Potentialtheory

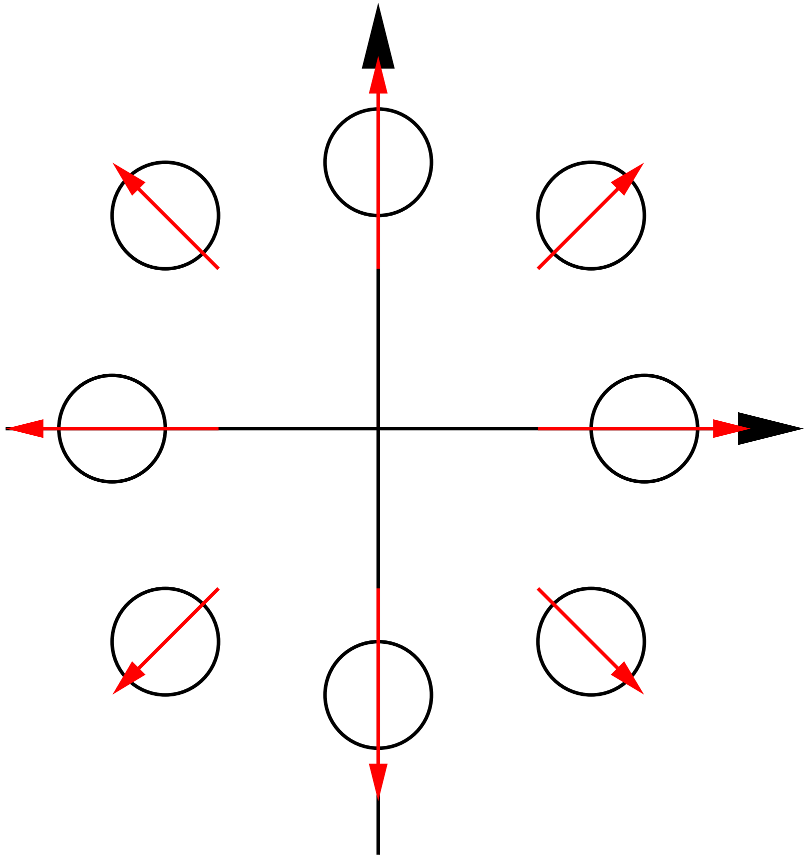
Assumptions: frictionless, rotationless
2-dimensional (plane)
incompressible, steady flow

no rotation: $\vec{\omega} = \vec{0}$

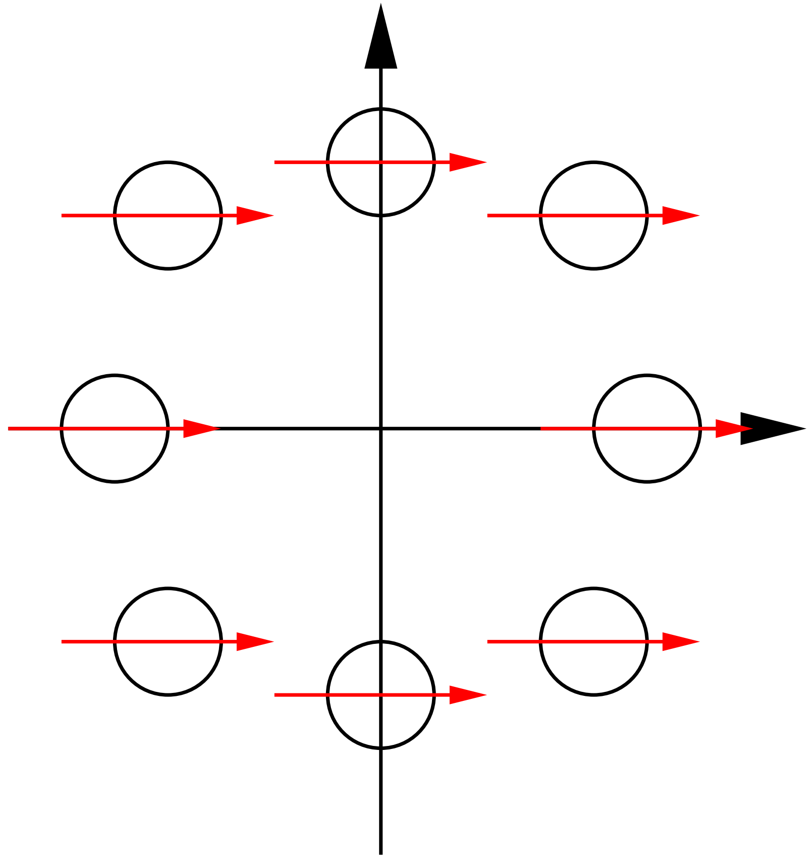
$$\vec{\omega} = \frac{1}{2}\,\text{rot}\,\vec{v} = \frac{1}{2}\nabla \times \vec{v} = \frac{1}{2}\begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix}$$

2-dimensional flow $\omega_x = \omega_y = 0$

$$\rightarrow \omega_z = \frac{1}{2}(v_x - u_y) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$



rotational flow



rotation free

Problem 4.1: rot(grad f) = $\vec{0}$

Potential theory

if $\omega_z = 0 \rightarrow$ a function Φ exists with the attribute

$$\vec{v} = \nabla \underbrace{\Phi}_{\text{Potential}} \rightarrow \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \Phi}{\partial x} \\ \\ \dfrac{\partial \Phi}{\partial y} \end{pmatrix}$$

$\rightarrow$ continuity (2-d, steady, incompressible)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \vec{v} \rightarrow \boxed{\nabla^2 \Phi = \Delta \Phi = 0}$$

$$\rightarrow \boxed{\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0}$$

Potential theory

lineare differential equation

$\rightarrow$ the principle of Superposition can be used

if Φ_1, Φ_2 are solutions of the equation, then $C_1 \cdot \Phi_1$, $C_2 \cdot \Phi_2$ und $C_1 \cdot \Phi_1 + C_2 \cdot \Phi_2$ are also solution of the equation

Stream function : $u = \frac{\partial \Psi}{\partial y}; \; v = -\frac{\partial \Psi}{\partial x}$ fullfills the continuity equation

$$\omega = 0 \; \boxed{\nabla^2 \Psi = \Delta \Psi = 0}$$

$\Phi_x = \Psi_y \, ; \Phi_y = -\Psi_x$ $\rightarrow$ lines of constant Φ and Ψ are perpendicular to each other

Potential theory

$\Phi = \text{const} \ \rightarrow$ Isopotential lines

$\Psi = \text{const} \ \rightarrow$ Streamlines

Φ and Ψ are used to describe flowfields around solid bodies

The contour is a special stream line

$\rightarrow$ velocity vector is parallel to the wall

But: The no-slip condition cannot be fulfilled
(frictionless, no roatation)

