r}{R}\right)^{2/7} \frac{r}{R} \ d\left(\frac{r}{R}\right) = \frac{50}{49}$$

10.4

a) $2300 \leq Re \leq 10^5$:
$$\lambda = \frac{0,316}{\sqrt[4]{Re}}$$

$$\bar{u}_m = \frac{\eta}{\rho \ D} \ Re$$

$$\lambda = \frac{8 \ \tau_w}{\rho \ \bar{u}_m^2} \implies \tau_w = \frac{\lambda \ \rho \ \bar{u}_m^2}{8} = 2,22 \ N/m^2$$

b)
$$\frac{\bar{u}_m}{u_*} = \frac{\bar{u}_{max}}{u_*} - 4,07$$

$$\lambda = \frac{8 \ \tau_w}{\rho \ \bar{u}_m^2} = 8\left(\frac{u_*}{\bar{u}_m}\right)^2$$

$$\Longrightarrow \quad \frac{\bar{u}_m}{\bar{u}_{max}} \;=\; \frac{1}{1 \;+\; 4,07\,\sqrt{\lambda/8}} \;=\; 0,84$$

c) $$\frac{y u_*}{\nu} \;=\; 5 \;=\; \frac{\bar{u}}{u_*} \quad \text{(viscous sublayer)} \quad \Longrightarrow \quad \bar{u} \;=\; 5\, u_*$$

$$\text{with} \quad u_* \;=\; \sqrt{\frac{\tau_w}{\rho}} \quad \text{and} \quad \tau_w \;=\; \frac{\lambda\;\rho\;\bar{u}_m^2}{8} \quad \text{follows}$$

$$\bar{u} \;=\; 5\,\sqrt{\frac{\lambda}{8}}\;\bar{u}_m \;=\; 0,236\; m/s \qquad (\text{for } y \;=\; 0,11\; mm \;\;,\;\; \lambda \text{ from diagram})$$

$$\frac{y u_*}{\nu} \;=\; 50 \qquad \text{(log. velocity distribution)}$$

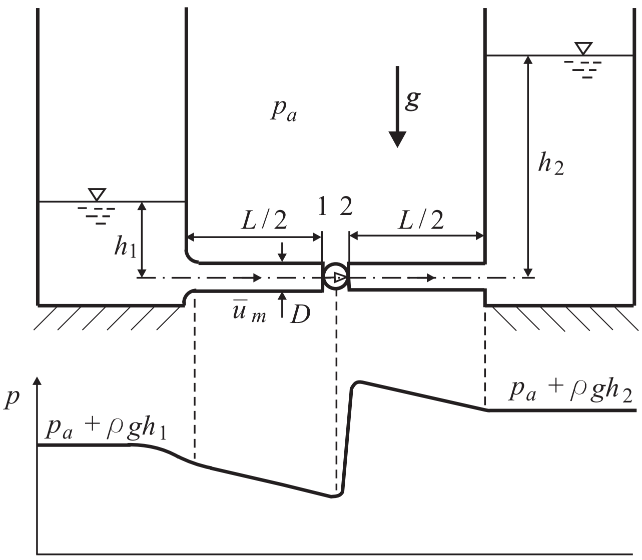
$$\frac{\bar{u}}{u_*} \;=\; 2,5\;\ln\left(\frac{y u_*}{\nu}\right) \;+\; 5,5$$

$$\Longrightarrow \quad \bar{u} \;=\; 0,720\; m/s \qquad (\text{for } y \;=\; 1,1\; mm)$$

d) $$l \;=\; 0,4\, y \;=\; 0,4\,\frac{y u_*}{\nu}\,\frac{\nu}{\sqrt{\lambda/8}\;\bar{u}_m} \;=\; 0,85\, mm$$

10.5

a)



b) Bernoulli with losses:

$$p_a + \rho\, g\, h_1 \;=\; p_1 + \left(1 \;+\; \lambda\,\frac{L}{2D}\right)\,\frac{\rho}{2}\,\bar{u}_m^2$$

$$\dot{V} \;=\; \bar{u}_m \, \frac{\pi \; D^2}{4}$$

$$Re \;=\; \frac{\bar{u}_m \; D}{\nu} \;=\; 8 \cdot 10^5$$

$$\frac{k_s}{D} \;=\; 0,002$$

$$\lambda \;=\; 0,024 \qquad \text{(from Moody} - \text{diagram)}$$

$$\Longrightarrow \quad p_1 \;=\; 1,22 \cdot 10^5 \;\, N/m^2$$

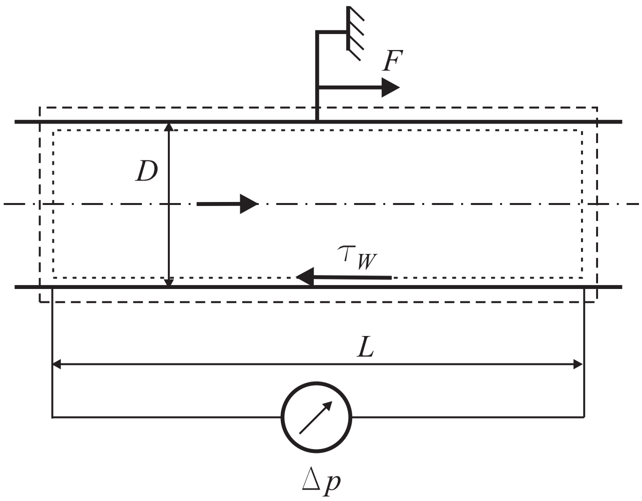
c) Bernoulli with losses:

$$p_2 \;=\; p_a + \rho\, g\, h_2 + \lambda\, \frac{L}{2D}\, \frac{\rho}{2}\, \bar{u}_m^2 \;=\; 3,77{\cdot}10^5 \; N/m^2$$

d)

$$P \;=\; \dot{V}\,(p_2 \;-\; p_1) \;=\; 160,5\, kW$$

10.6



a)

$$\Delta p \;=\; \lambda\, \frac{L}{D}\, \frac{\rho}{2}\, \bar{u}_m^2$$

$$\dot{V} \;=\; \bar{u}_m \, \frac{\pi \; D^2}{4}$$

$$\Longrightarrow \quad \lambda \;=\; \frac{\pi^2 \;\Delta p \; D^5}{8 \;\rho \; L \; \dot{V}^2} \;=\; 0,0356$$

b)

$$Re \;=\; \frac{\rho \;\bar{u}_m \; D}{\eta} \;=\; 1,8 \cdot 10^5$$

$$\frac{k_s}{D} = 0,0083 \quad \text{(from Moody diagram)}$$

$$\Longrightarrow \quad k_s = 4,2\ mm$$

c)
momentum equation for the inner control surface:

$$\Delta p\ \frac{\pi\ D^2}{4} - \tau_W\ \pi\ D\ L = 0$$

$$\Longrightarrow \quad \tau_W = \Delta p\ \frac{D}{4L} = 16\ N/m^2$$

momentum equation for the outer control surface:

$$F = -\Delta p\ \frac{\pi\ D^2}{4} = -2517\ N$$

d)

$$\lambda = 0,016 \quad \text{(from diagram)}$$

$$\Longrightarrow \quad \Delta p = 5,8 \cdot 10^3\ N/m^2$$

10.7

$$\frac{P_1}{P_2} = \frac{\left(1 + \lambda_1\ \dfrac{L}{D_1}\right)\ \dfrac{\rho}{2}\ \bar{u}_{m1}^2}{\left(1 + \lambda_2\ \dfrac{L}{D_2}\right)\ \dfrac{\rho}{2}\ \bar{u}_{m2}^2}$$

$$\dot{V} = \bar{u}_m\ \frac{\pi\ D^2}{4}$$

$$\frac{1}{\sqrt{\lambda}} = 2,0\ \log\left(\frac{D}{k_s}\right) + 1,14$$

$$\Longrightarrow \quad \frac{P_1}{P_2} = \frac{1 + \lambda_1\ \dfrac{L}{D_1}}{1 + \lambda_2\ \dfrac{L}{D_2}}\left(\frac{D_2}{D_1}\right)^4 = 39,2$$

10.8

PB: pipe bundle ; C: Channel

$$\lambda_{PB}\ \frac{L}{D}\ \frac{\rho}{2}\ \bar{u}_{mPB}^2 \;=\; \lambda_C\ \frac{L_K}{d_h}\ \frac{\rho}{2}\ \bar{u}_{mC}^2$$

$$d_h \;=\; a$$

$$Re_C \;=\; \frac{\rho\ a}{\eta}\ \frac{\dot{V}}{a^2} \;=\; 10^5 \qquad\qquad \lambda_C \;=\; 0,018$$

$$Re_{PB} \;=\; \frac{\rho\ D}{\eta}\ \frac{\dot{V}}{100\ \dfrac{\pi\ D^2}{4}} \;=\; 1,27\cdot 10^4 \qquad \lambda_{PB} \;=\; 0,030$$

$$L_K \;=\; 13,57\ m$$

11 Similarity theory

11.1 The frequency f depends on the following quantities:

$$\begin{array}{llcl}
\text{incoming velocity } u_\infty & [u_\infty] & \doteq & \dfrac{m}{s} \\
\text{density } \rho & [\rho] & \doteq & \dfrac{kg}{m^3} \\
\text{kin. viscosity } \nu & [\nu] & \doteq & \dfrac{m^2}{s} \\
\text{diameter of the cylinder } D & [D] & \doteq & m \\
\Longrightarrow f = F(u_\infty, \rho, \nu, D) & [f] & \doteq & \dfrac{1}{s}
\end{array}$$

5 variables exist with 3 different dimension $(s, m, kg) \Longrightarrow$ from Π-theorem the problem has $5 - 3 = 2$ dimensionless parameters.

The quantities D, ρ, u_∞ are chosen as reference quantities. They include all basic dimensions and are linearly independent
Determining the dimensionless parameters:

$$\pi_1 = f \cdot D^\alpha \cdot u_\infty^\beta \cdot \rho^\gamma$$

$$L\ [m] : 0 = 0 + 1 \cdot \alpha + 1 \cdot \beta + (-3) \cdot \gamma$$

$$3\gamma = \alpha + \beta$$

$$t\ [s] : 0 = -1 + 0 \cdot \alpha + (-1) \cdot \beta + 0 \cdot \gamma$$

$$\beta = -1$$

$$M\ [kg] : 0 = 0 + 0 \cdot \alpha + 0 \cdot \beta + 1 \cdot \gamma$$

$$\gamma = 0$$

$$\Longrightarrow \alpha = 1; \quad \beta = -1; \quad \gamma = 0$$

$$\Longrightarrow \boxed{\pi_1 = f \cdot \frac{D}{u_\infty}}$$

$$\pi_2 = \nu \cdot D^\alpha \cdot u_\infty^\beta \cdot \rho^\gamma$$

$$[m] : 0 = 2 + 1 \cdot \alpha + 1 \cdot \beta + (-3) \cdot \gamma$$

$$3\gamma = 2 + \alpha + \beta$$

$$[s] : 0 = -1 + 0 \cdot \alpha + (-1) \cdot \beta + 0 \cdot \gamma$$

$$\beta = -1$$

$$[kg] : 0 = 0 + 0 \cdot \alpha + 0 \cdot \beta + 1 \cdot \gamma$$

$$\gamma = 0$$

$$\Longrightarrow \alpha = -1; \quad \beta = -1; \quad \gamma = 0$$

$$\Longrightarrow \quad \boxed{\pi_2 = \nu \cdot \frac{1}{D \cdot u_\infty}}$$

The dimensionless parametersusually can be expressed in terms of wellknown parameters or combination of them.

$$\pi_1 = \frac{f \cdot D}{u_\infty} = Sr \qquad \text{Strouhalnumber}$$

$$\pi_2 = \frac{\nu}{D \cdot u_\infty} = \frac{1}{Re} \quad \text{Reynoldsnumber.}$$

The functional relationship for this problem is

$$Sr = F(Re)$$

Only one variation of parametyers is necessary.

Hint: It is possible to choose the dynamic viscosity ν instead of the kinematic viscosity η. It is not necessary to use the density ρ, since $\nu = \dfrac{\eta}{\rho}$ is the relevant quantity. In this case 4 variables and 2 basic dimensions are existing. In both cases the result is the same.

11.2

a)

$$L^3\, t^{-1} = (M\, L^{-2}\, t^{-2})^\alpha (M\, L^{-1}\, t^{-1})^\beta\, L^\gamma$$

$$\alpha = 1 \qquad \beta = -1 \qquad \gamma = 4$$

$$\Longrightarrow \quad \dot{V} \sim \frac{\Delta p\; D^4}{L \eta}$$

b)

$$\lambda = \frac{D}{L} \frac{\Delta p}{\frac{\rho}{2} u_m^2}$$

$$\frac{\Delta p \ D}{L} \sim \frac{\dot{V} \ \eta}{D^3} \sim \frac{u_m \ \eta}{D} \qquad (\text{with } \dot{V} \sim u_m \cdot D^2)$$

$$\lambda \sim \frac{1}{Re}$$

11.3

$$\mu = f(\frac{y}{x^{1/4}}, \ g, \ \eta, \ \rho); \quad [g] = \frac{m}{s^2}; \quad [\eta] = \frac{kg}{m \ s}; \quad [\rho] = \frac{kg}{m^3}$$

$$\text{Ansatz}: \qquad \mu = \frac{y}{x^{1/4}} \cdot g^{\alpha} \cdot \eta^{\beta} \cdot \rho^{\gamma}$$

$$\begin{aligned} \Longrightarrow M &: \ 0 = 0 + \beta + \gamma \\ L &: \ 0 = \frac{3}{4} + \alpha - \beta - 3\gamma \\ t &: \ 0 = 0 - 2\alpha - \beta \end{aligned}$$

$$\Longrightarrow \quad \alpha = \frac{1}{4}; \qquad \beta = -\frac{1}{2}; \qquad \gamma = \frac{1}{2}$$

$$\Longrightarrow \quad \mu = \frac{y}{x^{1/4}} \cdot g^{1/4} \cdot \eta^{-1/2} \cdot \rho^{1/2} = \frac{y}{x} \left(\frac{g \ x^3 \ \rho^2}{\eta^2} \right)^{1/4}$$

$$\Longrightarrow \quad \frac{T - T_\infty}{T_W - T_\infty} = \tilde{F} \left(\frac{y}{x} \left(\frac{g \ x^3 \ \rho^2}{\eta^2} \right)^{1/4} \right)$$

11.4

$$\frac{F_{wL}}{F_{wW}} = \frac{c_{wL} \ \frac{\pi}{4} \ D_L^2 \ \frac{\rho_L}{2} \ u_{\infty L}^2}{c_{wW} \ \frac{\pi}{4} \ D_W^2 \ \frac{\rho_W}{2} \ u_{\infty W}^2} = \frac{c_{wL} \ \eta_L^2 \ \frac{\rho_W}{2} \ Re_L^2}{c_{wW} \ \eta_W^2 \ \frac{\rho_L}{2} \ Re_W^2}$$

$$Re_L = Re_W \ : \ c_{wL} = c_{wW}$$

$$\frac{F_{wL}}{F_{wW}} = 0,281$$

11.5 a) $Re = Re' : v' = \sqrt{\frac{A}{A'}}\, v$ as small as possible in order to use incompressible equations:

$$A' = A_m : \quad v' = 77,46\ m/s$$

b)
$$P = \frac{F_w}{F_w^{'}} F_w^{'} v \ = \ \frac{c_w \ \frac{\rho}{2}\ v^2\ A}{c_w^{'} \ \frac{\rho}{2}\ v^{'2}\ A_m} F_w^{'}\ v$$

$$Re = Re^{'} : \quad c_w = c_w^{'}$$

$$P = F_w^{'}\ v = 24,3\ kW$$

11.6

a)
$$Re = Re^{'} : \quad \frac{D\ v\ \rho}{\eta} = \frac{D^{'}\ v^{'}\ \rho^{'}}{\eta^{'}}$$

$$\text{with} \quad \dot{\mathrm{V}} = \mathrm{v}\ \frac{\mathrm{D}^2\ \pi}{4} \quad \Longrightarrow \quad \frac{\dot{\mathrm{V}}\ \rho\ 4}{\mathrm{D}\ \eta\ \pi} = \frac{\dot{\mathrm{V}}'\ \rho'\ 4}{\mathrm{D}'\ \eta'\ \pi}$$

$$\Longrightarrow \quad \dot{V}^{'} \ = \ \dot{V} \cdot \frac{\eta^{'}\ \rho\ D^{'}}{\eta\ \rho^{'}\ D} = 0,5\ m^3/s$$

$$Sr = Sr^{'} : \quad \frac{n\ D}{v} = \frac{n^{'}\ D^{'}}{v^{'}}$$

$$\Longrightarrow n^{'} = n \cdot \frac{v^{'}}{v} \cdot \frac{D}{D^{'}} = n \cdot \frac{v\ D^{'2}}{v\ D^2} \cdot \frac{D^3}{D^{'3}}$$
$$= n \cdot \frac{\dot{V}^{'}}{\dot{V}} \cdot \left(\frac{D}{D^{'}}\right)^3 = 13,\overline{3}\ \frac{1}{s}$$

b)
$$Eu = Eu^{'} : \quad \Delta p_0 = \frac{\rho\ v^2}{\rho^{'}\ v^{'2}} \cdot \Delta p_0^{'} = \frac{\rho\ \dot{V}^2}{\rho^{'}\ \dot{V}^{'2}} \cdot \left(\frac{D^{'}}{D}\right)^4 \cdot \Delta p_0^{'} = 527,34\ \frac{N}{m^2}$$

c)
$$P = \dot{V}\ \Delta p_0 = 15,82\ kW$$
$$P^{'} = \dot{V}^{'}\ \Delta p_0^{'} = 15\ kW$$

$$M = \frac{P}{2\ \pi\ n} = 201\ Nm$$
$$M^{'} = \frac{P^{'}}{2\ \pi\ n^{'}} = 179\ Nm$$

11.7

a) $$Fr_L = Fr_L^{'}: \quad v^{'} = v \cdot \sqrt{\frac{L^{'}}{L}} = 0,75\ m/s$$

b) $$c_W = c_W^{'}$$

(Drag coefficient with respect to the cross section perpendicular to the incoming velocity)

$$F_W = \frac{\frac{\rho}{2}\ v^2\ B\ H}{\frac{\rho}{2}\ v^{'2}\ B^{'}\ H^{'}}\ F_W^{'}\ =\ 1,64 \cdot 10^4\ N$$

c) $$c_W = 1,12$$

d) $$\frac{v}{\sqrt{g\ h}}\ =\ \frac{v^{'}}{\sqrt{g\ h^{'}}}\ :\ h\ =\ 16\,h^{'}\ =\ 0,4\,m$$

11.8

a)

The reference values are chosen in such a way that the dimensionless quantities are of the order of magnitude O(1).

$$\bar{u}\ =\ \frac{u}{u_{ref}} \qquad \bar{p}\ =\ \frac{p}{\Delta p} \qquad \bar{x}\ =\ \frac{x}{L} \qquad \bar{y}\ =\ \frac{y}{h} \qquad \bar{\eta}\ =\ \frac{\eta}{\eta_{ref}}$$

Introducing into PDE:

$$\frac{\partial(\Delta p \cdot \bar{p})}{\partial(L \cdot \bar{x})}\ =\ \bar{\eta} \cdot \eta_{ref}\ \frac{\partial^2(u_{ref} \cdot \bar{u})}{\partial(\bar{y} \cdot h)^2}$$

with $\bar{\eta} = 1$:

$$\frac{\Delta p}{L} \cdot \frac{\partial \bar{p}}{\partial \bar{x}} = \frac{\eta_{ref} \cdot u_{ref}}{h} \cdot \frac{\partial \dfrac{\partial \bar{u}}{\partial (\bar{y}\ h)}}{\partial \bar{y}} = \frac{\eta_{ref} \cdot u_{ref}}{h^2} \cdot \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\Longrightarrow \quad \frac{\partial \bar{p}}{\partial \bar{x}} = \underbrace{\frac{L \cdot \eta_{ref} \cdot u_{ref}}{h^2 \cdot \Delta p}}_{\text{1stparameter}} \cdot \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$2 \text{ terms} \Longrightarrow 1 \text{ parameter}$$

b)

$$f(L,\ \eta_{ref},\ u_{ref},\ h,\ \Delta p) = 0$$

$$5 \text{ variables and } 3 \text{ basic dimensions} \Longrightarrow 2 \text{ parameters}$$

The Π-theorem says something about the maximum number of dimensionless parameters. The method of differential equations gives more informations about the actual problem and the terms can be neglected. The resulting number of parameters can be smaller.

11.9

$$\bar{u} = \frac{u}{v_1} \qquad \bar{v} = \frac{v}{v_1} \qquad \bar{p} = \frac{p}{\Delta p_1} \qquad \bar{\rho} = \frac{\rho}{\rho_1} \qquad \bar{\eta} = \frac{\eta}{\eta_1} \qquad \bar{x} = \frac{x}{L_1} \qquad \bar{y} = \frac{y}{L_1} \qquad \bar{t} = \frac{t}{t_1}$$

$$\bar{\rho} \left(Sr\ \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\ \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\ \frac{\partial \bar{u}}{\partial \bar{y}} \right) = -\, Eu\ \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\bar{\eta}}{Re} \left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} \right)$$

$$Sr = \frac{L_1}{v_1\ t_1} \qquad Eu = \frac{\Delta p_1}{\rho_1\ v_1^2} \qquad Re = \frac{\rho_1\ v_1\ L_1}{\eta_1}$$

Comment: usually, the election of the refernce values depends on the actual problem.

11.10

a)

$$\frac{\lambda\ T_R}{l^2} \frac{\partial^2\ \bar{T}}{\partial \bar{y}^2} + \eta \frac{u_R^2}{l^2} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 = 0 \quad \Bigg| \ : \ \frac{\lambda\ T_R}{l^2}$$

$$K^* = \frac{\eta\ u_R^2}{\lambda\ T_R}$$

b)

$$K^* = \lambda^\alpha\, \eta^\beta\, T_R^\gamma\, u_R^\delta\, l^\varepsilon$$

$$\text{choose } \beta = 1$$

$$\begin{aligned} kg &: \; 0 = \alpha + 1 \\ m &: \; 0 = \alpha - 1 + \delta + \varepsilon \\ s &: \; 0 = -3\alpha - 1 - \delta \\ K &: \; 0 = -\alpha + \gamma \end{aligned}$$

$$\begin{aligned} \alpha = \gamma &= -1 \\ \delta &= 2 \\ \varepsilon &= 0 \end{aligned}$$

$$K^* = \frac{\eta \; u_R^2}{\lambda_p \; T_R}$$

c)

$$K^* = \frac{\eta \; c_p}{\lambda} \frac{u_R^2}{c_p \; T_R} = \frac{\eta \; c_p}{\lambda} \frac{u_R^2}{\gamma \; R \; T}(\gamma - 1)$$

$$c_p = \frac{\gamma \; R}{\gamma \; - \; 1}$$

$$K^* = Pr \cdot Ma^2(\gamma \; - \; 1))$$

11.11

$$-\frac{\partial p}{\partial x} \; + \; \rho \; g \; + \; \frac{\eta}{r} \; \frac{d}{dr} \left(r \; \frac{du}{dr} \right) \; = \; 0$$

$$-\frac{\Delta p}{L} \; \frac{d \; \bar{p}}{d \; \bar{x}} \; + \; \rho \; g \; \bar{\rho} \; \bar{g} \; + \; \frac{\eta}{R^2} \; u_\infty \; \frac{\bar{\eta}}{\bar{r}} \; \frac{d}{d \; \bar{r}} \left(\bar{r} \; \frac{d \; \bar{u}}{d \; \bar{r}} \right) \; = \; 0$$

$$-\underbrace{\frac{\Delta p}{L} \; \frac{R^2}{\eta \; u_\infty}}_{K_1^*} \; \frac{d \; \bar{p}}{d \; \bar{x}} \; + \; \underbrace{\frac{\rho \; g \; R^2}{\eta \; u_\infty}}_{K_2^*} \; \bar{\rho} \; \bar{g} \; + \; \frac{\bar{\eta}}{\bar{r}} \; \frac{d}{d \; \bar{r}} \left(\bar{r} \; \frac{d \; \bar{u}}{d \; \bar{r}} \right) \; = \; 0$$

Einflußgrößen: $\Delta p, \; \rho, \; u_m, \; \eta, \; l, \; R, \; g$

$$K = \begin{array}{cccc} \Delta p^1 & \rho^{\,\beta} & u_m^\gamma & D^\delta \\ \left[\dfrac{kg}{s^2\ m}\right] & \left[\dfrac{kg}{m^3}\right] & \left[\dfrac{m}{s}\right] & [m] \end{array}$$

kg	1	$+\beta$			$\beta = -1$
m	-1	-3β	$+\gamma$	$+\delta$	$\gamma = -2,\ \delta = 1-3+2=0$
s	2		$-\gamma$		$K_1 = \dfrac{\Delta p}{\rho\ u_m^2} = Eu$
	$\eta\,[kg/s\ m]$				
kg	1	$+\beta$			$\beta = -1,\ \gamma = -1$
m	-1	-3β	$+\gamma$	$+\delta$	$\delta = 1-3+1 = -1$
s	-1		$-\gamma$		$K_2 = \dfrac{\eta}{\rho\ u_m\ D} = \dfrac{1}{Re}$
	$l\,[m]$				
kg		$+\beta$			$\beta = 0,\ \gamma = 0$
m	1	-3β	$+\gamma$	$+\delta$	$\delta = 1$
s			$-\gamma$		$K_3 = \dfrac{l}{D}$
	$g\,[m/s^2]$				
kg		$+\beta$			$\beta = 0,\ \gamma = -2$
m	1	-3β	$+\gamma$	$+\delta$	$\delta = -1+2 = 1$
s	-2		$-\gamma$		$K_4 = \dfrac{g\ D}{u_m^2} = \dfrac{1}{F_r^2}$

$$K_1^* = Eu \cdot Re \cdot \frac{R}{L}$$

$$K_2^* = \frac{Re}{F_r^2}$$

11.12

a)
$$\rho \frac{dw}{dt} = -\frac{\partial p}{\partial z} - \rho\ g + \eta \nabla^2 w$$

Reference values: $l,\ u_\infty$, constants: $\eta,\ \rho,\ g$

$$\Longrightarrow \frac{\rho\ u_\infty^2}{l}\ \frac{d\bar{w}}{d\bar{t}} = -\frac{\rho u_\infty^2}{l}\ \frac{\partial \bar{p}}{\partial \bar{z}} - \rho\ g + \eta \frac{u_\infty}{l^2} \nabla^2 \bar{w} \quad \Bigg| \cdot \frac{1}{\dfrac{\rho\ u_\infty^2}{l}}$$

$$\Longrightarrow \frac{d\bar{w}}{d\bar{t}} = -\frac{\partial \bar{p}}{\partial \bar{z}} - \frac{1}{Fr^2} + \frac{1}{Re}\ \nabla^2 \bar{w}$$

with $Fr^2 = \dfrac{u_\infty^2}{g\ l}$ and $Re = \dfrac{u_\infty\ l}{\eta\ /\ \rho}$

b) $$Fr = Fr^{'} \longrightarrow \frac{u_\infty^2}{g\ l} = \frac{u_\infty^{'2}}{g\ l^{'}} \Longrightarrow u_\infty^{'} = u_\infty \sqrt{\frac{l'}{l}} = \frac{u_\infty}{\sqrt{10}}$$

$$Re = Re^{'} \longrightarrow \frac{u_\infty\ l}{\nu} = \frac{u_\infty^{'}\ l^{'}}{\nu^{'}} \Longrightarrow \nu^{'} = \nu \frac{u_\infty^{'}}{u_\infty} \frac{l^{'}}{l} = \frac{\nu}{10\ \sqrt{10}}$$

c) $$c_W = \frac{F_W}{\frac{\rho}{2}\ u_\infty^2\ A} \quad \text{with } A = l \cdot H$$

$$c_W = c_W^{'} \Longrightarrow \frac{F}{\frac{\rho}{2}\ u_\infty^2\ A} = \frac{F^{'}}{\frac{\rho^{'}}{2}\ u_\infty^{'2}\ A^{'}}$$

$$\Longrightarrow F = F^{'}\ \frac{\rho}{\rho^{'}}\ \frac{u_\infty^2}{u_\infty^{'2}}\ \frac{A}{A^{'}} = 100\ F^{'}\ \frac{\rho}{\rho^{'}}\ \frac{u_\infty^2}{u_\infty^{'2}}$$

$$P = F \cdot u_\infty = 100\ F^{'}\ \frac{\rho}{\rho^{'}}\ \frac{u_\infty^3}{u_\infty^{'2}}$$

11.13

a) $$c_p \left(\rho\ u\ \frac{\partial T}{\partial x} + \rho\ v\ \frac{\partial T}{\partial y}\right) + \left(\rho\ u\ \frac{\partial \frac{u^2}{2}}{\partial x} + \rho\ v\ \frac{\partial \frac{u^2}{2}}{\partial y}\right)$$

$$= \eta\ u\ \frac{\partial^2\ u}{\partial\ y^2} + \eta \left(\frac{\partial u}{\partial y}\right)^2 + \lambda\ \frac{\partial^2\ T}{\partial\ y^2}$$

reference values: $\rho_\infty,\ u_\infty,\ T_\infty,\ L,\ \eta_\infty,\ c_{p\ \infty},\ \lambda_\infty$

$$\Longrightarrow \bar{u} = \frac{u}{u_\infty};\ \ \bar{v} = \frac{v}{u_\infty};\ \ \bar{T} = \frac{T}{T_\infty};\ \ \bar{\rho} = \frac{\rho}{\rho_\infty};\ \ \bar{x} = \frac{x}{L};$$

$$\bar{y} = \frac{y}{L};\ \ \bar{\eta} = \frac{\eta}{\eta_\infty};\ \ \bar{c_p} = \frac{c_p}{c_{p\infty}};\ \ \bar{\lambda} = \frac{\lambda}{\lambda_\infty}$$

introduce:

$$\Longrightarrow a_1\ \bar{c_p} \left(\bar{\rho}\ \bar{u}\ \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{\rho}\ \bar{v}\ \frac{\partial \bar{T}}{\partial \bar{y}}\right) + a_2 \left(\bar{\rho}\ \bar{u}\ \frac{\partial \frac{\bar{u}^2}{2}}{\partial \bar{x}} + \bar{\rho}\ \bar{v}\ \frac{\partial \frac{\bar{u}^2}{2}}{\partial \bar{y}}\right)$$

$$= a_3\ \bar{\eta}\ \bar{u}\ \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + a_4\ \bar{\eta} \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 + a_5\ \bar{\lambda}\ \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}$$

with

$$a_1 = \frac{c_{p\infty}\,\rho_\infty\,u_\infty\,T_\infty}{L}; \quad a_2 = \frac{\rho_\infty\,u_\infty^3}{L}; \quad a_3 = a_4 = \frac{\eta_\infty\,u_\infty^2}{L^2}; \quad a_5 = \frac{\lambda_\infty\,T_\infty}{L^2}$$

dimensionsless form via division of the equation with e. g. a_1:

b) $\Longrightarrow$ dimensionless parameters

$$(i) \quad K_1 = \frac{a_2}{a_1} = \frac{\rho_\infty\,u_\infty^3}{L}\,\frac{L}{c_{p\infty}\,\rho_\infty\,u_\infty\,T_\infty} = \frac{u_\infty^2\,\gamma\,R}{\gamma\,R\,c_\infty^2}\,(\gamma{-}1) = (\gamma{-}1)\,M_\infty^2$$

$$(ii) \quad K_2 = \frac{a_3}{a_1} = \frac{a_4}{a_1} = \frac{\eta_\infty\,u_\infty^2}{L^2}\,\frac{L}{c_{p\infty}\,\rho_\infty\,u_\infty\,T_\infty} = \frac{\eta_\infty}{u_\infty\,\rho_\infty\,L}\,(\gamma{-}1)\,\frac{u_\infty^2}{c_\infty^2} = \frac{1}{Re}\,(\gamma{-}1)\,M_\infty^2$$

$$(iii) \quad K_3 = \frac{a_5}{a_1} = \frac{\lambda_\infty\,T_\infty}{L^2}\,\frac{L}{c_{p\infty}\,\rho_\infty\,u_\infty\,T_\infty} = \frac{\eta_\infty}{u_\infty\,\rho_\infty\,L}\,\frac{\lambda_\infty}{\eta_\infty\,c_{p\infty}} = \frac{1}{Re}\,\frac{1}{Pr}$$

c) $$K_1 = K_2 = 0 \Longrightarrow (\gamma{-}1) = 0 \Longrightarrow \gamma = 1$$

11.14

$$\rho_{ref} = \rho_w\,; \quad \eta_{ref} = \eta_w\,; \quad \Delta p_{ref} = p_0 - p_a\,; \quad l_{ref} = D$$

$$u_{ref} = \sqrt{\frac{p_0 - p_a}{\rho_w}}\,\left[\frac{m}{s}\right] \quad \text{from Bernoulli} \qquad t_{ref} = \frac{D}{u_{ref}} = \frac{D\,\sqrt{\rho_w}}{\sqrt{p_0 - p_a}}\,[s]$$

Dimensionless parameters:

$$\bar\rho = \frac{\rho}{\rho_w}; \quad \bar\eta = \frac{\eta}{\eta_w}; \quad \bar x = \frac{x}{D}; \quad \bar y = \frac{y}{D}; \quad \bar p = \frac{p}{p_0 - p_a};$$

$$\bar u = \frac{u}{\sqrt{\dfrac{p_0 - p_a}{\rho_w}}}; \quad \bar v = \frac{v}{\sqrt{\dfrac{p_0 - p_a}{\rho_w}}}; \quad \bar t = \frac{t}{\dfrac{D\,\sqrt{\rho_w}}{\sqrt{p_0 - p_a}}}$$

introduced in PDE:

$$\frac{\partial\bar u}{\partial\bar t}\cdot\frac{p_0 - p_a}{\rho_w\,D} + \left(\bar u\,\frac{\partial\bar u}{\partial\bar x} + \bar v\,\frac{\partial\bar u}{\partial\bar y}\right)\cdot\frac{p_0 - p_a}{\rho_w\,D}$$

$$= \; -\frac{1}{\bar\rho}\,\frac{\partial \bar p}{\partial \bar x}\cdot\frac{p_0 \;-\; p_a}{\rho_w\;D} \;+\; \frac{\eta_w}{\rho_w}\cdot\sqrt{\frac{p_0 \;-\; p_a}{\rho_w}}\;\frac{1}{D^2}\cdot\frac{\bar\eta}{\bar\rho}\left(\frac{\partial^2\bar u}{\partial \bar x^2} \;+\; \frac{\partial^2\bar u}{\partial \bar y^2}\right) \qquad \Bigg|\;\cdot\;\frac{1}{\dfrac{p_0 \;-\; p_a}{\rho_w\;D}}$$

$$\Longrightarrow \frac{\partial \bar u}{\partial \bar t} \;+\; \bar u\,\frac{\partial \bar u}{\partial \bar x} \;+\; \bar v\,\frac{\partial \bar u}{\partial \bar y} \;=\; -\frac{1}{\bar\rho}\,\frac{\partial \bar p}{\partial \bar x} \;+\; K\cdot\frac{\bar\eta}{\bar\rho}\left(\frac{\partial^2\bar u}{\partial \bar x^2} \;+\; \frac{\partial^2\bar u}{\partial \bar y^2}\right)$$

$$\text{with } K \;=\; \frac{\eta_w}{\rho_w}\cdot\sqrt{\frac{\rho_w}{p_0 \;-\; p_a}}\;\frac{1}{D}$$

b)

$$u_{ref} \;=\; \sqrt{\frac{p_0 \;-\; p_a}{\rho_w}} \;\Longrightarrow\; K \;=\; \frac{1}{\rho_w}\,\frac{\eta_w}{u_{ref}\;D} \;=\; \frac{1}{Re}$$

$$K \;=\; \frac{\text{Friction forces}}{\text{Inertia forces}}$$

11.15

a)