$\rightarrow$ Drag forces and shear stresses cannot be
berechnet computed

Potential theory

- complex numbers

$$z = x + i\,y = r\,e^{i\varphi} = r(\cos\varphi + i\,\sin\varphi)$$

$$\begin{matrix} x = r\cos\varphi & & r = \sqrt{x^2 + y^2} \\ & \leftrightarrow & \\ y = r\sin\varphi & & \varphi = \tan^{-1}\left(\dfrac{y}{x}\right) \end{matrix}$$

- complex velocity

$$w = u + i\,v$$

- conjugate complex velocity

$$\bar{w} = u - i\,v$$

Potential theory

complex Potential function

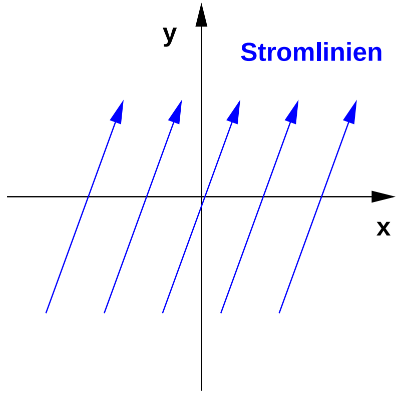
complex stream function

$$F(z) = \int \bar{w}\, dz \;=\; \Phi(x,y) \;+\; i\,\Psi(x,y)$$

$\longrightarrow$ Laplace equation $\boxed{\left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}\right) \;+\; i \left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right) = 0}$

$$\bar{w} = u - iv = \frac{dF}{dz}$$

Singularities

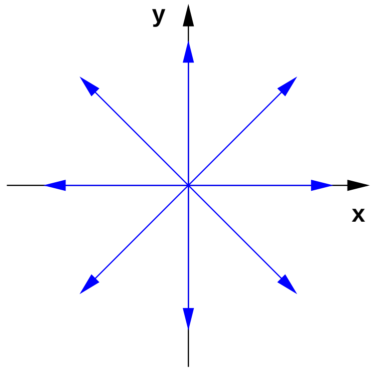


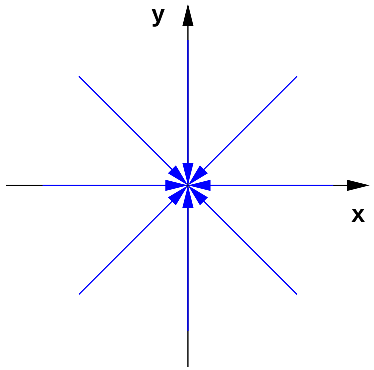
Parallel flow: $F(z) = (u_\infty - i\,v_\infty)\,z$

$$\Phi = u_\infty x + v_\infty y \qquad \Psi = u_\infty y - v_\infty x$$

$$u = u_\infty \qquad v = v_\infty$$

Source, Sink: $F(z) = \dfrac{E}{2\pi} \ln z$





$$\Phi = \frac{E}{2\pi} \ln r \qquad \Psi = \frac{E}{2\pi}\varphi$$

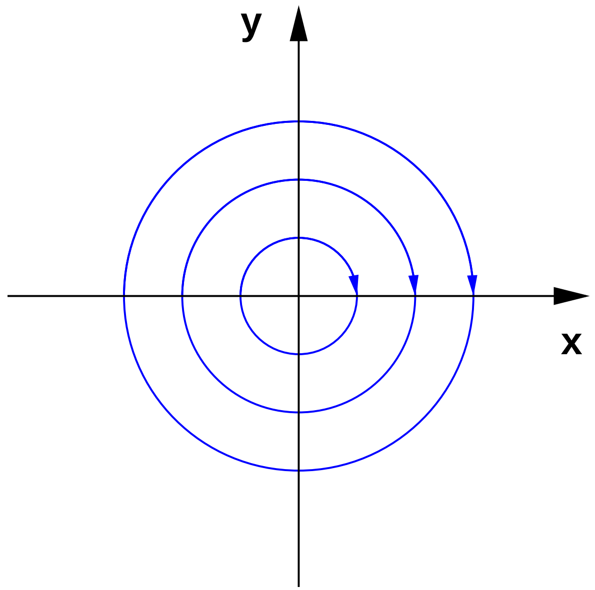
$$u = \frac{E}{2\pi}\frac{x}{x^2+y^2} \qquad v = \frac{E}{2\pi}\frac{y}{x^2+y^2}$$

Singularities

Potential vortex: $F(z) = \frac{\Gamma}{2\pi} i \, \ln z$

$$\Phi = -\frac{\Gamma}{2\pi} \tan^{-1}\left(\frac{y}{x}\right) \qquad \Psi = \frac{\Gamma}{2\pi} \ln \sqrt{x^2 + y^2}$$

$$u = \frac{\Gamma}{2\pi} \frac{y}{x^2 + y^2} \qquad v = -\frac{\Gamma}{2\pi} \frac{x}{x^2 + y^2}$$

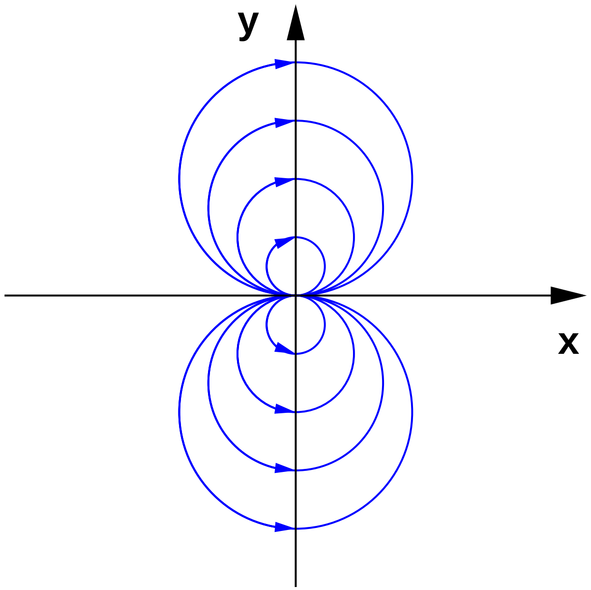


Singularities

Dipole: $F(z) = \dfrac{m}{z}$

$$\Phi = \frac{mx}{x^2 + y^2} \qquad \Psi = -\frac{my}{x^2 + y^2}$$

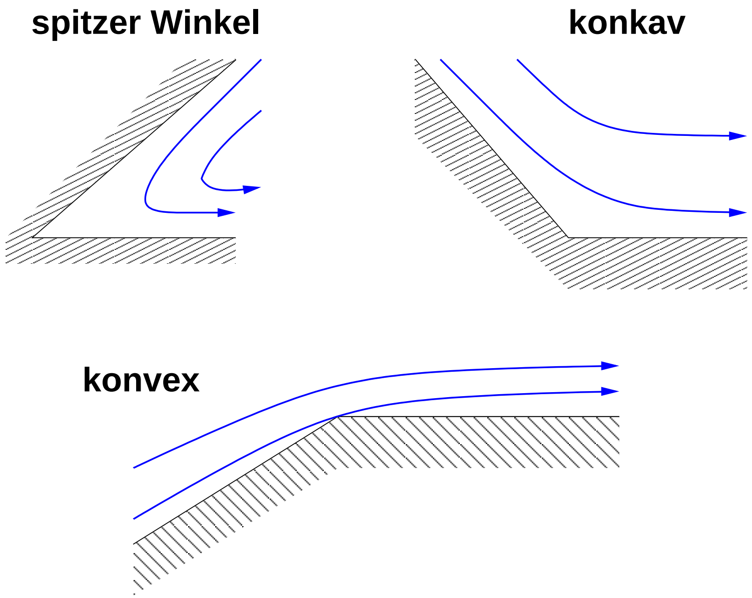
$$u = m\frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad v = -m\frac{2xy}{(x^2 + y^2)^2}$$