$$\text{conti:}\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \;=\; 0$$

$$\text{momentum:}\quad \rho\left(u\,\frac{\partial u}{\partial x} \;+\; v\,\frac{\partial u}{\partial y}\right) \;=\; \eta\,\frac{\partial^2 u}{\partial y^2}$$

$$\text{qnergy:}\quad \rho\;c_p\left(u\,\frac{\partial T}{\partial x} \;+\; v\,\frac{\partial T}{\partial y}\right) \;=\; \lambda\,\frac{\partial^2 T}{\partial y^2}$$

dimensionless parameters:

$$\bar u \;=\; \frac{u}{u_\infty};\;\; \bar v \;=\; \frac{v}{u_\infty};\;\; \bar\rho \;=\; \frac{\rho}{\rho_\infty};\;\; \bar x \;=\; \frac{x}{L};\;\; \bar y \;=\; \frac{y}{L};$$

$$\bar\eta \;=\; \frac{\eta}{\eta_\infty};\;\; \bar{c_p} \;=\; \frac{c_p}{c_{p_\infty}};\;\; \bar T \;=\; \frac{T}{T_\infty};\;\; \bar\lambda \;=\; \frac{\lambda}{\lambda_\infty}$$

b)

$$\text{conti:}\quad \frac{u_\infty}{L}\left(\frac{\partial \bar u}{\partial \bar x} \;+\; \frac{\partial \bar v}{\partial \bar y}\right) \;=\; 0 \qquad\qquad \Longrightarrow \text{no parameter}$$

$$\text{momentum:}\quad \rho_\infty\,\frac{u_{\infty^2}}{L}\,\bar\rho\left(\bar u\,\frac{\partial \bar u}{\partial \bar x} \;+\; \bar v\,\frac{\partial \bar u}{\partial \bar y}\right) \;=\; \eta_\infty\,\bar\eta\,\frac{u_\infty}{L^2}\,\frac{\partial^2\bar u}{\partial \bar{y^2}}$$

$$\bar\rho\left(\bar u\,\frac{\partial \bar u}{\partial \bar x} \;+\; \bar v\,\frac{\partial \bar u}{\partial \bar y}\right) \;=\; \bar\eta\,\frac{\partial^2\bar u}{\partial \bar{y^2}}\underbrace{\left(\frac{\eta_\infty\;u_\infty\;L}{L^2\;\rho_\infty\;u_\infty^2}\right)}_{K_1}$$

$$K_1 \;=\; \frac{\eta_\infty}{L\;\rho_\infty\;u_\infty} \;=\; \frac{1}{Re}$$

$$\text{energy:}\quad \frac{\rho_\infty\ c_{p_\infty}\ u_\infty\ T_\infty}{L}\ \bar{\rho}\ \bar{c_p}\left(\bar{u}\ \frac{\partial \bar{T}}{\partial \bar{x}}\ +\ \bar{v}\ \frac{\partial \bar{T}}{\partial \bar{y}}\right)\ =\ \frac{\lambda_\infty\ T_\infty}{L^2}\ \bar{\lambda}\ \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}$$

$$\bar{\rho}\ \bar{c_p}\left(\bar{u}\ \frac{\partial \bar{T}}{\partial \bar{x}}\ +\ \bar{v}\ \frac{\partial \bar{T}}{\partial \bar{y}}\right)\ =\ \underbrace{\frac{\lambda_\infty\ T_\infty\ L}{L^2\ \rho_\infty\ c_{p_\infty}\ u_\infty\ T_\infty}}_{K_2}\left(\bar{\lambda}\ \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\right)$$

$$K_2\ =\ \frac{\lambda_\infty}{L\ \rho_\infty\ c_{p_\infty}\ u_\infty}\ \frac{\eta_\infty}{\eta_\infty}\ =\ \frac{1}{Pr}\cdot\frac{1}{Re}$$

c) dimensionsless PDE.:

$$\text{conti:}\qquad \frac{\partial \bar{u}}{\partial \bar{x}}\ +\ \frac{\partial \bar{v}}{\partial \bar{y}}\ =\ 0$$

$$\text{momentum:}\quad \bar{\rho}\left(\bar{u}\ \frac{\partial \bar{u}}{\partial \bar{x}}\ +\ \bar{v}\ \frac{\partial \bar{u}}{\partial \bar{y}}\right)\ =\ \frac{1}{Re}\ \bar{\eta}\ \left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)$$

$$\text{energy:}\ \ \bar{\rho}\ \bar{c_p}\left(\bar{u}\ \frac{\partial \bar{T}}{\partial \bar{x}}\ +\ \bar{v}\ \frac{\partial \bar{T}}{\partial \bar{y}}\right)\ =\ \frac{1}{Pr}\cdot\frac{1}{Re}\ \bar{\lambda}\left(\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\right)$$

konstante fluid properties: $\bar{\rho}\ =\ \bar{c_p}\ =\ \bar{\lambda}\ =\ \bar{\eta}\ =\ 1$

Comparison between momentum and energy equation:
By replacing T with u and proposing

$$\boxed{\mathrm{Pr}\ =\ 1}$$

the energy and the momentum equation are the same

$$\text{i.e.}\qquad \frac{\eta_\infty\ c_{p_\infty}}{\lambda_\infty}\ =\ 1$$

14 Potential flows

14.1

a) Parallel flow x-direction: $F_1(z) = u_\infty z$

Source in $x = -a$: $F_2(z) = \dfrac{E}{2\pi} \ln((x+a) + iy)$

Sink in $x = +a$: $F_3(z) = \dfrac{-E}{2\pi} \ln((x-a) + iy)$

Superposition: $F(z) = u_\infty z + \dfrac{E}{2\pi} \ln((z+a) - \ln(z-a))$

$$
\begin{aligned}
\bar{w} = &\quad u - iv = \frac{dF}{dz} = u_\infty + \frac{E}{2\pi}\left(\frac{1}{z+a} - \frac{1}{z-a}\right) \\
= &\quad u_\infty + \frac{E}{2\pi}\left(\frac{-2a}{z^2-a^2}\right) \qquad \text{with } \ b = x^2 - y^2 - a^2 \\
= &\quad u_\infty - \frac{Ea}{\pi}\frac{b - 2ixy}{(b-2ixy)(b+2ixy)} \\
= &\quad u_\infty - \frac{Ea}{\pi}\frac{x^2 - y^2 - a^2 - 2ixy}{(x^2-y^2-a^2)^2 + 4(xy)^2} \\
u = &\quad u_\infty - \frac{Ea}{\pi}\frac{x^2 - y^2 - a^2}{(x^2-y^2-a^2)^2 + 4(xy)^2} \\
v = &\quad -\frac{Ea}{\pi}\frac{2xy}{(x^2-y^2-a^2)^2 + 4(xy)^2}
\end{aligned}
$$

for $u = 0, v = 0:\ y = 0, x_s = \pm\sqrt{\dfrac{Ea}{\pi u_\infty} + a^2}$

b) The contour is a streamline $\psi = const$ crossing the stagnation points.

$$
\psi = u_\infty y + \frac{E}{2\pi}\arctan\frac{y}{x+a} - \frac{E}{2\pi}\arctan\frac{y}{x-a} \qquad \text{(from table)}
$$

for $y = 0,\ x = \pm x_s$ follows $\psi = 0$

$\Longrightarrow$ Equation for the contour

$$
u_\infty y + \frac{E}{2\pi}\arctan\frac{y}{x+a} - \frac{E}{2\pi}\arctan\frac{y}{x-a} = 0
$$

Comment: This equation additionally describes the stagnation streamlines on the x-axis.

14.2

	$\bigtriangledown \cdot \vec{v}$	$\lvert \bigtriangledown \times \vec{v} \rvert$
a)	$4xy$	$y^2 - x^2$
b)	2	0
c)	0	-2
d)	0	0

The streamfunction exists for c) and d), the potential exists for b) and d).

Determination of the stramfunction:

c)

$$\begin{aligned}
\psi &= \int u dy + f(x) = \frac{y^2}{2} + f(x) \\
v &= -\frac{\partial \psi}{\partial x} = -f'(x) = -x \\
\psi &= \frac{1}{2}(x^2 + y^2) + c
\end{aligned}$$

d)

$$\psi = \frac{1}{2}(y^2 - x^2) + c$$

Determination of the potential:

b)

$$\begin{aligned}
\phi &= \int u dx + f(y) = \frac{x^2}{2} + f(y) \\
v &= \frac{\partial \phi}{\partial y} = f'(y) = y \\
\phi &= \frac{1}{2}(x^2 + y^2) + c
\end{aligned}$$

d)

$$\phi = xy + c$$

14.3

a) Condition for the stagnation point: $u = 0, v = 0$

$$u = \frac{\partial\psi}{\partial y} = -\frac{\Gamma}{2\pi}\left(\frac{y}{(x-a)^2+y^2} + \frac{2y}{(x+a)^2+y^2}\right)$$

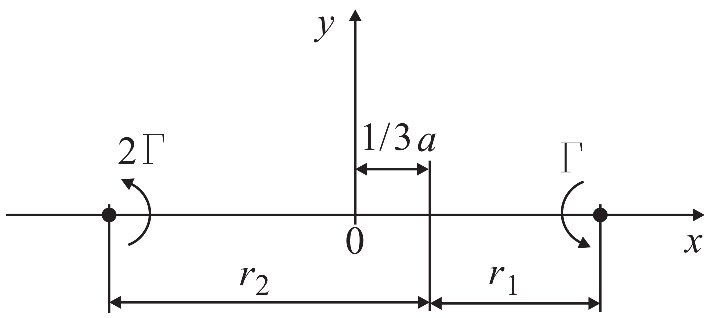
$$u = -\frac{\Gamma}{2\pi}y\left(\frac{1}{(x-a)^2+y^2} + \frac{2}{(x+a)^2+y^2}\right) = 0 \quad \text{wenn } y = 0$$

$$v = -\frac{\partial\psi}{\partial x} = \frac{\Gamma}{2\pi}\left(\frac{x-a}{(x-a)^2+y^2} + \frac{2(x+a)}{(x+a)^2+y^2}\right)$$

for $y = 0:\quad v = \frac{\Gamma}{2\pi}\left(\frac{1}{x-a} + \frac{2}{x+a}\right) = \frac{\Gamma}{2\pi}\frac{3x-a}{x^2-a^2} \quad \star)$

$v = 0$, if $3x - a = 0 \quad \rightarrow \quad$ stagnation point: $\quad x_s = \frac{a}{3}, \quad y_s = 0$

alternatively:



$$v_\theta = \frac{\Gamma}{2\pi r} : v_{\theta_1} = v_{\theta_2} \rightarrow \frac{\Gamma}{2\pi r_1} = \frac{2\Gamma}{2\pi r_2} \rightarrow r_2 = 2r_1$$

$$2a = r_1 + r_2 \rightarrow 2a = 3r_1 \rightarrow r_1 = \frac{2}{3}a \rightarrow x_s = \frac{a}{3}$$

b)

$c_p(x, y = 0) = 1 - \frac{u^2 + v^2}{u^2_{(0,0)} + v^2_{(0,0)}} \qquad$ mit $u = 0$ for $y = 0$.

$\Longrightarrow c_p(x, y = 0) = 1 - \frac{v^2}{v^2_{(0,0)}},$

$v_{(0,0)} = \frac{\Gamma}{2\pi}\frac{1}{a}$ (see $\star$) d.h. $c_{p_{(0,0)}} = 0$

with $\star$) $c_p(x, y=0) = 1 - \left(\dfrac{\dfrac{3x-a}{x^2-a^2}}{\dfrac{1}{a}} \right)^2$

$\Longrightarrow c_p(x, y=0) = 1 - \left(\dfrac{3xa - a^2}{x^2 - a^2} \right)^2$ for $x \neq a, -a$

14.4

a)

$$F(z) = \frac{2}{3}\frac{u_\infty}{\sqrt{L}} z^{\frac{3}{2}} + \frac{E}{2\pi} \ln z$$

$$= \frac{2}{3}\frac{u_\infty}{\sqrt{L}} r^{\frac{3}{2}} e^{i\frac{3}{2}\theta} + \frac{E}{2\pi}(\ln r + \ln e^{i\theta})$$

$$= Re(r, \theta) + iIm(r, \theta) = \phi(r, \theta) + i\psi(r, \theta)$$

$$\Longrightarrow \phi(r, \theta) = \frac{2}{3}\frac{u_\infty}{\sqrt{L}} r^{\frac{3}{2}} \cos\frac{3}{2}\theta + \frac{E}{2\pi} \ln r$$

$$\Longrightarrow \psi(r, \theta) = \frac{2}{3}\frac{u_\infty}{\sqrt{L}} r^{\frac{3}{2}} \sin\frac{3}{2}\theta + \frac{E}{2\pi}\theta$$

b)

$$v_r = \frac{\partial\phi}{\partial r} = u_\infty \sqrt{\frac{r}{L}} \cos\frac{3}{2}\theta + \frac{E}{2\pi r}$$

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -u_\infty \sqrt{\frac{r}{L}} \sin\frac{3}{2}\theta$$

c) stagnation point at

$$(x = -L, y = 0) \quad \rightarrow \quad \left(r = L, \theta = \frac{2}{3}\pi \right)$$

$$\Longrightarrow v_\theta = 0 = -u_\infty \sqrt{\frac{L}{L}} \underbrace{\sin\pi}_{=0}$$

$$\Longrightarrow v_r = 0 = u_\infty \sqrt{\frac{L}{L}} \underbrace{\cos\pi}_{=-1} + \frac{E}{2\pi L}$$

$$\Longrightarrow u_\infty = \frac{E}{2\pi L} \rightarrow E = 2\pi u_\infty L$$

d)

$$\psi\left(r = L, \theta = \frac{2}{3}\pi\right) = \frac{2}{3} u_\infty \frac{L^{\frac{3}{2}}}{\sqrt{L}} \underbrace{\sin\pi}_{=0} + \frac{E}{2\pi}\frac{2}{3}\pi = \frac{E}{3}$$

$$\Longrightarrow \psi\left(r = L, \theta = \frac{2}{3}\pi\right) = \frac{2}{3}\pi u_\infty L \stackrel{!}{=} \psi_k(r,\theta)$$

$$\Longrightarrow \frac{2}{3}\pi u_\infty L = \frac{2}{3} u_\infty \frac{r^{\frac{3}{2}}}{L^{\frac{1}{2}}} \sin\frac{3}{2}\theta + u_\infty L\theta$$

$$\Longrightarrow r_k(\theta) = L\left(\frac{\pi - \frac{3}{2}\theta}{\sin\frac{3}{2}\theta}\right)^{\frac{2}{3}}$$

14.5

a) $|\bigtriangledown \times \vec{v}| = 0 :$ Potential exists.

b)

$$u = \frac{U}{L}x, \qquad v = -\frac{U}{L}y$$

stagnation points:

$u = v = 0 : \quad x = y = 0$

pressure coefficient:

$$c_p = \frac{p - p_{ref}}{\frac{\rho}{2}\vec{v}_{ref}^2} = 1 - \frac{u^2 + v^2}{u_{ref}^2 + v_{ref}^2} = 1 - \frac{x^2 + y^2}{x_{ref}^2 + y_{ref}^2}$$

Isotach:

$$\vec{v}^2 = u^2 + v^2 = \left(\frac{U}{L}\right)^2 (x^2 + y^2)$$

$$x^2 + y^2 = \left(\frac{\vec{v}L}{U}\right)^2$$

Circles around the origin with radius $\dfrac{|\vec{v}|L}{U}$

c)

$$u_1 = 4\ m/s, \qquad v_1 = -4\ m/s, \qquad |\vec{v}_1| = 5.66\ m/s$$

$$p_1 = p_{ref} + c_{p1}\frac{\rho}{2}\vec{v}_{ref}^2 = 0.86 \cdot 10^5\ N/m^2$$

d)

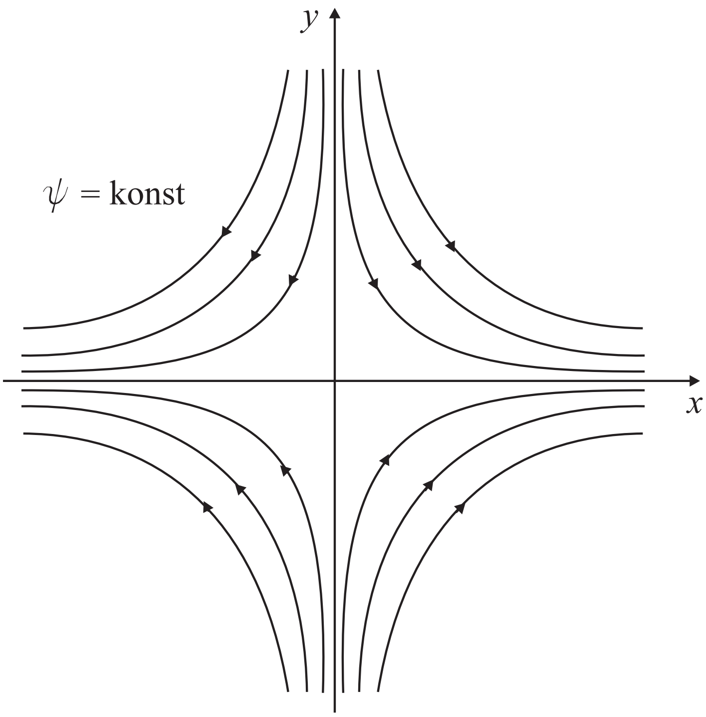
$$t = \int_{x_1}^{x_2} \frac{dx}{u} = \frac{L}{U}\ln\frac{x_2}{x_1}$$
$$x_2 = 5.44\ m$$

$$\psi = const: \quad x_1 y_1 = x_2 y_2, \qquad y_2 = 0.74\ m$$

e)

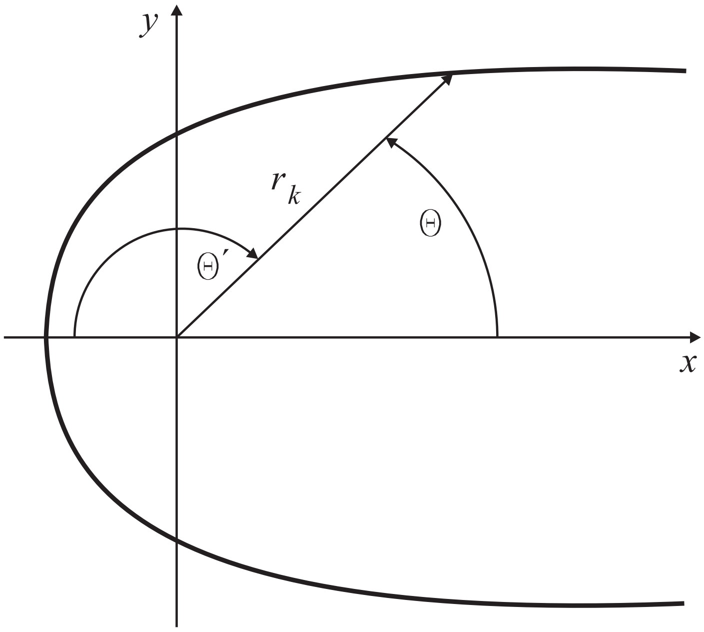
$$p_1 - p_2 = (c_{p1} - c_{p2})\frac{\rho}{2}\vec{v}_{ref}^2 = 0.442 \cdot 10^5\ N/m^2$$

f)



14.6

a)



$$\psi = u_\infty y + \frac{E}{2\pi}\theta + c = u_\infty \left[y + \frac{h}{\pi} \arctan\left[\frac{y}{x}\right]\right] + c$$

$$u = u_\infty \left[1 + \frac{h}{\pi}\frac{x}{x^2+y^2}\right]$$

$$v = u_\infty \frac{h}{\pi}\frac{y}{x^2+y^2}$$

stagnation point: $u = v = 0: \quad x_s = -\dfrac{h}{\pi}, y_s = 0$

$$u(x_s, h) = u_\infty \frac{\pi^2}{1+\pi^2}$$

$$v(x_s, h) = u_\infty \frac{\pi}{1+\pi^2}$$

b) Contour: Streamline crossing the stagnation point

$$r_k = \frac{h}{\pi}\ \frac{\pi-\theta}{\sin\theta} = \frac{h}{\pi}\ \frac{\theta'}{\sin\theta} \qquad \text{with} \qquad \theta' = \pi - \theta$$

c)

$$c_p = 1 - \frac{u^2 + v^2}{u_\infty^2} = -\frac{h}{\pi}\frac{2x + \frac{h}{\pi}}{x^2 + y^2}$$

$$c_{pk} = \frac{\sin(2\theta')}{\theta'} - \left(\frac{\sin\theta'}{\theta'}\right)^2$$

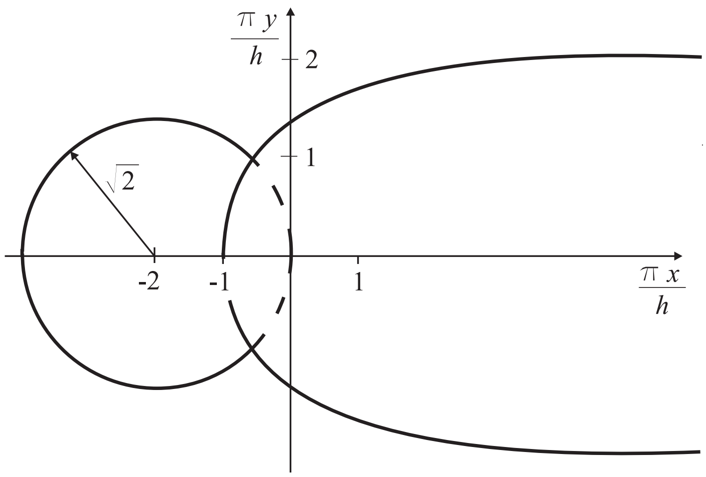
d)

$$c_p = const: \qquad \left[x + \frac{h}{\pi c_p}\right]^2 + y^2 = (1 - c_p)\left(\frac{h}{\pi c_p}\right)^2$$

Circles around $\left(-\frac{h}{\pi c_p}, 0\right)$ with radius $\frac{h\sqrt{1 - c_p}}{\pi c_p}$

e)

$$c_p = \frac{1}{2}: \qquad \left(x + \frac{2h}{\pi}\right)^2 + y^2 = 2\left(\frac{h}{\pi}\right)^2$$



f)

$$\frac{\sqrt{u^2 + v^2}}{u_\infty} = k$$

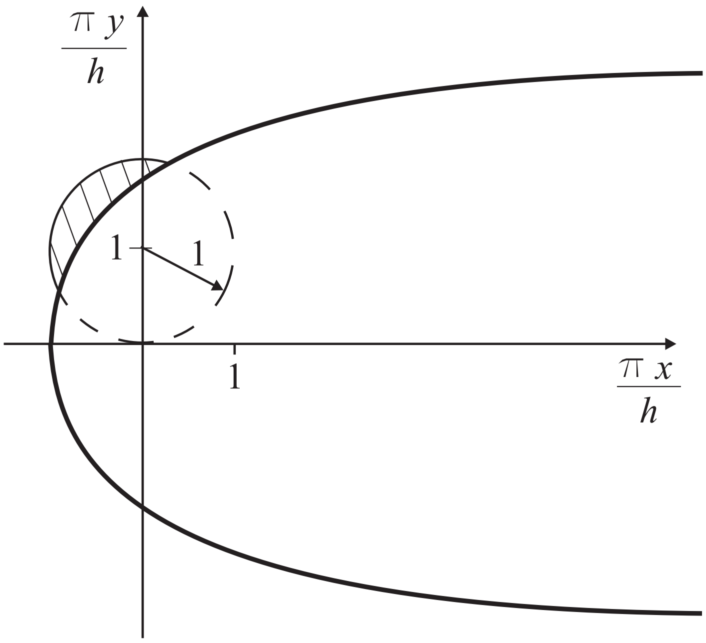
$$\left(x - \frac{\frac{h}{\pi}}{k^2 - 1}\right)^2 + y^2 = \left(\frac{\frac{kh}{\pi}}{k^2 - 1}\right)^2$$

Circles around $\left(\frac{h}{\pi(k^2 - 1)}, 0\right)$ with radius $\frac{kh}{\pi(k^2 - 1)}$

g)

$$v = u_\infty \frac{h}{\pi}\frac{y}{x^2 + y^2} > \frac{u_\infty}{2}$$

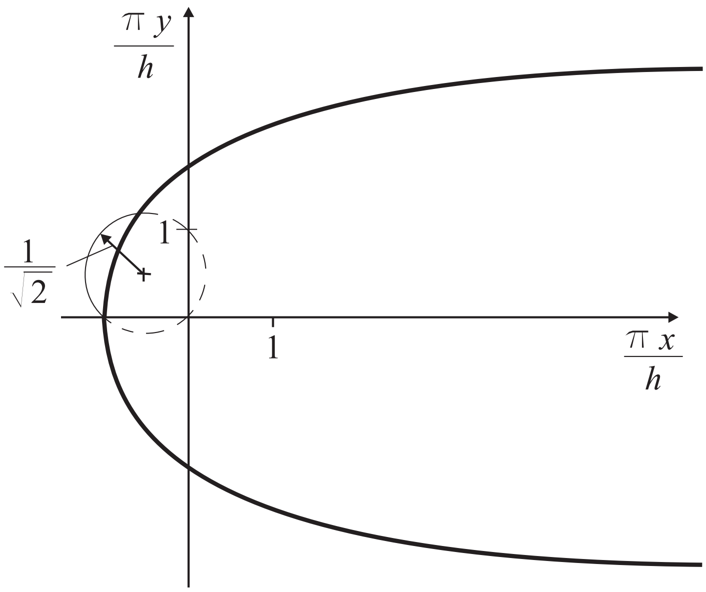
$$x^2 + \left(y - \frac{h}{\pi}\right)^2 < \left(\frac{h}{\pi}\right)^2$$



h)

$$\tan\alpha = \frac{v}{u} = 1$$

$$\left(x + \frac{h}{2\pi}\right)^2 + \left(y - \frac{h}{2\pi}\right)^2 = \frac{1}{2}\left(\frac{h}{\pi}\right)^2$$



i) Acceleration on the x-axis:

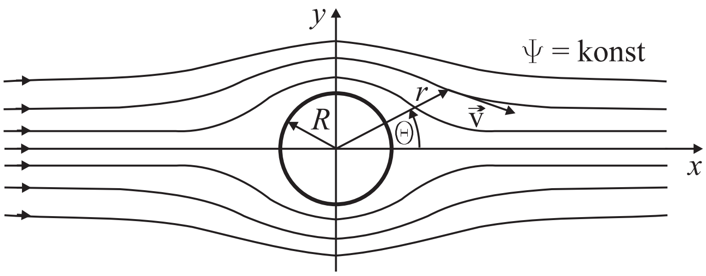
$$b = \frac{du}{dt} = u\frac{\partial u}{\partial x} = -u_\infty \frac{h}{\pi}\left(\frac{1}{x^2} + \frac{h}{\pi}\frac{1}{x^3}\right)$$

$$\frac{db}{dx} = 0: \quad x_{max} = -\frac{3}{2}\frac{h}{\pi}$$

$$b_{max} = -\frac{4}{27}\frac{\pi}{h}u_\infty^2$$

14.7

a)



$$\psi = 0: \quad y = 0$$

$$x^2 + y^2 = R^2$$

$$r = \sqrt{x^2 + y^2} \to \infty: \quad \psi \to u_\infty y (\text{Parallel flow})$$

Streamfunction describes the flow around a cylinder.

b)

$$c_p = 1 - \frac{v_r^2 + v_\theta^2}{u_\infty^2}$$

$$v_r = u_\infty \left[1 - \left(\frac{R}{r}\right)^2\right] \cos\theta$$

$$v_\theta = -u_\infty \left[1 + \left(\frac{R}{r}\right)^2\right] \sin\theta$$

$$r = R: c_p = 1 - 4\sin^2\theta$$

c)

$$\triangle t = \int_{-3R}^{-2R} \frac{dx}{u(x,0)}$$

$$u(x,0) = u_\infty \left(1 - \frac{R^2}{x^2}\right)$$

$$\triangle t = \frac{1}{u_\infty}\left[x + \frac{R}{2}\ln\frac{x-R}{x+R}\right]_{-3R}^{-2R} = \frac{R}{u_\infty}(1 + 0.5\ln 1.5)$$

14.8

a)

$$\rho g h_\infty + \frac{\rho}{2}u_\infty^2 = \rho g h(\theta) + \frac{\rho}{2}\vec{v}^2$$

$$r = R : \vec{v}^2 = v_\theta^2 = 4u_\infty^2 \sin^2\theta$$

$$h(\theta) - h_\infty = \frac{u_\infty^2}{2g}(1 - 4\sin^2\theta)$$

b) Stagnation points: $\theta = 0$ and $\theta = \pi$

$$h = h_\infty + \frac{u_\infty^2}{2g} = 6.05\ m$$

c)

$$\theta_{min} = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$h_{min} = h_\infty - \frac{3u_\infty^2}{2g} = 5.85\ m$$

14.9

a)

$$F(z) = u_\infty z + 0.8\frac{u_\infty H^2}{z} = u_\infty(x+iy) + 0.8\frac{u_\infty H^2}{x^2+y^2}(x - iy)$$

$$\Longrightarrow \psi = u_\infty y - 0.8\frac{u_\infty H^2}{x^2+y^2}y$$

$$\psi_k = \psi(x=0, y=H) = u_\infty H - 0.8\frac{u_\infty H^2}{H^2}H = 0.2Hu_\infty$$

Contour equation:

$$0.2Hu_\infty = u_\infty y - 0.8\frac{u_\infty H^2}{x^2+y^2}y$$

$$\Longrightarrow \frac{y}{H} - 0.8\frac{H^2}{x^2+y^2}\frac{y}{H} = 0.2$$

b)

$$\frac{P}{\dot{V}} = \frac{\rho}{2}\vec{v}^2 \quad\Longrightarrow\quad P = \frac{\rho}{2}(u^2+v^2)\frac{\dot{V}}{B} \quad \text{(based on the width)}$$

$$u = \frac{\partial\psi}{\partial y} = u_\infty - 0,8H^2u_\infty\frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2} = u_\infty - 0.8H^2u_\infty\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$v = -\frac{\partial\psi}{\partial x} = -0.8H^2u_\infty y\frac{-2x}{(x^2+y^2)^2} = 1.6u_\infty H^2\frac{xy}{(x^2+y^2)^2}$$

I. far before the mountain ($x \to -\infty$)

$u = u_\infty, v = 0$ mit $\dfrac{\dot{V}}{B} = Hu_\infty \to P_I = \dfrac{\rho}{2}Hu_\infty^3$

II. on the top ($x = 0, y$)

$$u = u_\infty + 0.8u_\infty\frac{H^2}{y^2}; \quad v = 0$$

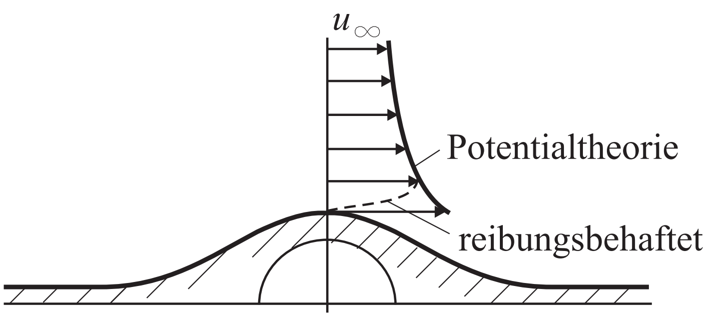
$$P = \frac{\rho}{2}\int_H^{2H} u^3(y)dy = \frac{\rho}{2}\int_H^{2H}(u_\infty + 0.8u_\infty\frac{H^2}{y^2})^3dy$$

$$P = \frac{\rho}{2}u_\infty^3\int_H^{2H}(1 + 3\cdot 0.8\frac{H^2}{y^2} + 3\cdot 0.8^2\frac{H^4}{y^4} + 0.8^3\frac{H^6}{y^6})dy$$

$$= \frac{\rho}{2} u_\infty^3 \left(y - 3 \cdot 0.8 \frac{H^2}{y} - 3 \cdot 0.8^2 \frac{H^4}{3y^3} - 0.8^3 \frac{H^6}{5y^5} \right) \Bigg|_H^{2H}$$

$$P = 2.86 \frac{\rho}{2} H u_\infty^3$$

c) Parallel flow + dipole $\hat{=}$ cylinder flow



14.10

a)

$$\psi = u_\infty r \sin\theta \left[1 - \left(\frac{R}{r} \right)^2 \right] - \frac{\Gamma}{2\pi} \ln\ r$$

$$v_r = u_\infty \left[1 - \left(\frac{R}{r} \right)^2 \right] \cos\theta$$

$$v_\theta = -u_\infty \left[1 + (\frac{R}{r})^2 \right] \sin\theta + \frac{\Gamma}{2\pi r}$$

$$r = R: \quad v_\theta|_{\text{Wirbel}} = v_t = \frac{\Gamma}{2\pi R}$$

$$\Gamma = 2\pi R v_t$$

b) Flow field $v_t = u_\infty$:

$$\psi = u_\infty \left[r \sin\theta \left(1 - \frac{R^2}{r^2} \right) - R \ln\ r \right]$$

$$v_r = u_\infty \left[1 - \frac{R^2}{r^2}\right] \cos\theta$$

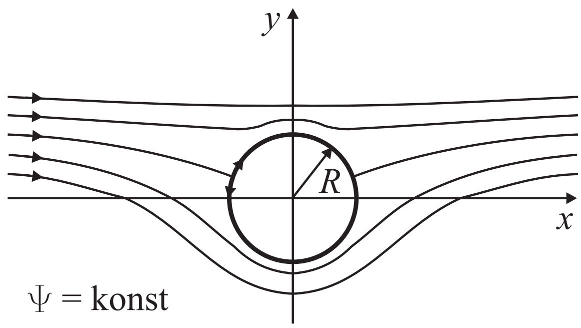
$$v_\theta = u_\infty \left[\frac{R}{r} - \left(1 + \frac{R^2}{r^2}\right) \sin\theta\right]$$

contour:

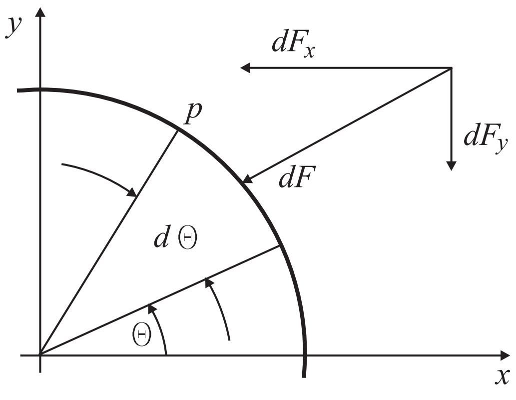
$$\psi = \psi_K = \psi(r = R, \theta = 0) = u_\infty(-R\ \ln R)$$

$$\Rightarrow u_\infty \left[r\ \sin\left(1 - \frac{R^2}{r^2}\right) - R \ln\left(\frac{r}{R}\right)\right] = 0 \quad ()$$

2 stagnation points on the contour ($r = R$):
$\theta_s = \frac{\pi}{6}, \frac{5\pi}{6}$; no free stagnation points



c)



$$dF_x = -pLR \cos\theta d\theta$$

$$dF_y = -pLR \sin\theta d\theta$$

$$p = p_\infty + c_p \frac{\rho}{2} {u_\infty}^2$$

$$r = R: \quad c_p = 1 - \left[\frac{v_t}{u_\infty} - 2\sin\theta\right]^2$$

$$F_x = -LR\int_0^{2\pi}\left\{\frac{\rho}{2}{u_\infty}^2\left[1-\left(\frac{v_t}{u_\infty} - 2\sin\theta\right)^2\right] + p_\infty\right\}\cos\theta d\theta = 0$$

$$\begin{aligned} F_y &= -LR\int_0^{2\pi}\left\{\frac{\rho}{2}u_\infty^2\left[1-\left(\frac{v_t}{u_\infty} - 2\sin\theta\right)^2\right] + p_\infty\right\}\sin\theta d\theta \\ &= -2\pi\rho LRv_t u_\infty = -\rho u_\infty \Gamma L \end{aligned}$$