Singularities

Corner flow: $F(z) = \dfrac{a}{n} z^n \qquad (n \in \mathbf{R},\ a \in \mathbf{C})$

$$\Phi = \frac{a}{n} r^n \cos n\varphi \qquad\qquad \Psi = \frac{a}{n} r^n \sin n\varphi$$

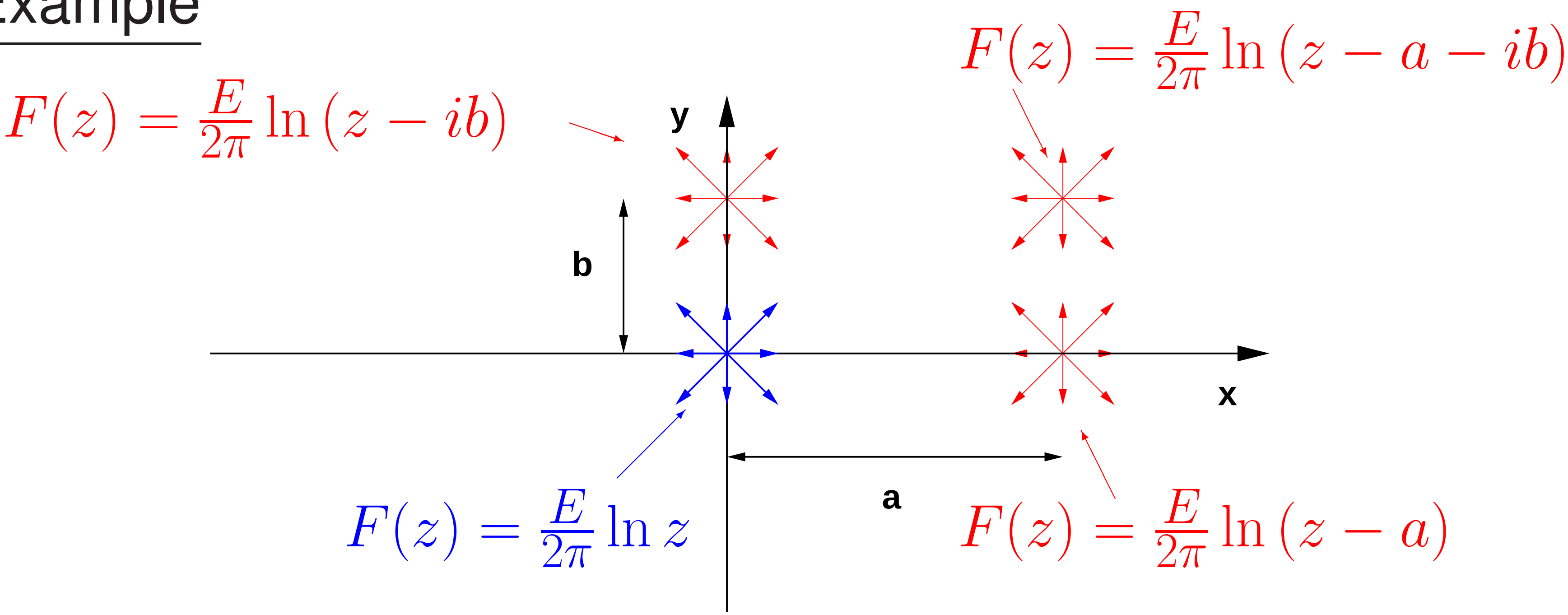


Singularities

The center of the singularities in the origin of the coordinate system

$\rightarrow$ Offset

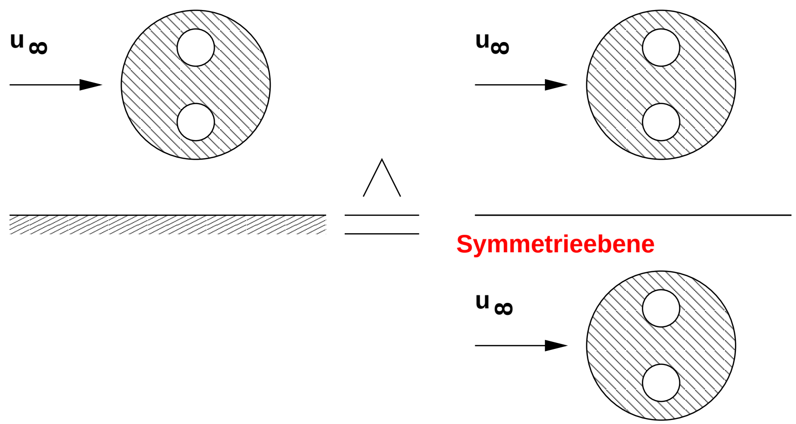
Example



Potential theory

Simulation of walls

by mirroring

- 

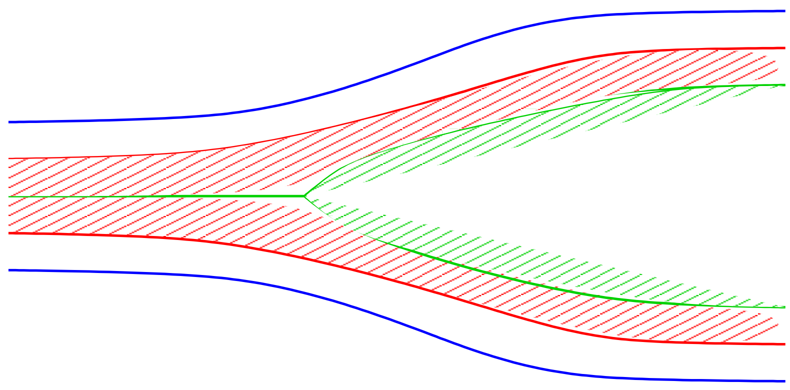
- Usually, contours are represented by stagnation point streamlines
  - localisation of stagnation points ($u = v = 0$)
  - Computation of Ψ in the stagnation point
  - sketch of the stremlines

$$\Psi_k(x, y) = \Psi_k(x_s, y_s) = const.$$

$$\rightarrow y_s = f(x_s)$$

$$\rightarrow r_s = f^{\text{‘}}(y_s)$$

- streamlines do not intersect
  $\rightarrow$ any streamline can represent a contour
  usually $u_w \neq 0$



- Bernoulli equation is valid

$$p_0 = p_\infty + \frac{1}{2}\rho(u_\infty^2 + v_\infty^2) = p + \frac{1}{2}\rho(u^2 + v^2) = const.$$

– computation of $c_p = \dfrac{p - p_{\text{ref}}}{\frac{\rho}{2}u_{\text{ref}}^2} = \dfrac{\frac{1}{2}\rho u_{\text{ref}}^2 - \frac{1}{2}\rho \vec{v}^2}{\frac{\rho}{2}u_{\text{ref}}^2} = 1 - \dfrac{\vec{v}^2}{u_{\text{ref}}^2}$

A planar flow is described by the stream function $\psi = (\frac{U}{L})xy$. The pressure in $x_{ref} = 0, y_{ref} = 1\ m$ is $p_{ref} = 10^5\ N/m^2$.

$$U = 2\ m/s \qquad L = 1\ m \qquad \rho = 10^3\ kg/m^3$$

a) Proof, if the flow has a potential!