14.11

a)

$$F = Az^{\frac{2}{3}} = A\left(re^{i\theta}\right)^{\frac{2}{3}} = Ar^{\frac{2}{3}}\left(\cos\frac{2}{3}\theta + i\sin\frac{2}{3}\theta\right) = \phi + i\psi$$

$$\psi = Ar^{\frac{2}{3}}\sin\frac{2}{3}\theta$$

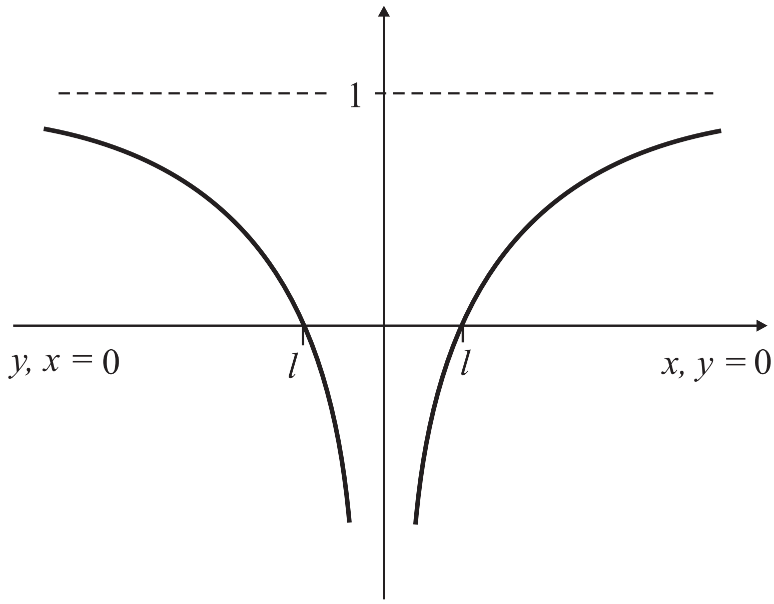
$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{2}{3}Ar^{-\frac{1}{3}}\cos\frac{2}{3}\theta$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -\frac{2}{3}Ar^{-\frac{1}{3}}\sin\frac{2}{3}\theta$$

$$\theta = 0, r = l : v_r = \frac{2}{3}Al^{-\frac{1}{3}} = u_1 \longrightarrow A = \frac{3}{2}u_1 l^{\frac{1}{3}}$$

b)

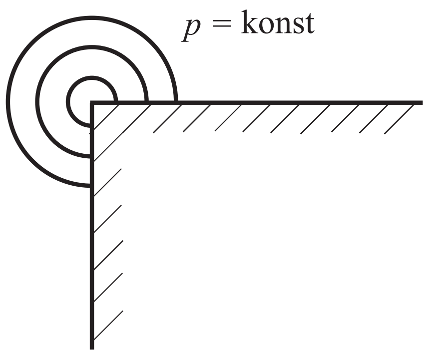
$$c_p = 1 - \frac{\vec{v}^2}{u_1^2} = 1 - \left(\frac{r}{l}\right)^{-\frac{2}{3}}$$



c)
Lines $p = const.$:

$r = const. \to c_p = const. \to p = const.$

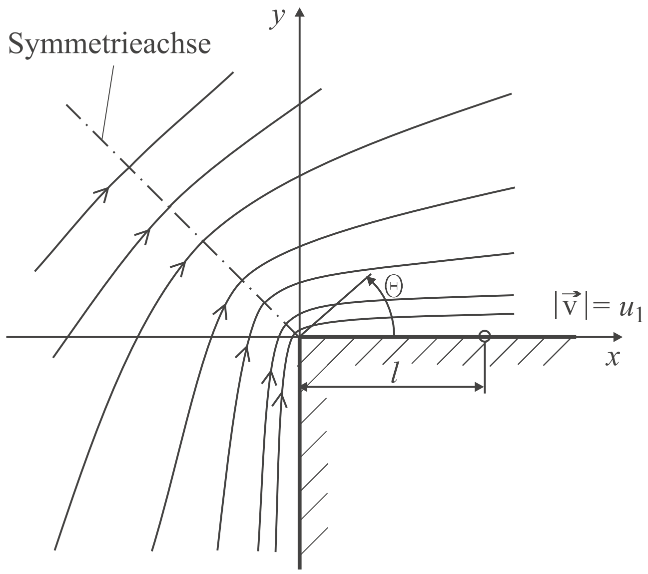
Circles around the origin



d)
Streamlines: $\psi = Ar^{\frac{2}{3}} \sin \dfrac{2}{3}\theta = const.$

$$\Longrightarrow \quad r_s^{\frac{2}{3}} = \frac{\psi}{A \sin \frac{2}{3}\theta} \quad \Longrightarrow \quad r_s(\theta) = \left(\frac{\psi}{A \sin \frac{2}{3}\theta} \right)^{\frac{3}{2}}$$

e)



14.12

a)

Condition: $\omega = 0 \rightarrow \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0$

$$\frac{\partial^2\psi(x,y)}{\partial y^2} = 0 \rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(yu_\infty L \frac{\partial}{\partial x}\left(\frac{1}{h(x)}\right)\right) = 0$$

$$\frac{\partial}{\partial x}\left(-yu_\infty L \frac{h'(x)}{h^2(x)}\right) = 0 \rightarrow \frac{d}{dx}\left(\frac{h'(x)}{h^2(x)}\right) = 0$$

integrate 2 times: $-\dfrac{1}{h(x)} = C_1 x + C_2$

B.C.:

$$x = L, \quad h = L \quad \Longrightarrow \quad -\frac{1}{L} = C_1 L + C_2$$

$$x = 3L, \quad h = \frac{1}{3}L \quad \Longrightarrow \quad -\frac{3}{L} = 3C_1 L + C_2$$

$$\Longrightarrow \quad C_2 = 0, \quad C_1 = -\frac{1}{L^2} \quad \Longrightarrow \quad h(x) = \frac{L^2}{x}$$

b)

$$u = \frac{\partial\psi}{\partial y}; \quad v = -\frac{\partial\psi}{\partial x}; \quad \psi = \frac{u_\infty}{L} xy$$

$$u = u_\infty \frac{x}{L}; \qquad v = -u_\infty \frac{y}{L}$$

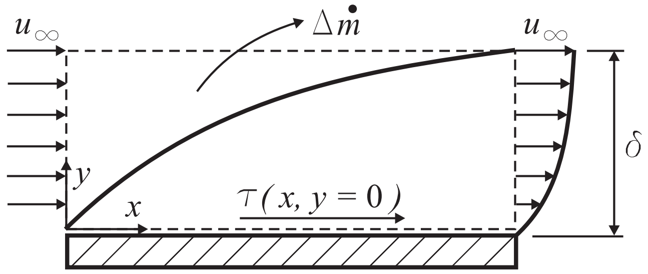
c)

$$\dot{V} = \psi_{(y=h)} - \psi_{(y=-h)}|_{x=L}$$

$$\dot{V} = (h_1 + h_1)Bu_\infty$$
$$\dot{V} = 2U_\infty LB$$

15 Laminar boundary layers

15.1



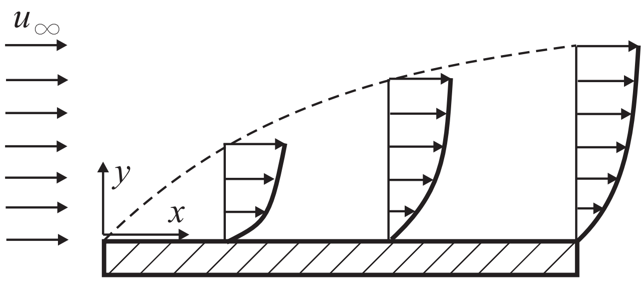
$$-\rho u_\infty^2 \delta + \rho \int_0^\delta u^2 dy + \triangle \dot{m} u_\infty = \int_0^x \tau(x',\ y = 0) dx'$$

$$\triangle \dot{m} = \rho \int_0^\delta (u_\infty - u) dy$$

$$\int_0^\delta \frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty}\right) dy = \delta_2 = - \int_0^x \frac{\tau(x',\ y = 0)}{\rho u_\infty^2} dx'$$

15.2

a) $Re_L = 3.33 \cdot 10^5$: laminar boundary layer

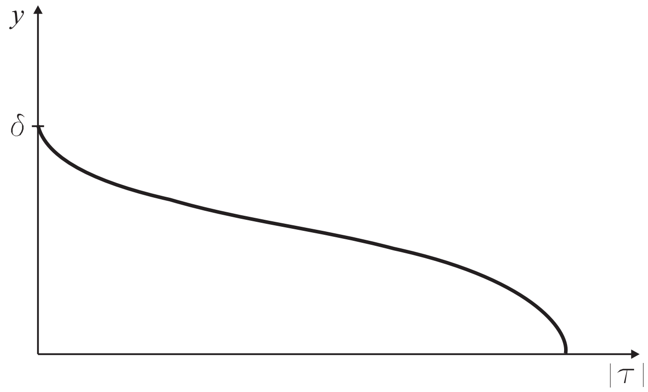


b)

$$y = 0 : u = v = 0$$

$$y \to \infty : u \to u_\infty$$

c) from boundary layer equation :



$\dfrac{\partial \tau}{\partial y} = 0$ for $y = 0$ and $y = \delta$

d) from Blasius solution:

$$\delta(x = L) = \frac{5L}{\sqrt{Re_L}} \qquad c_f = \frac{0.664}{\sqrt{Re_x}}$$

$$\delta(x = L) = 4.33\ mm$$

$$F_w = 2\int_0^L B\tau_w dx = \rho u_\infty^2 B \int_0^L c_f dx = 0.144\ N$$

15.3

Determination of the coefficients $a_0,\ a_1,\ a_2$ by using the boundary conditions:

B.C.1: no slip $u(x, y = 0) = 0$

B.C.2: boundary layer edge $u(x, y = \delta) = u_\infty$

R.B 3: at the wall (from x-momentum) $\eta \dfrac{\partial^2 u(x, y = 0)}{\partial y^2} = \dfrac{\partial p}{\partial x} = 0$

If additional boundary conditions are necessary, a steady transition at $\dfrac{y}{\delta} = 1$ ac be assumed, i.e.

$\left.\dfrac{\partial^n u}{\partial y^n}\right|_{y=\delta} = 0$ with $n \geq 1$

$\Longrightarrow$ from B.C.1 follows $a_0 = 0$

from B.C.2 follows $a_1 + a_2 = 1$

from B.C.3 follows

$$\frac{\partial^2 u}{\partial y^2} = u_\infty \frac{1}{\delta^2} \frac{\partial^2 (u/u_\infty)}{\partial (y/\delta)^2} = 2u_\infty \frac{1}{\delta^2} a_2 = 0$$

$$\Longrightarrow a_2 = 0, \quad a_1 = 1$$

$$\Longrightarrow \frac{u(x,y)}{u_\infty} = \frac{y}{\delta} \qquad \text{linear distribution.}$$

Permutation of the order of boundary conditions, e,g,:

B.C.1 $u(x, y = 0) = 0$

B.C.2 $u(x, y = \delta) = u_\infty$

B.C.3 $\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0$

$\Longrightarrow a_0 = 0; \quad a_1 = 2; \quad a_2 = -1$ parabolic distribution

Since the approximation of the velocity profiles is not an exact solution of the boundary layer equations, the boundary conditions are not satisfiable in all cases. Hence, it has to be paid attention, that the physical boundary conditionsa re satisfied first.

Computation of the boundary layer thickness $\delta(x)$:

$$\tau_w = \eta \left.\frac{\partial u}{\partial y}\right|_{y=0} = \eta \frac{u_\infty}{\delta} \left.\frac{\partial u/u_\infty}{\partial y/\delta}\right|_{y=0} = \eta \frac{u_\infty}{\delta}$$

$$u(u_\infty - u) = u_\infty \frac{y}{\delta}\left(u_\infty - u_\infty \frac{y}{\delta}\right) = u_\infty^2 \left(\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right)$$

with the equation

$$B \int_0^x \tau_w(x) dx = B\rho \int_0^{\delta(x)} u(u_\infty - u) dy$$

$$\Longrightarrow \int_0^x \eta \frac{u_\infty}{\delta(x)} dx = \rho u_\infty^2 \int_0^1 \delta \left(\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2\right) d\left(\frac{y}{\delta}\right)$$

$$\eta u_\infty \int_0^x \frac{dx}{\delta(x)} = \rho u_\infty^2 \delta(x) \left(\frac{1}{2} \left(\frac{y}{\delta}\right)^2 - \frac{1}{3} \left(\frac{y}{\delta}\right)^3 \right) \Bigg|_0^1$$

$$= \rho u_\infty^2 \delta(x) \frac{1}{6}$$

Differentiating gives

$$\frac{1}{\delta(x)} = \frac{1}{6} \frac{\rho}{\eta} u_\infty \frac{d\delta(x)}{dx}$$

$$\Longrightarrow dx = \frac{1}{6} \frac{\rho}{\eta} u_\infty \delta(x) d\delta(x)$$

Integration:

$$x = \frac{1}{12} \frac{\rho}{\eta} u_\infty \delta^2(x)$$

$$\Longrightarrow \delta(x) = \sqrt{\frac{12\nu x}{u_\infty}}$$

$$\Longrightarrow \delta(x) = \sqrt{12} \sqrt{\frac{\nu x}{u_\infty}} \approx 3,5 \sqrt{\frac{\nu x}{u_\infty}}$$

Compare with Blasius solution: $\delta(x) = 5.2 \sqrt{\dfrac{\nu x}{u_\infty}}$

15.4

a) $\dfrac{\partial p}{\partial x}$ in the flow:

frictionless outer flow: $\rho u_a \dfrac{\partial u_a}{\partial x} = -\dfrac{\partial p}{\partial x}$

$$\Longrightarrow \frac{\partial p}{\partial x} = -\rho u_a \frac{\partial u_a}{\partial x} = -\rho(u_{a_1} - C(x - x_1)^2) \cdot (-2C(x - x_1))$$

$$\frac{\partial p}{\partial x} = +2\rho C(x - x_1)(u_{a_1} - C(x - x_1)^2)$$

$$\frac{\partial p}{\partial x}\bigg|_{x_1} = 0$$

$$\frac{\partial p}{\partial x}\bigg|_{x_2} = 2\rho C(x_2 - x_1)(u_{a_1} - C(x_2 - x_1)^2)$$

b) 4 B.C:

I $\frac{y}{\delta} = 0 \quad u = 0 \quad H.B. \Longrightarrow a_0 = 0$

II $\frac{y}{\delta} = 1 \quad u = u_a$

III $\frac{y}{\delta} = 0 \quad \frac{\partial p}{\partial x} = \eta \frac{\partial^2 u}{\partial y^2}$ at the wall

IV $\frac{y}{\delta} = 1 \quad \frac{\partial u}{\partial y} = 0$

$$u = \left(a_1 \left(\frac{y}{\delta}\right) + a_2 \left(\frac{y}{\delta}\right)^2 + a_3 \left(\frac{y}{\delta}\right)^3 \right) u_a$$

$$\frac{\partial u}{\partial y} = \left(a_1 \frac{1}{\delta} + 2a_2 \frac{y}{\delta^2} + 3a_3 \frac{y^2}{\delta^3} \right) u_a$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\qquad + 2a_2 \frac{1}{\delta^2} + 6a_3 \frac{y}{\delta^3} \right) u_a$$

II: $\quad 1 = a_1 + a_2 + a_3$

IV: $\quad 0 = a_1 + 2a_2 + 3a_3$

III: $\quad \frac{\partial p}{\partial x} = \eta\, u_a \frac{2a_2}{\delta^2} \quad \Longrightarrow \quad a_2 = \frac{1}{2} \frac{\delta^2}{\eta\, u_a} \frac{\partial p}{\partial x}$

$$\Longrightarrow \quad a_2(x) = \frac{\delta^2}{2\eta} \frac{2\, \rho\, C\, (x - x_1)(u_{a_1} - C\, (x - x_1)^2)}{(u_{a_1} - C \cdot (x - x_1)^2)}$$

$$a_2(x) = \frac{\delta^2\, \rho\, C}{\eta} (x - x_1)$$

II / IV: $\quad a_3 = -\frac{1}{2} - \frac{1}{2} a_2$

$$a_1 = \frac{3}{2} - \frac{1}{2} a_2$$

therefore:

$$a_1(x) \;=\; \frac{3}{2} \;-\; \frac{1}{2}\,\frac{\delta^2\;\rho\;C}{\eta}\;(x-x_1)$$

$$a_3(x) \;=\; -\;\frac{1}{2} \;-\; \frac{1}{2}\,\frac{\delta^2\;\rho\;C}{\eta}\;(x-x_1)$$

15.5

a)

$$\frac{d\delta_2}{dx} + \frac{1}{u_a}\frac{du_a}{dx}(2\delta_2+\delta_1) + \frac{\tau(y=0)}{\rho u_a^2} = 0$$

- $$\frac{\delta_1}{\delta_0} = \int_0^1\left(1-\frac{u}{u_a}\right)d\left(\frac{y}{\delta}\right) = \int_0^1\left(1-\left(\frac{y}{\delta_0}\right)^{\frac{1}{2}}\right)d\left(\frac{y}{\delta}\right) = \left(\frac{y}{\delta}-\frac{2}{3}\left(\frac{y}{\delta}\right)^{\frac{3}{2}}\right)_0^1$$

$$\Longrightarrow \delta_1 = \frac{1}{3}\;\delta_0$$

- $$\frac{\delta_2}{\delta_0} = \int_0^1\frac{u}{u_a}\left(1-\frac{u}{u_a}\right)d\left(\frac{y}{\delta}\right) = \int_0^1\left(\left(\frac{y}{\delta}\right)^{1/2}-\left(\frac{y}{\delta}\right)\right)d\left(\frac{y}{\delta}\right) = \left(\frac{2}{3}\left(\frac{y}{\delta}\right)^{3/2}-\frac{1}{2}\left(\frac{y}{\delta}\right)^2\right)_0^1$$

$$\Longrightarrow \delta_2 = \frac{1}{6}\;\delta_0$$

- $\dfrac{d\delta_2}{dx} = 0$, since $\delta_0 = const.$

into the above equation: $\rho\;u_a\;\dfrac{du_a}{dx}(2\delta_2+\delta_1) = \tau_w$

from the x-momentum equation for $y=\delta_0$:

$$\rho u_a\frac{du_a}{dx} = -\frac{dp}{dx}$$

$$\Longrightarrow \tau_w = -\frac{dp}{dx}(2\delta_2+\delta_1)$$

with $$\frac{dp}{dx} = p_0\frac{d}{dx}\left(1-k\left(\frac{x}{l}\right)^2\right) = -p_0\;k\;\frac{2x}{l^2}$$

$$\Longrightarrow \tau_w = p_0k\frac{2x}{l^2}\frac{2}{3}\delta_0 = \frac{4}{3}p_0k\frac{\delta_0 x}{l^2}$$

15.6

conti.: $\underbrace{\dfrac{\partial u}{\partial x}}_{=0} + \dfrac{\partial v}{\partial y} = 0 \rightarrow v = v_a = konst$

momentum equation:

$$\frac{d\vec{I}}{dt} = \int \rho\ \vec{u}\ (\vec{u}\ \cdot \vec{n})\ dA = \sum\ \vec{F}_{KV} = \vec{F}_R$$

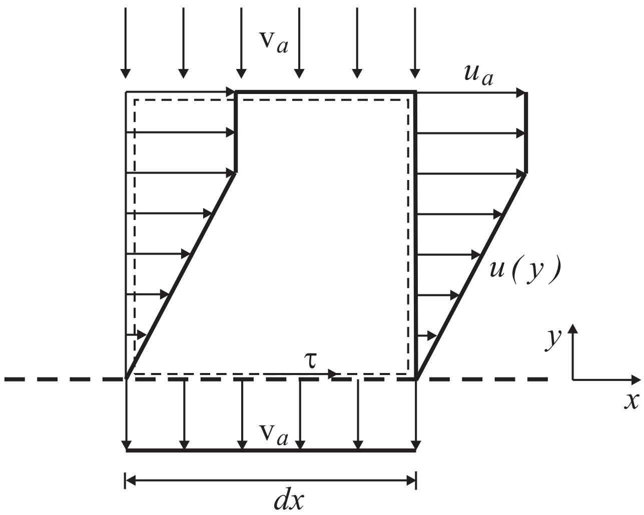
momentum balance x-direction:

$$-\rho\ v_a\ u_a\ dx\ B - \rho\ u^2(y)\ dy\ B + \rho\ u^2(y)\ dy\ B + \rho\ v_a\ 0\ dx\ B = \tau_{y=0}\ dx\ B$$

$$-\rho\ v_a\ u_a\ dx\ B = \tau_{y=0}\ dx\ B$$

with

$$\tau_{y=0} = -\eta \frac{du}{dy} = -\eta \frac{u_a}{\delta} \quad \Longrightarrow \quad \rho v_a = \frac{\eta}{\rho} \quad \Longrightarrow \quad v_a = \frac{\eta}{\rho\delta}$$



15.7

a) boundary conditions:

$$\frac{y}{\delta} = 0: \qquad \frac{u}{u_a} = 0, \qquad \frac{v}{u_a} = 0$$

$$\frac{y}{\delta} = 1: \qquad \frac{u}{u_a} = 1$$

from boundary layer equation

$$\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = \eta \frac{\partial^2 u}{\partial y^2}:$$

$$\frac{y}{\delta} = 0: \qquad u = v = 0: \qquad \frac{\partial^2 (u/u_a)}{\partial (y/\delta)^2} = 0$$

$$\frac{y}{\delta} = 1: \qquad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0: \qquad \frac{\partial^2 (u/u_a)}{\partial (y/\delta)^2} = 0$$

frictionless outer flow:

$$\frac{y}{\delta} = 1: \qquad \tau \sim \frac{\partial(u/u_a)}{\partial(y/\delta)} = 0$$

$$\frac{u}{u_a} = 2\left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^3 + \left(\frac{y}{\delta}\right)^4$$

b)

$$\frac{\delta_1}{\delta} = \int_0^1 \left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \frac{3}{10}$$

$$\frac{\delta_2}{\delta} = \int_0^1 \frac{u}{u_a}\left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \frac{37}{315}$$

von Kármán integral equation

$$\frac{d\delta_2}{dx} + \frac{\tau(y=0)}{\rho u_a^2} = 0$$

$$\tau(y=0) = -\frac{\eta u_a}{\delta} \left.\frac{d(u/u_a)}{d(y/\delta)}\right|_{y/\delta = 0} = -2\frac{\eta u_a}{\delta}$$

Integration: $\dfrac{\delta}{x} = \dfrac{5.84}{\sqrt{Re_x}}$

$$c_w = \frac{2}{L}\int_0^L \frac{\tau_w}{\rho u_a^2} dx = -\frac{2}{L}\int_0^L \frac{\tau(y=0)}{\rho u_a^2} dx = \frac{1.371}{\sqrt{Re_L}}$$

15.8

a)

$$\frac{u(x,y)}{u_a(x)} = a_0 + a_1 \frac{y}{\delta_0}$$

B.C. $y = 0 \implies u = 0 \implies a_0 = 0$

$y = \delta_0 \implies u = u_a(x) \implies a_1 = 1$

$$\implies \frac{u(x,y)}{u_a(x)} = \frac{y}{\delta_0}$$

b)

$$\frac{d\delta_2}{dx} + \frac{1}{u_a}\frac{du_a}{dx}(2\delta_2 + \delta_1) = \frac{\tau_w}{\rho u_a^2}$$

displacement thickness $\frac{\delta_1}{\delta_0} = \int_0^1 \left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta_0}\right) = 1/2$

momentum thickness $\frac{\delta_2}{\delta_0} = \int_0^1 \left(1 - \frac{u}{u_a}\right) \frac{u}{u_a} d\left(\frac{y}{\delta_0}\right) = \frac{1}{6} \neq f(x) \implies \frac{d\delta_2}{dx} = 0$

$$\tau_w = \eta \left.\frac{\partial u}{\partial y}\right|_{y=0} = \eta \frac{u_a}{\delta} \left.\frac{\partial\left(\frac{u}{u_a}\right)}{\partial\left(\frac{y}{\delta}\right)}\right|_{y=0} = \eta \frac{u_a}{\delta}$$

von Kármán integral equation $\frac{du_a}{dx}$

$$\implies \frac{du_a}{dx} = \frac{6}{5} \frac{\eta}{\rho} \frac{1}{\delta_0^2}$$

$$\implies u_a(x) = \frac{6}{5} \frac{\eta}{\rho} \frac{x}{\delta_0^2} + C$$

B.C.: $x = 0 \implies u_a(x) = 0 \implies C = 0$

$$\implies u_a(x) = \frac{6}{5} \frac{\eta}{\rho} \frac{x}{\delta_0^2}$$

c)

$$F = \int_0^L \tau_w(x) B dx, \qquad \tau_w = \eta \frac{u_a(x)}{\delta_0}$$

$$\implies F = \frac{6}{5} \frac{\eta^2}{\rho} \frac{B}{\delta_0^3} \int_0^L x dx = \frac{3}{5} \frac{\eta^2}{\rho} \frac{BL^2}{\delta_0^3}$$

15.9

a) Solution see 15.7

A)

$$\frac{\delta_1}{\delta} = \frac{3}{8}$$

$$\frac{\delta_2}{\delta} = \frac{39}{280}$$

$$\frac{\delta}{x} = \frac{4.641}{\sqrt{Re_x}}$$

$$c_w = \frac{1.293}{\sqrt{Re_L}}$$

B)

$$\frac{\delta_1}{\delta} = 1 - \frac{2}{\pi} = 0.363$$

$$\frac{\delta_2}{\delta} = \frac{2}{\pi} - \frac{1}{2} = 0.137$$

$$\frac{\delta}{x} = \frac{\sqrt{\frac{2\pi^2}{4-\pi}}}{\sqrt{Re_x}} = \frac{4.795}{\sqrt{Re_x}}$$

$$c_w = \frac{2\sqrt{2-\pi/2}}{\sqrt{Re_L}} = \frac{1.310}{\sqrt{Re_L}}$$

b)

A)

$$\delta(x = L) = 3.28\ mm$$

$$F_w = c_w \frac{\rho}{2} u_a^2 2LB = 0.91\ N$$

B)

$$\delta(x = L) = 3.39\ mm$$

$$F_w = 0.93\ N$$

16 Turbulent boundary layers

16.1

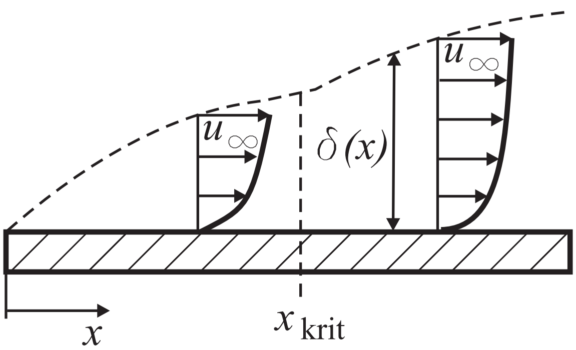
a)

$$x_{krit} = \frac{\nu Re_{crit}}{u_\infty} = 0,167\ m$$

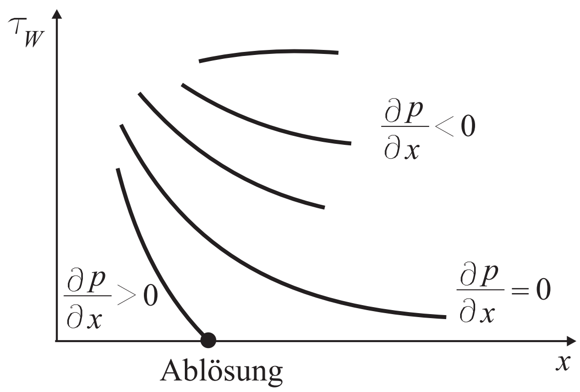
b)

$$\begin{aligned} \xi &= \frac{y}{x}\sqrt{Re_x} = 1,095 \\ \frac{u}{u_\infty} = 0,36 \quad &: \quad u = 16,2\ m/s \\ \text{(see Diagram} \quad \xi &= f(u/u_\infty), \quad \text{script chapter 15.4)} \\ x = 0,15\ m \quad &: \quad Re_x = 4,5 \cdot 10^5 \\ \frac{y}{x}\sqrt{Re_x} = 1,095 \quad &: \quad y = 2,45 \cdot 10^{-4}\ m \end{aligned}$$

c)



d)



16.2

a)

with the product rule

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial(uv)}{\partial y}\right)=\rho\left(\frac{\partial u}{\partial t}+2u\frac{\partial u}{\partial x}+u\frac{\partial v}{\partial y}+v\frac{\partial u}{\partial y}\right)=$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)+\rho u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

with $\partial u/\partial x+\partial v/\partial y=0$ (conti.) follows

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial(uv)}{\partial y}\right)=\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right).$$

b)

$$u=\overline{u}+u^{'},\quad v=\overline{v}+v^{'},\quad p=\overline{p}+p^{'}$$

$$\overline{\rho\left(\frac{\partial(\bar{u}+u^{'})}{\partial t}+\frac{\partial(\bar{u}+u^{'})^2}{\partial x}+\frac{\partial((\bar{u}+u^{'})(\bar{v}+v^{'}))}{\partial y}\right)}=$$

$$\overline{f_x-\frac{\partial(\bar{p}+p^{'})}{\partial x}+\eta\left(\frac{\partial^2(\bar{u}+u^{'})}{\partial x^2}+\frac{\partial^2(\bar{u}+u^{'})}{\partial y^2}\right)}$$

The computational rules 2 and 4 result with $\overline{f}_x=f_x$

$$\rho\left(\frac{\overline{(\overline{u}+u^{'})}}{\partial t}+\frac{\partial\overline{(\overline{u}+u^{'})^2}}{\partial x}+\frac{\partial\overline{((\overline{u}+u^{'})(\overline{v}+v^{'}))}}{\partial y}\right)=$$

$$\overline{f}_x-\frac{\partial\overline{(\overline{p}+p^{'})}}{\partial x}+\eta\left(\frac{\partial^2\overline{(\overline{u}+u^{'})}}{\partial x^2}+\frac{\partial^2\overline{(\overline{u}+u^{'})}}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial\overline{\overline{u}}}{\partial t}+\frac{\partial\overline{u^{'}}}{\partial t}+\frac{\partial\overline{(\overline{u}+u^{'})^2}}{\partial x}+\frac{\partial\overline{((\overline{u}+u^{'})(\overline{v}+v^{'}))}}{\partial y}\right)=$$

$$f_x - \frac{\partial \overline{\overline{p}}}{\partial x} - \frac{\partial \overline{p'}}{\partial x} + \eta \left(\frac{\partial^2 \overline{\overline{u}}}{\partial x^2} + \frac{\partial^2 \overline{u'}}{\partial x^2} + \frac{\partial^2 \overline{\overline{u}}}{\partial y^2} + \frac{\partial^2 \overline{u'}}{\partial y^2} \right)$$

with $\overline{u'} = 0$ and $\overline{p'} = 0$ follows

$$\rho \left(\frac{\partial \overline{u}}{\partial t} + \frac{\partial \overline{(\overline{u} + u')^2}}{\partial x} + \frac{\partial \overline{((\overline{u} + u')(\overline{v} + v'))}}{\partial y} \right) = f_x - \frac{\partial \overline{p}}{\partial x} + \eta \frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2}.$$

Regard

$$\frac{\partial \overline{(\overline{u} + u')^2}}{\partial x} \qquad \text{und} \qquad \frac{\partial \overline{(\overline{u} + u')(\overline{v} + v')}}{\partial y}$$

follows

$$\overline{\frac{\partial (\bar{u} + u')^2}{\partial x}} = \frac{\partial \overline{(u^{\bar{2}} + 2\bar{u}u' + u'^2)}}{\partial x} = \frac{\partial \overline{\overline{u}^2}}{\partial x} + \frac{\partial \overline{(2\bar{u}u')}}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x} = \frac{\partial \overline{u}^2}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x}$$

and

$$\frac{\partial \overline{(\overline{u} + u')(\overline{v} + v')}}{\partial y} = \frac{\partial \overline{(\overline{u}\overline{v} + \overline{u}v' + u'\overline{v} + u'v')}}{\partial y} =$$

$$\frac{\partial \overline{(\bar{u}\bar{v})}}{\partial y} + \frac{\partial \overline{(\overline{u}v')}}{\partial y} + \frac{\partial \overline{(u'\overline{v})}}{\partial y} + \frac{\partial \overline{(u'v')}}{\partial y} = \frac{\partial \bar{u}\bar{v}}{\partial y} + \frac{\partial \overline{u'v'}}{\partial y}.$$

put into the Reynolds averaged x-momentum equation

$$\rho \left(\frac{\partial \overline{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial (\bar{u}\bar{v})}{\partial y} + \frac{\partial \overline{(u'v')}}{\partial y} \right) = f_x - \frac{\partial \overline{p}}{\partial x} + \eta \left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} \right)$$

algebraic transformation

$$\rho \left(\frac{\partial \overline{u}}{\partial t} + \frac{\partial \bar{u}^2}{\partial x} + \frac{\partial (\bar{u}\bar{v})}{\partial y} \right) = f_x - \frac{\partial \overline{p}}{\partial x} + \eta \left(\frac{\partial^2 \overline{u}}{\partial x^2} + \frac{\partial^2 \overline{u}}{\partial y^2} \right) + \rho \left(-\frac{\partial \overline{u'^2}}{\partial x} - \frac{\partial \overline{(u'v')}}{\partial y} \right)$$

16.3

a)

$$\frac{F_{w1}}{F_{w2}} = \frac{c_{w1}}{c_{w2}}$$

u_∞	Re_1	$10^3\ c_{w1}$	Re_2	$10^3\ c_{w2}$	$\frac{F_{w1}}{F_{w2}}$
$0,4 m/s$	$4 \cdot 10^5$	$2,10$	$2 \cdot 10^5$	$2,97$	$0,707$
$0,8 m/s$	$8 \cdot 10^5$	$2,76$	$4 \cdot 10^5$	$2,10$	$1,313$
$1,6 m/s$	$1,6 \cdot 10^6$	$3,19$	$8 \cdot 10^5$	$2,76$	$1,156$

b) $Re_1 = 1,96 \cdot 10^5$

$c_{w1} = c_{w2} = 3,0 \cdot 10^{-3}$

1) $Re_2 = Re_1$: $u_{\infty 2} = 0,392\ m/s$

2) $Re_2 \approx 1,3 \cdot 10^6$: $u_{\infty 2} = 2,6\ m/s$

3) $Re_3 \approx 9 \cdot 10^6$: $u_{\infty 2} = 18\ m/s$

(see diagram, chapter 16.2)

17 Boundary layer separation

17.1 The coefficients $a_0(x),\ a_1(x),\ a_2 \cdot x,\ a_3(x)$ of the polynomial are determined first. At least 4 boundary conditions are necessary. Since $\dfrac{dp}{dx}$ is unknown the momentum equation at the wall gives no further information. The condition for separation gives an additional boundary condition

Separation occurs at $\left.\dfrac{\partial u/u_a}{\partial y/\delta}\right|_{y\,=\,0} = 0.$