Determine

b) the stagnation points, the pressure coefficient, and the lines of constant total velocity

c) the velocity and the pressure at $x_1 = 2m, y_1 = 2m$,

d) the coordinates of a particle at $t = 0.5s$, if it passes at $t = -0$ the point x_1, y_1,

e) the pressure difference between these two points.

f) Sketch the stream lines.

a) Given: stream function $\Psi = \frac{U}{L}xy$

Φ exists, if $\vec{\omega} = \vec{0}$

planar flow $\rightarrow$ 2-dimensional $\rightarrow \omega_x = \omega_y = 0$

$$\rightarrow \omega_z = \frac{1}{2}(v_x - u_y) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\left.\begin{aligned} u &= \frac{\partial \Psi}{\partial y} = \frac{U}{L}x \rightarrow \frac{\partial u}{\partial y} = 0 \\[2em] v &= -\frac{\partial \Psi}{\partial x} = -\frac{U}{L}y \rightarrow \frac{\partial v}{\partial x} = 0 \end{aligned}\right\} \qquad \omega_z = 0$$

$\rightarrow$ The flow is frictionless and the potential exist
Φ exists $\rightarrow$ computation of Φ

$$1.)\ u = \frac{\partial \Phi}{\partial x} \rightarrow \Phi = \int u\,dx \ + \ f_1(y) + C_1$$

$$2.)\ v = \frac{\partial \Phi}{\partial y} \rightarrow \Phi = \int v\,dy \ + \ f_2(x) + C_2$$

$$1.)\ \Phi(x,y) = \int \frac{U}{L} x dx + f_1(y) + C_1$$

$$2.)\ \Phi(x,y) = \int -\frac{U}{L} y dy + f_2(x) + C_2$$

14.5

$$1.)\ \ \Phi(x,y) = \frac{U}{L}\frac{x^2}{2} + f_1(y) + C_1$$

$$2.)\ \ \Phi(x,y) = -\frac{U}{L}\frac{y^2}{2} + f_2(x) + C_2$$

Comparison between 1.) and 2.)

$$\underbrace{\frac{U}{L}\frac{x^2}{2}} + \underbrace{f_1(y)} + C_1 = -\underbrace{\frac{U}{L}\frac{y^2}{2}} + \underbrace{f_2(x)} + C_2$$

$$f_1(y) = -\frac{U}{L}\frac{y^2}{2}; \qquad f_2(x) = \frac{U}{L}\frac{x^2}{2}; \qquad C_1 = C_2 = C$$

$$\rightarrow \Phi = \frac{U}{2L}(x^2 - y^2) + C$$

complex potential $F(z)$

$$
\begin{aligned}
F(z) &= F(x+iy) = \Phi(x,y) + i\Psi(x,y) \\
&= \frac{U}{2L}(x^2 - y^2) + i\frac{U}{L}xy \\
&= \frac{U}{2L}(x^2 + 2ixy - y^2) \\
&= \frac{U}{2L}z^2
\end{aligned}
$$

Sketch of the flow field

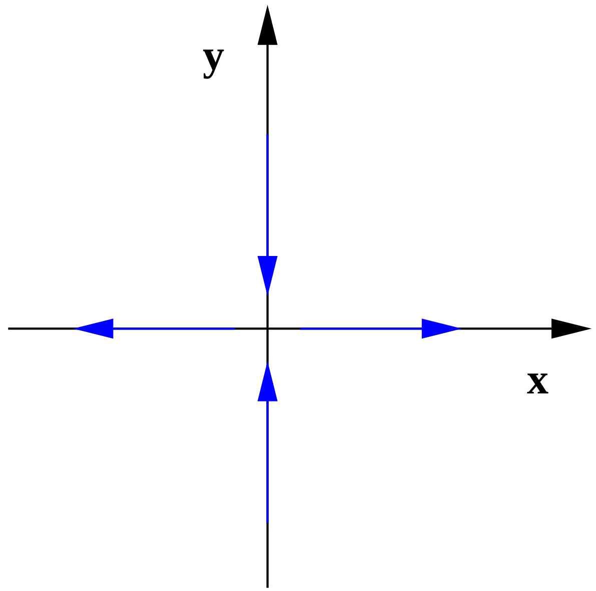
- Stagnation points $\rightarrow$ stagnation streamlines
- asymptotic streamlines
  $x, y \rightarrow \infty\,;\; x, y \rightarrow 0$
- direction of the flow

Stagnation points: $\vec{v} = \vec{0}\;:\; u = v = 0$

$$u = \frac{U}{L}x\,,\; v = -\frac{U}{L}y \rightarrow (x_s, y_s) = (0, 0)$$

Additionally: $u = 0$ on the y-axis

$v = 0$ on the x-axis



$\Psi = const.$

$$\Psi = \frac{U}{L}xy = const.$$

$$\rightarrow \; y = \frac{L}{U}const.\frac{1}{x} = \frac{C}{x} \text{ for } x \neq 0$$

$$x = \frac{L}{U}const.\frac{1}{y} = \frac{C}{y} \text{ for } y \neq 0$$

$\rightarrow$ Hyperbola

Stream lines: $\Psi = const.$

$$\Psi = \frac{U}{L} xy = const.$$

$$\rightarrow y = \frac{L}{U} const. \frac{1}{x} = \frac{C}{x} \text{ for } x \neq 0$$

$$x = \frac{L}{U} const. \frac{1}{y} = \frac{C}{y} \text{ for } y \neq 0$$

$\rightarrow$ Hyperbola

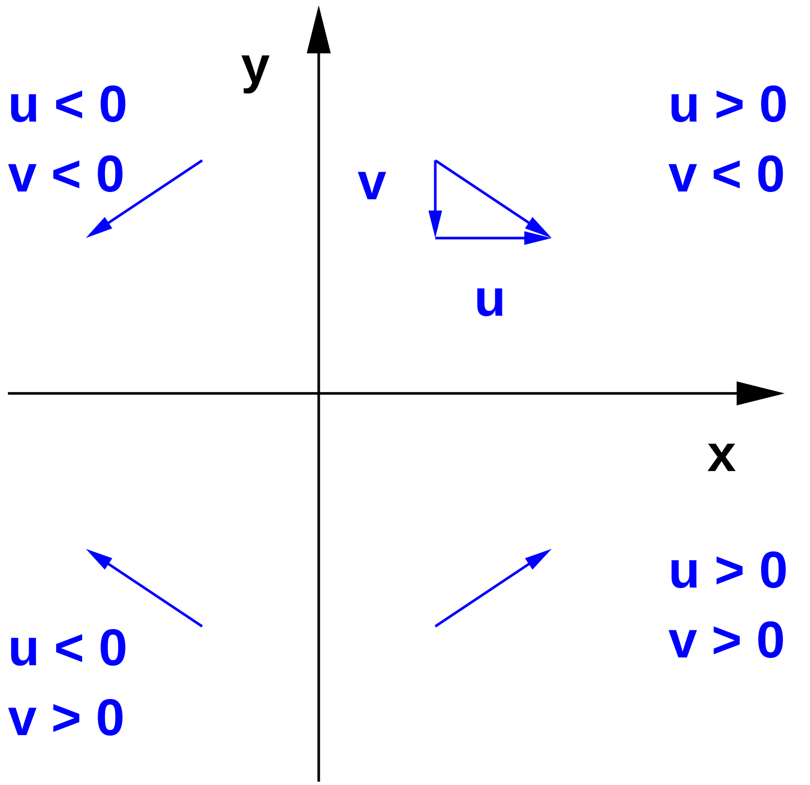
Stagnation stream line

$$\Psi_{sp} = \frac{U}{L} x_{sp} y_{sp} = 0 \quad | \quad \text{problem dependant}$$

$$\Psi = 0 \rightarrow \qquad x = 0 \text{ or } y = 0$$

$\rightarrow$ x-Axis and y-Axis are stagnation streamlines

Direction $u = \dfrac{U}{L}x\,,\;\; v = -\dfrac{U}{L}y$



Pressure coefficient

$$c_p = \frac{p - p_{\text{ref}}}{\frac{\rho}{2} v_{\text{ref}}^2} = 1 - \frac{\vec{v}^2}{\vec{v}_{\text{ref}}^2} = 1 - \frac{u^2 + v^2}{u_{\text{ref}}^2 + v_{\text{ref}}^2}$$

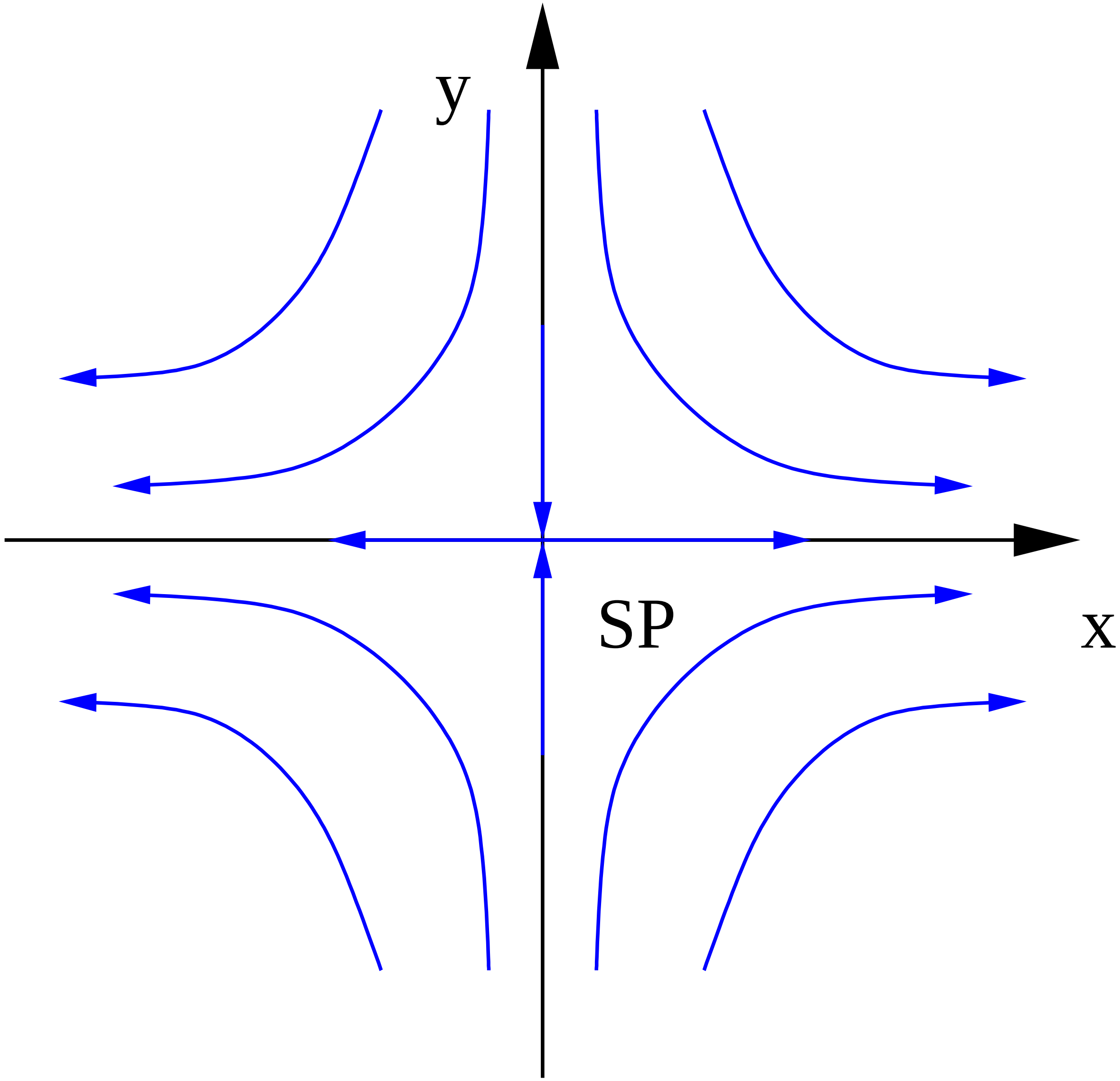
$$\left. \begin{aligned} u &= \frac{U}{L} x \\[2em] v &= -\frac{U}{L} y \end{aligned} \right\} c_p = 1 - \frac{x^2 + y^2}{x_{\text{ref}}^2 + y_{\text{ref}}^2}$$