Boundary conditions:

$$\begin{array}{llll} y=0 & \Longrightarrow u=0 & & \Longrightarrow a_0(x)=0 \\ y=\delta & \Longrightarrow u=u_a(x) & & \Longrightarrow 1=a_1(x)+a_2x+a_3(x) \\ y=\delta & \Longrightarrow \left.\dfrac{\partial u/u_a}{\partial y/\delta}\right|_{y\,=\,\delta} & =0 & \Longrightarrow 0=a_1(x)+2a_2x+3a_3(x) \end{array}$$

for $x = x_a$

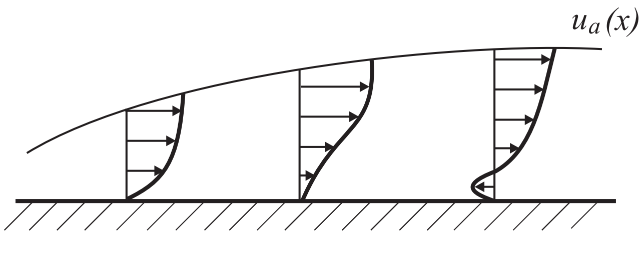
$$y=0 \quad \Longrightarrow \quad \left.\frac{\partial u/u_a}{\partial y/\delta}\right|_{y\,=\,0} = 0 \Longrightarrow 0 = a_1(x_a)$$

At $x = x_a$

$$a_2x_a + a_3(x_a) = 1 \quad ; \quad 2a_2x_a + 3a_3(x_a) = 0$$

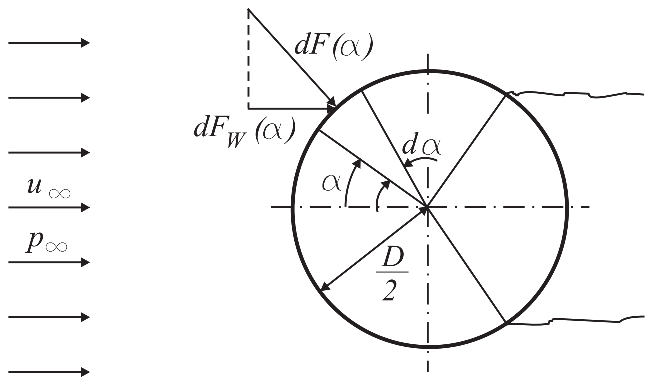
$$\Longrightarrow a_2 = \frac{3}{x_a} \qquad a_3(x_a) = -2$$

$$\Longrightarrow \frac{u(x_a, y/\delta(x_a))}{u_a(x_a)} = 3\left(\frac{y}{\delta(x_a)}\right)^2 - 2\left(\frac{y}{\delta(x_a)}\right)^3$$



$x = x_a$

17.2



$$c_w = \frac{2\int_0^\pi dF_w(\alpha)}{\frac{\rho}{2}u_\infty^2 DL}$$

$$dF_w(\alpha) = dF(\alpha)\cos(\alpha) = p(\alpha)L\frac{D}{2}\cos(\alpha)d\alpha$$

$$p(\alpha) = c_p(\alpha)\frac{\rho}{2}u_\infty^2 + p_\infty$$

$$0 \leq \alpha \leq \frac{2}{3}\pi : p(\alpha) = (1 - 4\sin^2\alpha)\frac{\rho}{2}u_\infty^2 + p_\infty$$

$$\frac{2}{3}\pi \leq \alpha \leq \pi : p(\alpha) = \left[1 - 4\sin^2\left(\frac{2}{3}\pi\right)\right]\frac{\rho}{2}u_\infty^2 + p_\infty = -2\frac{\rho}{2}u_\infty^2 + p_\infty$$

$$c_w = \sqrt{3}$$

17.3

a)

Determination of the coefficients a_i with the following boundary conditions:

(I) $u(y = 0) = 0$

(II) $u(y = \delta) = u_a$

(III) momentum at the wall: $y \to 0 : u \to 0, v \to 0, \dfrac{dp}{dx} = \eta \left.\dfrac{\partial^2 u}{\partial y^2}\right|_{y=0}$

(IV) $\tau(y = \delta) = 0$

$$\frac{dp}{dx} = \frac{dp_a}{dx} = -\rho u_a(x)\frac{du_a(x)}{dx} = -2\rho\frac{u_\infty^2}{R}\sin\frac{2x}{R}$$

Using the ansatz:

$$\frac{dp}{dx} = \eta \frac{u_a}{\delta^2} \left. \frac{\partial^2 \dfrac{u}{u_a}}{\partial \left(\dfrac{y}{\delta}\right)^2} \right|_{\frac{y}{\delta} = 0} = \frac{\eta u_a}{\delta^2} 2a_2 = \frac{\rho u_a^2}{25x} 2a_2 = -4\rho \frac{u_\infty^2}{R} \sin\frac{x}{R} \cos\frac{x}{R}$$

$$\Longrightarrow a_2 = -\frac{25}{2}\frac{x}{R} \cot\frac{x}{R}$$

Following the boundary conditions

$$\begin{aligned} a_0 &= 0 \\ a_1 + a_2 + a_3 &= 1 \\ a_1 + 2a_2 + 3a_3 &= 0 \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad a_3 &= -\frac{1}{2}(1 + a_2) \\ a_1 &= \frac{3}{2} - \frac{1}{2}a_2 \end{aligned}$$

b)
Separation: $\tau_w = 0$

$$\tau_w = \eta \left. \frac{\partial u}{\partial y} \right|_{y = 0} \overset{!}{=} 0 \qquad \left. \frac{\partial \dfrac{u}{u_a}}{\partial \left(\dfrac{y}{\delta}\right)} \right|_{\frac{y}{\delta}} \overset{!}{=} 0 \quad \Longrightarrow \quad a_1 = 0$$

$$-\frac{25}{2}\frac{x}{R} \frac{\cos\left(\dfrac{x}{R}\right)_A}{\sin\left(\dfrac{x}{R}\right)_A} = 3 \quad ;$$

$$\text{assumption: } \left(\frac{x}{R}\right)_A \approx \frac{\pi}{2} \quad \Longrightarrow \quad \left(\frac{x}{R}\right)_A \left(\frac{\pi}{2} - \left(\frac{x}{R}\right)_A\right) = -\frac{6}{25} \quad \Longrightarrow \quad \left(\frac{x}{R}\right)_A = 1.71 \quad ,$$

$$(\approx \frac{\pi}{2} \text{assumption is justifiabe})$$

$$\Theta_A = 98^o$$

17.4

a)

equilibrium of forces: $F_w = G$ (buoyancy can be neglected)

sphere:

$$c_w \frac{\rho_L}{2} v^2 \frac{\pi D_K^2}{4} = \rho \frac{\pi D_K^3}{6} g$$

$$v = Re \frac{\nu_L}{D_K}$$

$$D_K = \sqrt[3]{18 Re \frac{\rho_L}{\rho} \frac{\nu_L^2}{g}}$$

$$Re = 0,5: \quad D_{K_{max}} = 6,81 \cdot 10^{-2}\ mm$$

cylinder (Lnge L):

$$c_w \frac{\rho_L}{2} v^2 D_Z L = \rho \frac{\pi D_Z^2}{4} L g$$

$$D_Z = \sqrt[3]{\frac{16}{2 - lnRe} \frac{Re}{} \frac{\rho_L \nu_L^2}{\rho \quad g}}$$

$$Re = 0,5: \quad D_{Z_{max}} = 4,71 \cdot 10^{-2}\ mm$$

b)

$$v_K = 0,110\ m/s$$

$$v_Z = 0,159\ m/s$$

17.5

a)

$$G = F_{w1} = c_{w1} \frac{\rho_L}{2} v_1^2 \frac{\pi D^2}{4} \quad \text{(buoyancy neglected)}$$

$$c_{w1} = 0,4 \qquad (Re_1 = 3,03 \cdot 10^5)$$

b)

$$Re_2 = \frac{v_2 D}{\nu_L} = 4,2 \cdot 10^5$$

from diagram: $c_{w2} = 0,1$

$$F_{w2} = c_{w2} \frac{\rho_L}{2} v_2^2 \frac{\pi D^2}{4} = 1,95N < G$$

acceleration onto v_3, so that $G = F_{w3}$ ist:

$$G = c_{w3} \frac{\rho_L}{2} v_3^2 \frac{\pi D^2}{4}$$

from diagram: $c_{w3} = 0,1$

$$v_3 = 26,0 \ m/s$$

17.6

a)

$$\rho_K \frac{\pi D^3}{6} \frac{dv}{dt} = -c_w \frac{\rho_L}{2} v^2 \frac{\pi D^2}{4} - \rho_K \frac{\pi D^3}{6} g$$

steady sink velocity:

$$v_s^2 = \frac{4}{3} \frac{\rho_k D g}{\rho_L c_w}$$

$$-\frac{1}{g} \frac{dv}{1 + \left(\frac{v}{v_s} \right)^2} = dt = \frac{dz}{v}$$

$$H = -\frac{1}{g} \int_{v_0}^{0} \frac{v dv}{1 + \left(\frac{v}{v_s} \right)^2} = \frac{v_s^2}{2g} ln \left[1 + \left(\frac{v_0}{v_s} \right)^2 \right]$$

b)

$$T_H = -\frac{1}{g} \int_{v_0}^{0} \frac{dv}{1 + \left(\frac{v}{v_s} \right)^2} = \frac{v_s}{g} arctan \frac{v_0}{v_s}$$

c)

$$\rho_K \frac{\pi D^3}{6} \frac{dv}{dt} = -c_w \frac{\rho_L}{2} v^2 \frac{\pi D^2}{4} + \rho_K \frac{\pi D^3}{6} g$$

$$\frac{1}{g} \frac{dv}{1 - \left(\frac{v}{v_S} \right)^2} = dt = \frac{dz}{v}$$

$$H = \frac{1}{g}\int_0^{V_B} \frac{vdv}{1-\left(\dfrac{v}{v_S}\right)^2} = -\frac{v_s^2}{2g} ln\left[1-\left(\frac{v_B}{v_S}\right)^2\right]$$

$$v_B = \frac{v_S}{\sqrt{1+\left(\dfrac{v_S}{v_0}\right)^2}}$$

d)

$$T_B = \frac{1}{g}\int_0^{v_B} \frac{dv}{1-\left(\dfrac{v}{v_S}\right)^2} = \frac{v_S}{2g} ln\frac{v_S+v_B}{v_S-v_B}$$

e)

	$c_w = 0,4$		$c_w = 0$
	wooden sphere	metal sphere	
$H[m]$	$37,2$	$44,0$	45
$T_H[s]$	$2,64$	$2,96$	3
$v_B[m/s]$	$24,9$	$29,3$	30
$T_B[s]$	$2,81$	$2,98$	3

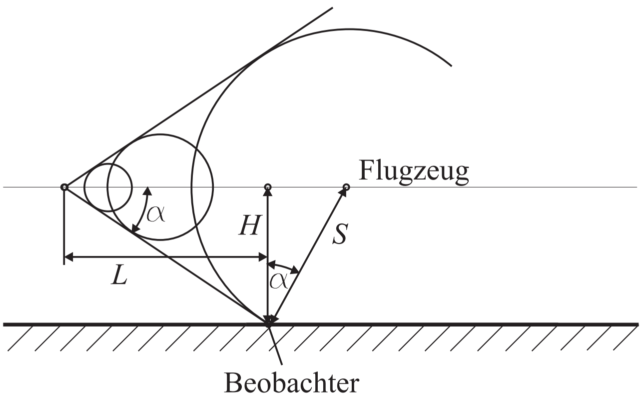
19 Compressible flows

19.1

a)

$$M = \frac{v}{\sqrt{\gamma RT}} = 2.0$$

b)



$$\begin{aligned} L =& \quad \frac{H}{\tan\alpha} \\ \alpha =& \quad \arcsin\left(\frac{1}{Ma}\right) \\ L =& \quad 1001\ m \end{aligned}$$

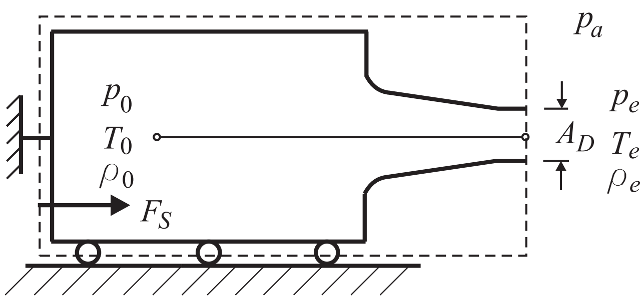
c)

$$\begin{aligned} \Delta t =& \quad \frac{S}{a} \\ \cos\alpha =& \quad \frac{H}{S} \\ \Delta t =& \quad 1.96\ s \text{ before passing the observer} \end{aligned}$$

19.2

$$\begin{aligned} \Delta t =& \quad \frac{b}{(v_B - v_A)\tan\alpha} \\ M_B =& \quad 2; \qquad \alpha = 30^o \\ \Delta t =& \quad 1.73\ s \end{aligned}$$

19.3



a)
momentum:

$$\rho_e v_e^2 A_D = (p_a - p_e)A_D + F_s$$

energy:

$$v_e^2 = 2c_p(T_0 - T_e)$$

$$c_p = \frac{\gamma R}{\gamma - 1}$$

$$\frac{F_s}{p_0 A_D} = \frac{2\gamma}{\gamma - 1}\frac{\rho}{\rho_0}\left(1 - \frac{T_e}{T_0}\right) - \frac{p_a}{p_0} + \frac{p_e}{p_0}$$

$$\frac{T}{T_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma-1} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\frac{F_s}{p_0 A_D} = \frac{2\gamma}{\gamma - 1}\left(\frac{p_e}{p_0}\right)^{\frac{1}{\gamma}} - \frac{\gamma+1}{\gamma-1}\,\frac{p_e}{p_0} - \frac{p_a}{p_0}$$

subcritical: $p_e = p_a$

supercritical: $p_e = p^* = 0.528 \;\cdot p_0$

b)

momentum:

$$F_s = \rho v_e^2 A_D$$

Bernoulli:

$$v_e^2 = \frac{2(p_0 - p_a)}{\rho}$$

$$\frac{F_s}{p_0 A_D} = 2\left(1 - \frac{p_a}{p_0}\right)$$

$\frac{p_a}{p_0}$	$\frac{F_s}{p_0 A_D}$	
	a) compr.	b) incompr.
1	0	0
0.6	0.66	0.8
0.2	1.07	1.6
0	1.27	2

19.4

Energy:

$$c_p T_1 + \frac{u_1^2}{2} = c_p T_2 + \frac{u_2^2}{2}$$

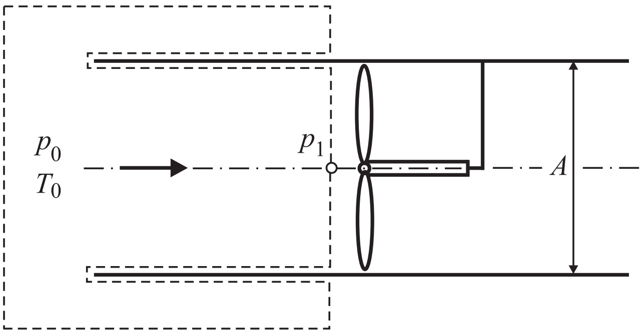
$$\frac{u_1}{u_2} = \frac{(\gamma+1)(M_1^2)}{2+(\gamma-1)M_1^2} \quad \text{(see script)}$$

$$c_p = \frac{\gamma R}{\gamma - 1}$$

$$T_2 - T_1 = \frac{(\gamma-1)(u_1^2 - u_2^2)}{2\gamma R} = 197.9\ K$$

19.5

$$\dot{m} = \rho_1 v_1 A = \sqrt{\frac{\gamma}{RT_1}} p_1 M_1 A$$



momentum:

$$\rho_1 v_1^2 A = (p_0 - p_1) A$$

$$M_1 = \sqrt{\frac{1}{\gamma}\left(\frac{p_0}{p_1} - 1\right)}$$

energy:

$$c_p T_0 = c_p T_1 + \frac{v_1^2}{2}$$

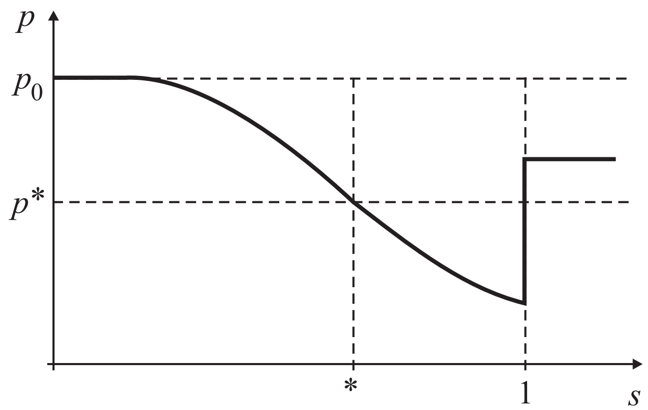
$$T_1 = \frac{T_0}{1 + \frac{\gamma-1}{2} M_1^2}$$

$$\dot{m} = 1.41\ kg/s$$

19.6

$$\begin{aligned}
\dot{m} =&\ \ \rho_1 u_1 A_1 = \frac{\rho_1}{\rho_0}\frac{p_0}{RT_0}\frac{u_1}{\sqrt{\gamma RT_1}}\sqrt{\gamma RT_0}\sqrt{\frac{T_1}{T_0}}A_1 \\
M_1 >&\ \ 1, \quad M_2 < 1; \quad p_2 = p_a \\
0 \rightarrow&\ \ 1 \ \text{ isentropic flow} \\
\frac{A^*}{A_1} =&\ \ \frac{1}{1.8} \qquad \text{(fron diagram)} \qquad \rightarrow \qquad M_1 = 2, \\
p_1 =&\ \ \frac{p_2}{1 + \dfrac{2\gamma}{\gamma+1}(M_1^2 - 1)} = 2.22 \cdot 10^4 \ N/m^2 \\
T_1 =&\ \ \frac{T_0}{1 + \dfrac{\gamma - 1}{2}M_1^2} \\
p_0 =&\ \ p_1\left(\frac{T_0}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = p_1\left(1 + \frac{\gamma-1}{2}M_1^2\right)^{\frac{\gamma}{\gamma-1}} = 1.74 \cdot 10^5 \ N/m^2 \\
\dot{m} =&\ \ \left(\frac{1}{1 + \dfrac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \quad M_1 \frac{p_0}{\sqrt{RT_0}}\sqrt{\gamma}A_1 = 4.43 \ kg/s
\end{aligned}$$

b)