Lines of constant velocity

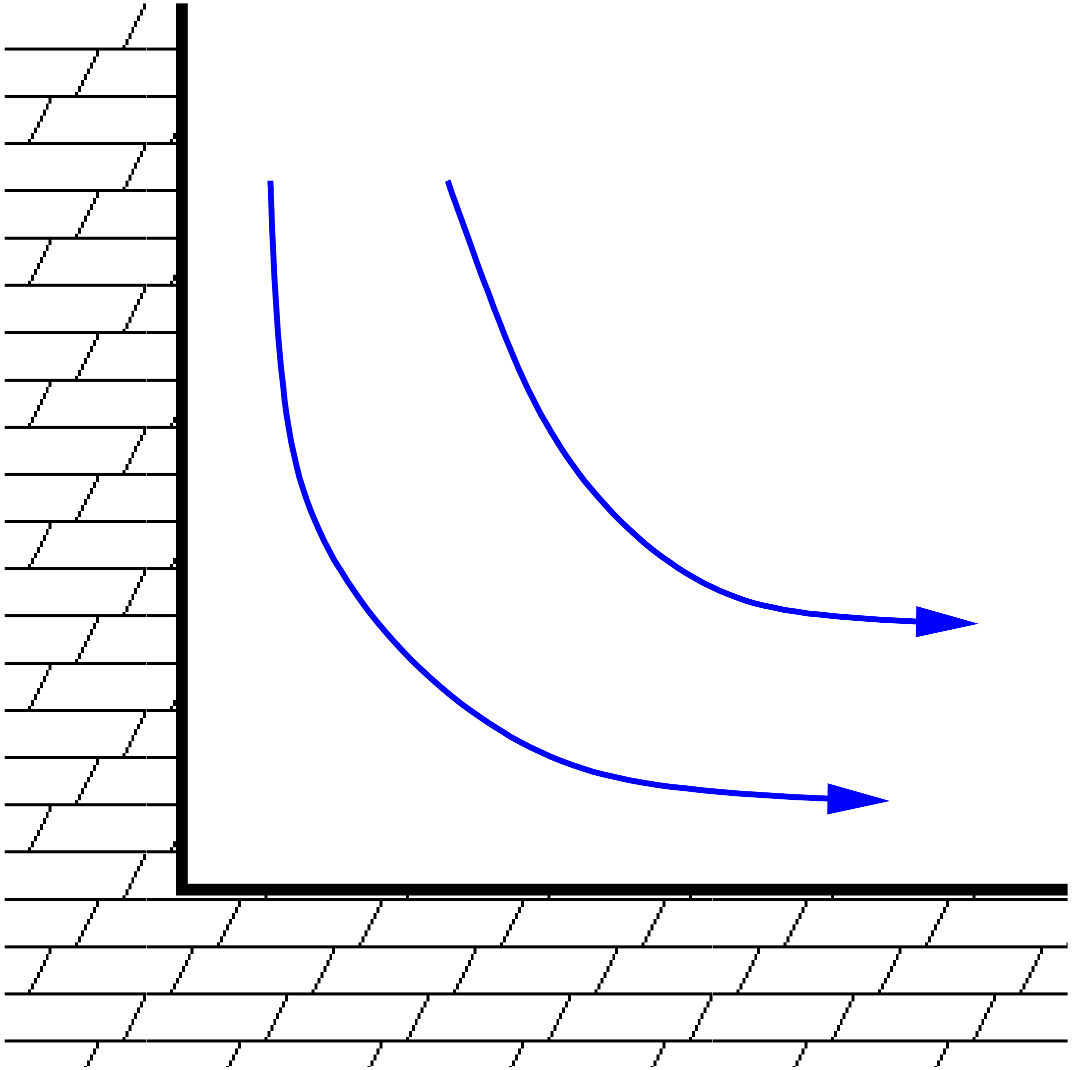
$$|\bar{v}| = \sqrt{u^2 + v^2} = const.$$

$$= \sqrt{\left(\frac{U}{L}x\right)^2 + \left(-\frac{U}{L}y\right)^2} \rightarrow x^2 + y^2 = \left(\frac{L\vec{v}}{U}\right)^2$$

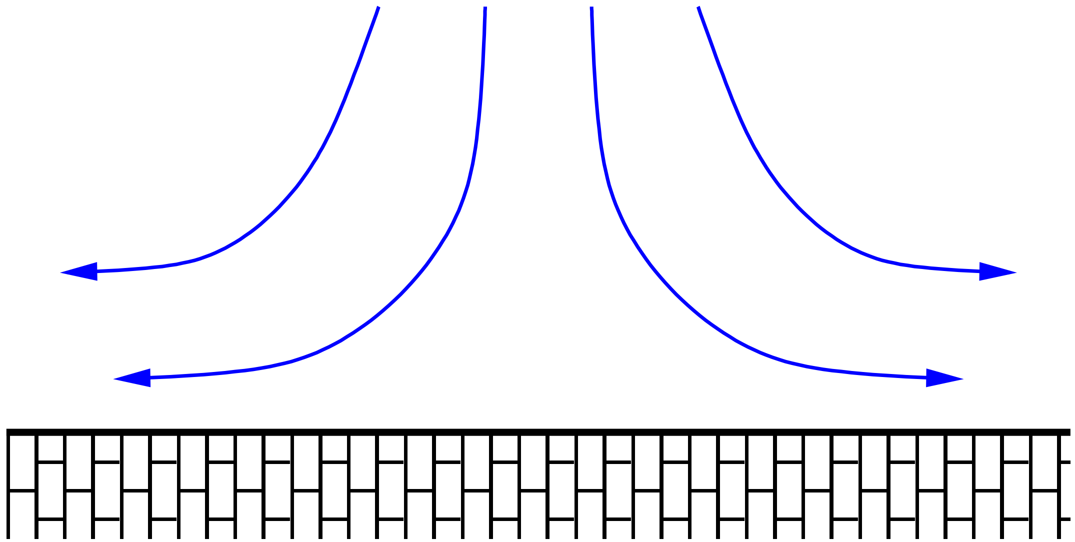
Circles with the radius $R = \dfrac{L\vec{v}}{U}$


y
SP
x

90° corner flow

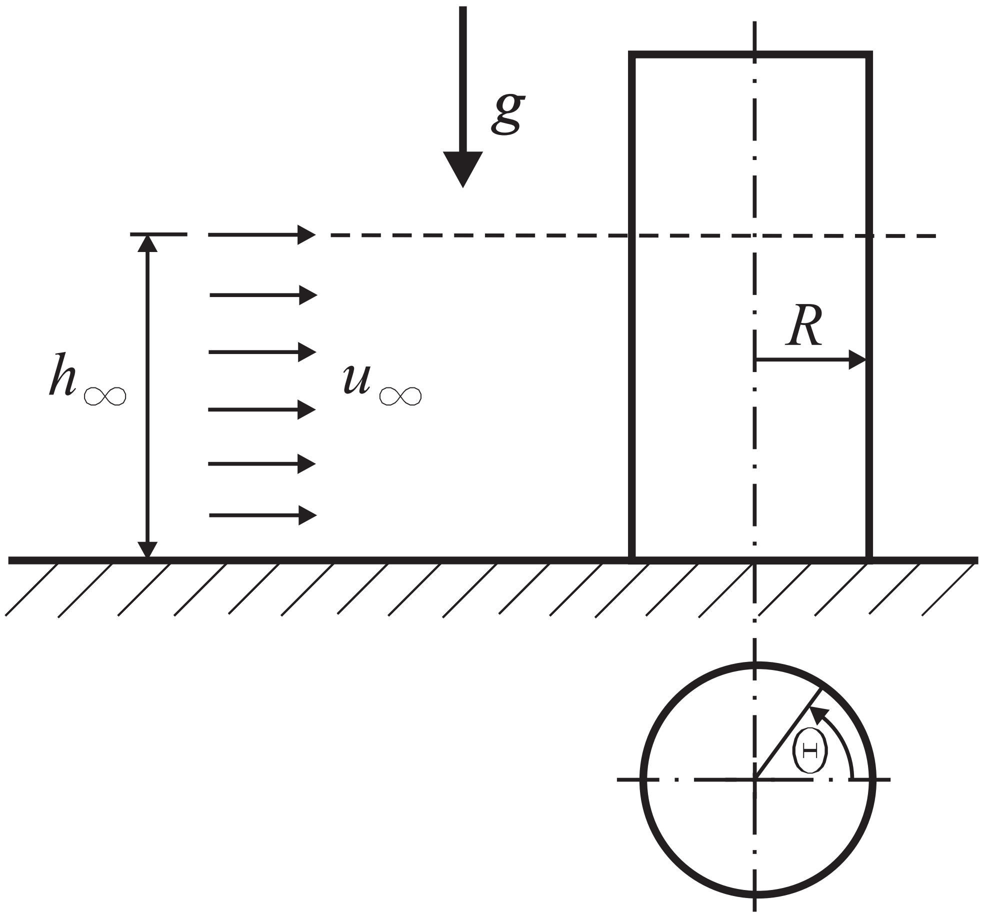


plane stagnation point flow



14.8

A bridge pylon with a circular cross-section is flown against with the velocity u_∞. Far away from the pylon the water depth is h_∞.



$$u_\infty = 1\ m/s \qquad h_\infty = 6\ m \qquad R = 2$$

$$m \qquad \rho = 10^3\ kg/m^3 \qquad g = 10\ m/s^2$$

Determine

a) the water depth at the pylon wall as a function of θ,
b) the water depth in the stagnation points,
c) the smallest water depth over the ground.

a) Circular cylinder: Dipole + Parallel flow

$$F(z) = u_\infty z + \frac{M}{2\pi z} = u_\infty z + \frac{R^2 u_\infty}{z}$$

$$\rho g h_\infty + \frac{\rho}{2} u_\infty^2 = \rho g h(\theta) + \frac{\rho}{2} \vec{v}^2$$

$$v_\theta = \frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -\left(u_\infty + \frac{R^2 u_\infty}{r^2}\right) \sin\theta$$

$$r = R : \vec{v}^2 = v_\theta^2 = 4u_\infty^2 \sin^2\theta$$

$$h(\theta) - h_\infty = \frac{u_\infty^2}{2g}(1 - 4\sin^2\theta)$$

b) Stagnation points: $\theta = 0$ and $\theta = \pi$

$$h = h_\infty + \frac{u_\infty^2}{2g} = 6.05\ m$$

c)

$$\theta_{min} = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$h_{min} = h_\infty - \frac{3u_\infty^2}{2g} = 5.85\ m$$

14.12

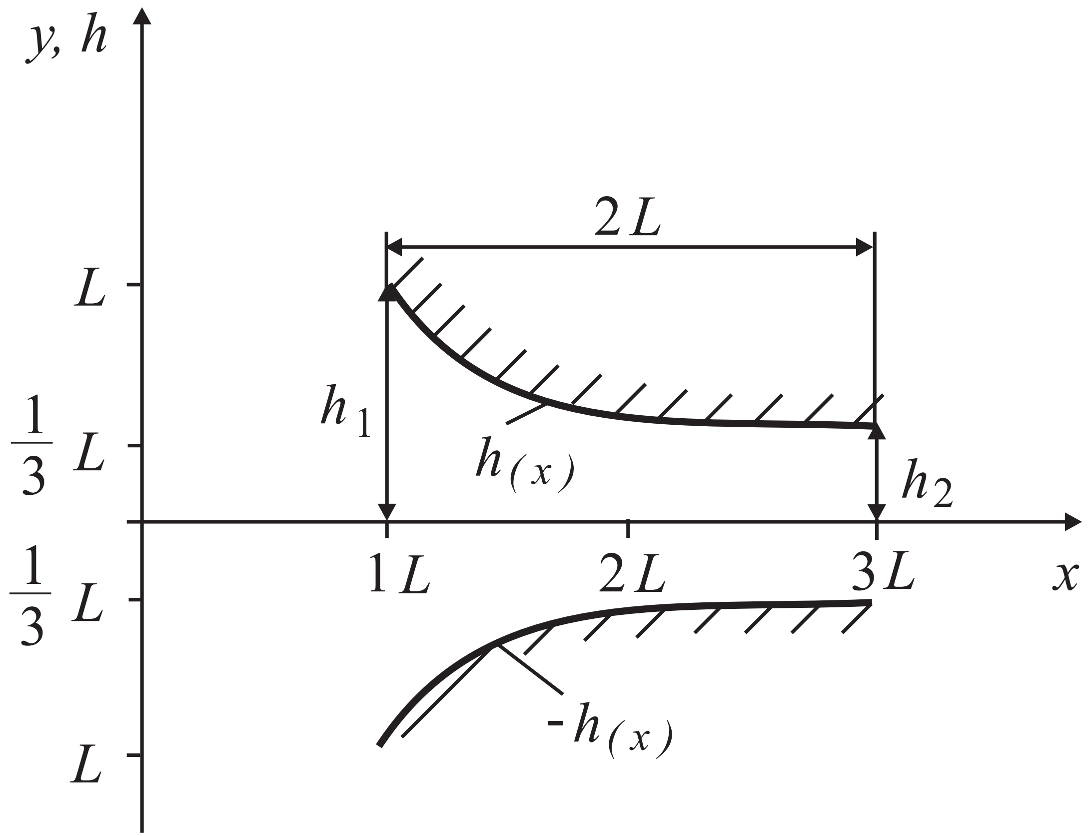
The stream function $\psi(x, y)$ for the flow of an incompressible fluid through the sketched plane nozzle is given.

$$\psi(x, y) = \frac{y}{h(x)} u_\infty L$$

Given: $u_\infty,\ L,\ B,\ h_1 = L,\ h_2 = \frac{1}{3}L$

a) Determine the upper and the lower contour $h(x)$ such that the flow can be described with the potential theory,
b) Compute the velocity distribution $u(x, y)$ and $v(x, y)$.
c) Determine the volume flux for a nozzle with the width B.