19.7

$$\text{from hint:} \quad \frac{A_2^*}{A_1^*} = \frac{M_{E_2}}{M_{E_1}}\left(\frac{\dfrac{2}{\gamma+1}\left(1 + \dfrac{\gamma-1}{2}M_{E_1}^2\right)}{\dfrac{2}{\gamma+1}\left(1 + \dfrac{\gamma-1}{2}M_{E_2}^2\right)}\right)^{\dfrac{1}{2}\dfrac{\gamma+1}{\gamma-1}} \qquad *)$$

with $M_{E_1} = 2$

Energy equation: $\quad c_pT + \dfrac{u^2}{2} = c_pT_0 \quad$ with $\quad c_p = \dfrac{\gamma R}{\gamma - 1}$

$$\Longrightarrow \quad T + \frac{1}{2}\frac{u^2}{c^2}\frac{c^2}{c_p} = T_0 \quad \Longrightarrow \quad T + \frac{1}{2}M^2\frac{\gamma RT}{\gamma R}(\gamma-1) = T_0$$

$$\Longrightarrow \quad T_0 = \left(1 - \frac{1}{2}M^2(\gamma-1)\right)T \quad \Longrightarrow \quad T_0 = (1+0.2M^2)T \qquad **)$$

$$\Longrightarrow \quad \frac{T_0}{T_{E_1}} = (1+0.2M_{E_1}^2) = 1.8$$

$$\left.\begin{array}{lll} \text{ground} & : & \dfrac{p_a}{p_0} = \left(\dfrac{T_{E_1}}{T_0}\right)^{\frac{\gamma}{\gamma-1}} \\ \text{altitude } H & : & \dfrac{\frac{1}{4}p_a}{p_0} = \left(\dfrac{T_{E_2}}{T_0}\right)^{\frac{\gamma}{\gamma-1}} \end{array}\right\} \quad \frac{T_0^{\frac{\gamma}{\gamma-1}}}{p_0} = \frac{T_{E_1}^{\frac{\gamma}{\gamma-1}}}{p_a} = \frac{T_{E_2}^{\frac{\gamma}{\gamma-1}}}{\frac{1}{4}p_a}$$

$$\Longrightarrow \quad T_{E_2} = (\frac{1}{4})^{\frac{\gamma-1}{\gamma}}T_{E_1} = 0.673 \qquad T_{E_1} = \frac{0.673}{1.8}T_0 \quad \Longrightarrow \quad \frac{T_0}{T_{E_2}} = 2.67$$

with $**)$ $\quad M_{E_2} = \sqrt{5\left(\dfrac{T_0}{T_{E_2}} - 1\right)} = 2.89$

into $*)$: $\dfrac{A_2^*}{A_1^*} = 0.44$

19.8

a)

$$M_2 = \frac{u_2}{a_2}. \qquad h_0 = const: \quad T_0 = T_{01} = T_{02}$$

Energy:

$$c_pT_0 = c_pT_2 + \frac{u_2^2}{2}$$

$$c_2 = \sqrt{\gamma RT_2}$$

$$c_p = \frac{\gamma}{\gamma-1}R$$

$$M_2 = \sqrt{\frac{2}{\gamma-1}\left(\frac{T_0}{T_2} - 1\right)}$$

$$\frac{T_0}{T_2} = \left(\frac{p_{02}}{p_2}\right)^{\frac{\gamma-1}{\gamma}} = 1.14$$

$$p_{02} = p_{01'} = \frac{p_{01'}}{p_0}p_0$$

$$M_2 < 1: \quad p_a = p_2$$

$$M_2 = 0.837$$

b)

$$u_2 = M_2\sqrt{\gamma R T_2} \qquad T_2 = \frac{T_2}{T_0}\cdot T_0 = 263.12\ K$$
$$\Longrightarrow u_2 = 272.2\ m/s$$

c)

$$\rho_{02} = \frac{p_{02}}{RT_{02}} = 1.838\ kg/m^3$$

d)

$$\dot{m} = \rho_2 u_2 A_2 \qquad \rho_2 = \frac{p_2}{R\cdot T_2} = 1.324\ kg/m^3$$
$$A_2 = \frac{\dot{m}}{\rho_2 u_2} = 0.555\ m^2$$

19.9

a)

$$\frac{T_2}{T_1} = 1.2$$
$$\frac{T_2}{T_1} = f(M_1 \sin\sigma)$$

Diagram or formula: $M_1 \sin\delta = 1.31$

$$\sigma = \arctan\left(\frac{u_{n1}}{u_{t1}}\right) = 53.13^o$$
$$\Longrightarrow M_1 = 1.64$$
$$M_1 = \frac{\mid \vec{v}_1 \mid}{c_1} = \sqrt{\frac{u_{n1}^2 + u_{t1}^2}{\gamma R T_1}}$$
$$\Longrightarrow T_1 = \frac{u_{n1}^2 + u_{t1}^2}{\gamma R M_1^2} = 229\ K$$

b)

momentum equation tangential to the shock + continuity:

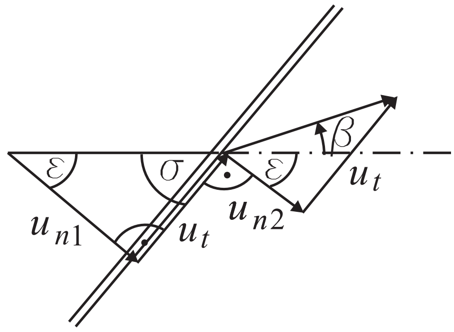
$$u_{t2} = u_{t1} = 300\ m/s$$
$$\frac{u_{n1}}{u_{n2}} = f(M_1,\ \sin\sigma)$$

diagram or formula:

$$\frac{u_{n1}}{u_{n2}} = 1.55$$

$$\Longrightarrow u_{n2} = \frac{u_{n2}}{u_{n1}} u_{n1} = 2.58\ m/s$$

$$M_2 = \frac{\sqrt{u_{n2}^2 + u_{t2}^2}}{\sqrt{\gamma R \frac{T_2}{T_1} T_1}} = 1.2$$



$$\tan(\beta + \varepsilon) = \frac{u_t}{u_{n2}}$$

$$\tan(\varepsilon) = \frac{u_t}{u_{n1}}$$

$$\Longrightarrow \beta = \arctan\frac{u_t}{u_{n2}} - \arctan\frac{u_t}{u_{n1}} = 12^o$$

or β from diagram $\beta = f(M_1,\ \sigma)$

c)

$$M_1 = const = 1.64$$

$$M_2 = 1$$

fro diagram $\beta = f(M_1,\ \sigma)$:

$$\beta = 15^o; \quad \sigma = 61^o$$

$$u_{n1} = M_1 \sin\sigma \sqrt{\gamma R T_1} = 440\ m/s$$

$$M_1 = \frac{\sqrt{u_{n1}^2 + u_{t1}^2}}{c_1}$$

$$\Longrightarrow u_{t1} = \sqrt{M_1^2 c_1^2 - u_{n1}^2} = 234\ m/s$$

19.10

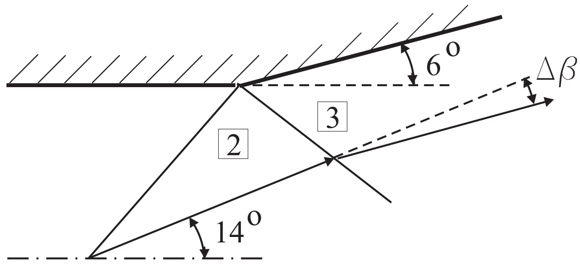
$$\sigma = 40^o \quad M_1 = 2.2 \quad \Longrightarrow \quad M_1 \sin\sigma_{12} = 1.41$$

Formula or diagramm $\beta = f(M_1,\ \sigma):\quad \beta_{12} = 14^o$

$$M_2^2 \sin^2(\sigma_{12} - \beta_{12}) = \frac{(\gamma - 1) M_1^2 \sin^2\sigma_{12} + 2}{2\gamma M_1^2 \sin^2\sigma_{12} - (\gamma - 1)} \Longrightarrow M_2 = 1.65$$

Formula or diagram $\frac{p_1}{p_2} = f(M_1, \sigma_{12}) \implies \frac{p_2}{p_1} = 2.17$

$$\frac{T_2}{T_1} = f(M_1,\ \sigma_{12}) \implies \frac{T_2}{T_1} = 1.26$$



$$\Delta\beta = 14^o - 6^o = 8^o = \beta_{23}$$

$$M_2 = 1.65$$

from diagram $\sigma = f(\beta_{23},\ M_2)$, schwacher Stoß:

$$\sigma_{23} = 46^o$$

using formula $M_3 = f(M_2,\ \sigma_{23},\ \beta_{23})$:

$$M_3 = 1.37$$

from diagramm $\frac{p_3}{p_2} = f(M_2,\ \sigma_{23}) \implies \frac{p_3}{p_2} = 1.53$

$$\frac{T_3}{T_2} = f(M_2,\ \sigma_{23}) \implies \frac{T_3}{T_2} = 1.13$$

19.11

The suction leads to an increasung pressure $p_k(t)$ in the boiler. The flow is undisturbed until a shock is located in the exit cross-section A_E of the measuring chamber.

$M_E = 2.3$

$$\frac{p_0}{p_{E\,Ausl}} = \left(1 + \frac{\gamma - 1}{2} M_E^2\right)^{\frac{\gamma}{\gamma-1}} = 12.5$$

with $p_0 = 1$ bar, follows $p_{E\,Ausl} = 0.08$ bar $= p_k(t = 0)$

Determining of the boiler pressure $p_k(\Delta t)$ at the end of the measuring time:

relations across the normal shock:

$$\frac{p_k(\Delta t)}{p_{E\,Ausl}} = 1 + \frac{2\gamma}{\gamma+1}(M_E^2 - 1) = 6.0$$

$$\Longrightarrow p_k(\Delta t) = 0.48\ bar$$

$$\dot{m} = \rho^* A^* c^* = \frac{\rho^*}{\rho_0}\rho_0 A^* \sqrt{\gamma R \frac{T^*}{T_0}}\sqrt{T_0}$$

$$= \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} \frac{p_0}{RT_0} A^* \sqrt{\gamma R \frac{T^*}{T_0}}\sqrt{T_0}$$

$$A^* = A_H$$

$$\Longrightarrow \dot{m} = 24.2\ kg/s$$

$$p_k(t=0) = \rho_k(t=0) R T_K$$

$$p_k(\Delta t) = \rho_k(\Delta t) R T_K$$

$$\Delta m = \dot{m}\Delta t = V_k\ [\rho(\Delta t) - \rho(t=0)]$$

$$\Longrightarrow \Delta t = \frac{V_k}{\dot{m}} \frac{1}{RT_K}\ [p_k(\Delta t) - p_k(t=0)\ ] = 20.6\ s$$

b)

$$\frac{p_k}{p_{E\,Ausl}} = 2$$

from diagram $\dfrac{p_k}{p_{E\,Ausl}} = f(M_E \sin\sigma)$:

$M_E \sin\sigma = 1.36$ with $M_E = 2.3$ follows $\sigma = 36.2^o$

β from diagram $\beta = f(M,\ \sigma)$: $\beta = 12^o$

from diagram $\dfrac{p_{01}}{p_{02}} = f(M_E \cdot \sin\sigma)$: $\dfrac{p_{01}}{p_{02}} = 1.03$

$$\frac{T_2}{T_1} = 1.22$$

with $p_{01} = 1$ bar follows $p_{02} = 0.97$ bar

$$T_1 = \frac{T_1}{T_0}\ T_0 = \frac{1}{1 + \dfrac{\gamma-1}{2}M_E^2}\ T_0 = 136\ K$$

$$\Longrightarrow \quad T_2 = 166\ K$$

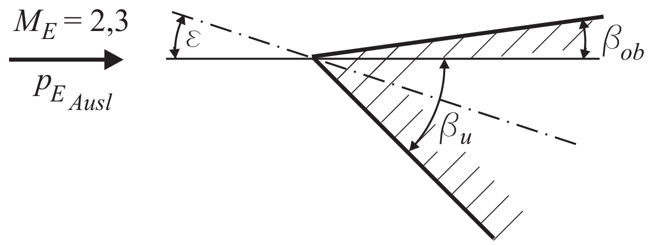
$$M_2^2 \sin^2(\sigma - \beta) = \frac{(\gamma-1)M_1^2 \sin^2\sigma + 2}{2\gamma M_1^2 \sin^2\sigma - (\gamma - 1)}$$

$$\Longrightarrow \quad M_2 = 1.83$$

$$\frac{T_{02}}{T_2} = 1 + \frac{\gamma - 1}{2}\, M_2^2 \quad \Longrightarrow \quad T_{02} = 277.2\ K$$

$$V_2 = c_2\ M_2 = \sqrt{\gamma R T_2}\ M_2 = 473\ m/s$$

c)



$$\beta_{ab} = \beta_k - \varepsilon = 20^o - \varepsilon$$

$$\beta_u = \beta_k + \varepsilon = 20^o + \varepsilon$$

$$\beta_k + \varepsilon \le \beta_{max}(M_E = 2.3) = 27.5^o \Longrightarrow \varepsilon \le 7.5^o$$

$$\beta_{ob} = 12.5^o M_E = 2.3$$

from diagram $\sigma = f(\beta,\ M)$: $\quad \sigma_{ob} = 37^o$

$$\Longrightarrow \quad M_E \sin\sigma_{ob} = 1.384$$

$$\beta_u = 27.5^o, \quad M_E = 2.3$$

from diagram $\sigma = f(\beta,\ M)$: $\quad \sigma_u = 62^o$

$$\Longrightarrow \quad M_E \sin\sigma_u = 2.03$$

$$p_u - p_{ob} = \left(\frac{p_u}{p_{E\,Ausl}} - \frac{p_{ob}}{p_{E\,Ausl}}\right) p_{E\,Ausl}$$

$\dfrac{p_u}{p_{E\,Ausl}}$ and $\dfrac{p_{ob}}{p_{E\,Ausl}}$ from diagram $\dfrac{p_2}{p_1} = f(M_1 \sin\sigma)$:

$$\Longrightarrow p_u - p_{ob} = 0.22\ bar$$

using $M_2^2 \sin^2(\sigma-\beta) = \dfrac{(\gamma-1)M_1^2\sin^2\sigma+2}{2\gamma M_1^2\sin^2\sigma-(\gamma-1)}$ follows $M_{ob} = 1.85$ and $M_u = 0.94$

19.12

a)

$M_\infty = 3; \quad \beta_1 = 15^o \quad \rightarrow \quad$ diagram $\sigma_1 = f(\beta_1,\ M_\infty)$: $\quad \sigma_1 = 33^o$

with $M_1^2\sin^2(\sigma_1-\beta_1) \ = \ \dfrac{(\gamma-1)M_\infty^2\sin^2\sigma_1+2}{2\gamma M_\infty^2\sin^2\sigma_1-(\gamma-1)}$ follows $\quad M_1 = 2.2$

$\beta_{12} = \beta_1 - \beta_2 = 5^o \quad \rightarrow \quad$ diagram $\sigma_2 = f(\beta_{12},\ M_1)$ follows $\quad \sigma_2 = 31^o$

using $M_2 = f(\beta_{12},\ \sigma_2,\ M_1)$ follows $M_2 = 2.02$

b)

$$c^* = \sqrt{\gamma R\frac{T^*}{T_0}\frac{T_0}{T_\infty}T_\infty} = 503.2\ m/s$$

$$\frac{u_n}{c^*} = 1.29$$

$$M_2^* = \sqrt{\frac{\gamma+1}{(\gamma-1)+\dfrac{2}{M_2^2}}} = 1.64$$

$$M_2\sin\sigma_3 = \frac{u_n}{c}$$

$$\Longrightarrow \quad M_2^*\sin\sigma_3 = \frac{u_n}{c^*}$$

$\Longrightarrow \quad \sigma_3 = 52^o \quad \rightarrow \quad$ Diagramm $\quad \beta = f(\sigma_3,\ M_2): \quad \beta = 20^o$

$$\beta_3 = \beta - \beta_2 = 10^o$$

using $M_3 = f(\beta, \delta_3, M_2)$ $\quad$ follows $\quad M_3 = 1.27$

$M_\infty \sin\sigma_1 = 1.63 \Longrightarrow$ diagram $\dfrac{p_1}{p_\infty} = f(M_\infty\sin\sigma_1) : \dfrac{p_1}{p_\infty} = 2.9$

$M_1 \sin\sigma_2 = 1.13 \Longrightarrow$ diagram $\dfrac{p_2}{p_1} = f(M_1\sin\sigma_2) : \dfrac{p_2}{p_1} = 1.4$

$M_2 \sin\sigma_3 = 1.59 \Longrightarrow$ diagram $\dfrac{p_3}{p_2} = f(M_2\sin\sigma_3) : \dfrac{p_3}{p_2} = 2.7$

$$p_3 = \frac{p_3}{p_2}\cdot\frac{p_2}{p_1}\cdot\frac{p_1}{p_\infty}\cdot p_\infty = 10.96\ bar$$

19.13

a)

$$M_1 = 1.6$$

$$\frac{p_1}{p_{01}} = \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{-\frac{\gamma}{\gamma-1}} = 0.235$$

$$p_2 = p_1$$

$$\frac{p_2}{p_{02}} = \frac{p_{01}}{p_{02}} \frac{p_1}{p_{01}} = 0.282$$

$$\frac{p_2}{p_{02}} = \left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{-\frac{\gamma}{\gamma-1}}$$

$$\Longrightarrow \quad M_2 = 1.48$$

b)

$$\frac{|\vec{V}_2|}{|\vec{V}_1|} = \frac{M_2 \cdot c_2}{M_1 \cdot c_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_{02}}\frac{T_{02}}{T_{01}}\frac{T_{01}}{T_1}}$$

$$T_{01} = T_{02}$$

$$\frac{T_2}{T_{02}} = \left(1 + \frac{\gamma - 1}{2} M_2^2\right)^{-1} = 0.69$$

$$\text{analogous} \qquad \frac{T_1}{T_{01}} = 0.66$$

$$\Longrightarrow \frac{|\vec{V}_2|}{|\vec{V}_1|} = 0.95$$

$$\frac{\rho_2}{\rho_1} = \frac{p_2/(RT_2)}{p_1/(RT_1)} = \frac{T_1}{T_2} = 0.96$$