14.12



a)

Condition: $\omega = 0 \rightarrow \dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0$

$$\frac{\partial^2\psi(x,y)}{\partial y^2} = 0 \rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(yu_\infty L\frac{\partial}{\partial x}\left(\frac{1}{h(x)}\right)\right) = 0$$

$$\frac{\partial}{\partial x}\left(-yu_\infty L\frac{h'(x)}{h^2(x)}\right) = 0 \rightarrow \frac{d}{dx}\left(\frac{h'(x)}{h^2(x)}\right) = 0$$

After 2 integrations: $-\dfrac{1}{h(x)} = C_1 x + C_2$

BC:

$$x = L, \quad h = L \quad \Longrightarrow \quad -\frac{1}{L} = C_1 L + C_2$$

$$x = 3L, \quad h = \frac{1}{3}L \quad \Longrightarrow \quad -\frac{3}{L} = 3C_1 L + C_2$$

$$\Longrightarrow \quad C_2 = 0, \quad C_1 = -\frac{1}{L^2} \quad \Longrightarrow \quad h(x) = \frac{L^2}{x}$$

b)

$$u = \frac{\partial \psi}{\partial y}; \quad v = -\frac{\partial \psi}{\partial x}; \quad \psi = \frac{u_\infty}{L} x y$$

$$u = u_\infty \frac{x}{L}; \quad v = -u_\infty \frac{y}{L}$$

c)

$$\dot{V} = \psi_{(y=h)} - \psi_{(y=-h)}\big|_{x=L}$$

$$\dot{V} = (h_1 + h_1)Bu_\infty$$
$$\dot{V} = 2U_\infty LB$$

The stream function is given $\psi = \psi_1 + \psi_2$

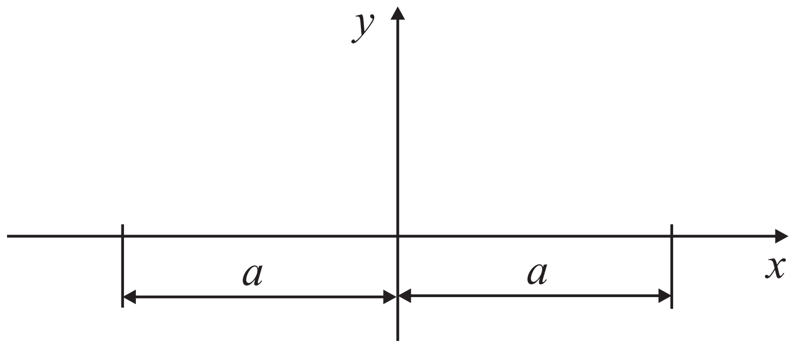
with $\psi_1 = -\frac{\Gamma}{2\pi} \ln \sqrt{(x-a)^2 + y^2}$

$\psi_2 = -\frac{2\Gamma}{2\pi} \ln \sqrt{(x+a)^2 + y^2}$

Given: $a,\ \Gamma > 0$

Determine

a) the coordinates of the stagnation point,
b) the pressure coefficient on the x-axis $c_p(x, y=0)$ such, that $c_p = 0$ in the origin of the coordinate system.



a) Condition for the stagnation point: $u = 0, v = 0$

$$u = \frac{\partial \psi}{\partial y} = -\frac{\Gamma}{2\pi}\left(\frac{y}{(x-a)^2+y^2} + \frac{2y}{(x+a)^2+y^2}\right)$$

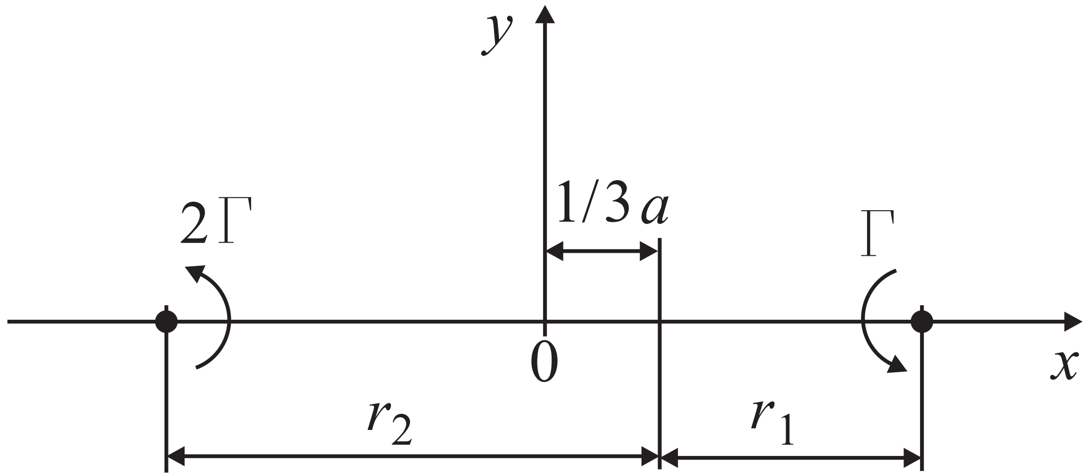
$$u = -\frac{\Gamma}{2\pi} y \left(\frac{1}{(x-a)^2+y^2} + \frac{2}{(x+a)^2+y^2}\right) = 0 \quad \text{if } y = 0$$

$$v = -\frac{\partial \psi}{\partial x} = \frac{\Gamma}{2\pi}\left(\frac{x-a}{(x-a)^2+y^2} + \frac{2(x+a)}{(x+a)^2+y^2}\right)$$

for $y = 0$: $\quad v = \frac{\Gamma}{2\pi}\left(\frac{1}{x-a} + \frac{2}{x+a}\right) = \frac{\Gamma}{2\pi}\frac{3x-a}{x^2-a^2} \quad \star)$

$v = 0$, if $3x - a = 0 \quad \rightarrow \quad$ stagnation point: $\quad x_s = \frac{a}{3}, \quad y_s = 0$

alternatively:



$$v_\theta = \frac{\Gamma}{2\pi r} : v_{\theta_1} = v_{\theta_2} \rightarrow \frac{\Gamma}{2\pi r_1} = \frac{2\Gamma}{2\pi r_2} \rightarrow r_2 = 2r_1$$

$$2a = r_1 + r_2 \rightarrow 2a = 3r_1 \rightarrow r_1 = \frac{2}{3}a \rightarrow x_s = \frac{a}{3}$$

b)

$$c_p(x, y=0) = 1 - \frac{u^2 + v^2}{u^2_{(0,0)} + v^2_{(0,0)}}$$ with $u = 0$ for $y = 0$.

$$\Longrightarrow c_p(x, y=0) = 1 - \frac{v^2}{v^2_{(0,0)}},\; v_{(0,0)} = \frac{\Gamma}{2\pi}\frac{1}{a}$$ (see $\star$) i.e. $c_{p_{(0,0)}} = 0$

with $\star$) $$c_p(x, y=0) = 1 - \left(\frac{\dfrac{3x - a}{x^2 - a^2}}{\dfrac{1}{a}} \right)^2$$

$$\Longrightarrow c_p(x, y=0) = 1 - \left(\frac{3xa - a^2}{x^2 - a^2} \right)^2$$ for $x \neq a, -a$