2^{nd} method to determine the similarity parameters

differential equation of a problem $\longrightarrow$ example: $\boxed{\dfrac{\partial p}{\partial x} = \eta \dfrac{\partial^2 u}{\partial y^2}}$

① introduction of dimensionless values using reference values

example: $u_{ref} = u_\infty$; $\eta_{ref} = \eta_{ref}$; $h_{ref} = h$

$p_{ref} = \Delta p$; $L_{ref} = L$ ....

$$\Rightarrow \bar{u} = \frac{u}{u_\infty} ; \bar{p} = \frac{p}{\Delta p} ; \bar{\eta} = \frac{\eta}{\eta_{ref}} ; \bar{x} = \frac{x}{L} ; \bar{y} = \frac{y}{h} ; \dots$$

$$\Rightarrow \boxed{\frac{\Delta p}{L} \frac{\partial \bar{p}}{\partial \bar{x}} = \frac{\eta_{ref} \cdot u_\infty}{h^2} \cdot \bar{\eta} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}}$$

② divide the equation by the coefficient of one term $\rightarrow$ dimensionless parameter

$$\boxed{\frac{\partial \bar{p}}{\partial \bar{x}} = \underbrace{\boxed{\frac{L}{\Delta p} \frac{\eta_{ref} \cdot u_\infty}{h^2}}}_{\pi} \cdot \bar{\eta} \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}}$$

2 terms $\rightarrow$ 1 parameter

! method of dimensional analysis works as well $\rightarrow$ gives maximum number of parameters

$\rightarrow$ method of differential equation may provide less parameters

$\rightarrow$ usually the parameters found can be expressed by known / common parameters like Re, Eu...

$$\frac{L}{\Delta p} \cdot \frac{\eta_{ref} \cdot u_\infty}{h^2} = \underset{\substack{\uparrow \\ 1/Re}}{\frac{\eta_{ref}}{\rho u_\infty h}} \cdot \underset{\substack{\uparrow \\ 1/Eu}}{\frac{\rho u_\infty^2}{\Delta p}} \cdot \underset{\substack{\uparrow \\ \text{geometry}}}{\frac{L}{h}}$$

example: The governing equations for a laminar boundary layer flow of a flat plate are:

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0} \qquad \text{(continuity)}$$

Hint: use the inflow conditions as reference

$$\boxed{\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \eta\frac{\partial^2 u}{\partial y^2}} \qquad \text{(momentum)}$$

$$\boxed{\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \lambda\frac{\partial^2 T}{\partial y^2}} \qquad \text{(energy)}$$

Determine the characterizing parameters of the problem and write them as functions of commonly known parameters.

$$\bar{u} = \frac{u}{u_\infty}\ ;\ \bar{v} = \frac{v}{u_\infty}\ ;\ \bar{\rho} = \frac{\rho}{\rho_\infty}\ ;\ \bar{x} = \frac{x}{L}\ ;\ \bar{y} = \frac{y}{L}\ ;\ \bar{\eta} = \frac{\eta}{\eta_\infty}\ ;\ \bar{c}_p = \frac{c_p}{c_{p\infty}}\ ;\ \bar{T} = \frac{T}{T_\infty}$$

$$\bar{\lambda} = \frac{\lambda}{\lambda_\infty}$$

$$\rightarrow \boxed{\frac{u_\infty}{L}\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}}\right) = 0} \Rightarrow \text{no parameter}$$

$$\rightarrow \boxed{\rho_\infty \frac{u_\infty^2}{L}\bar{\rho}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \eta_\infty \bar{\eta}\frac{u_\infty}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}}$$

$$\Rightarrow \bar{\rho}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \bar{\eta}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\cdot\underbrace{\left(\frac{\eta_\infty \cdot u_\infty \cdot L}{L^2 \rho_\infty \cdot u_\infty^2}\right)}_{K_1 = \frac{\eta_\infty}{L\rho_\infty u_\infty} = \frac{1}{Re}}$$

$$\rightarrow \boxed{\frac{\rho_\infty c_{p\infty} u_\infty T_\infty}{L}\bar{\rho}\bar{c}_p\left(\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}\right) = \frac{\lambda_\infty T_\infty}{L^2}\bar{\lambda}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}}$$

$$\Rightarrow \bar{\rho}\bar{c}_p\left(\bar{u}\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{T}}{\partial \bar{y}}\right) = \bar{\lambda}\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\cdot\underbrace{\left(\frac{\lambda_\infty T_\infty L}{L^2 \rho_\infty c_{p\infty} u_\infty T_\infty}\right)}$$

$$K_2 = \underbrace{\frac{\lambda_\infty}{c_{p\infty}\eta_\infty}}_{1/Pr}\cdot\underbrace{\frac{\eta_\infty}{L\rho_\infty u_\infty}}_{1/Re}$$

11.10

In a gas flow the heat transfer is determined from the viscous effects and from heat conduction. The influencing quantities are the heat conductivity $\lambda \left[\frac{kgm}{s^3K}\right]$, the dynamic viscosity $\eta \left[\frac{kg}{ms}\right]$ and the reference values for the temperature, the velocity, and the length. The physical relationship can be described with the energy equation

$$\boxed{\lambda\frac{\partial^2 T}{\partial y^2} + \eta\left(\frac{\partial u}{\partial y}\right)^2 = 0.}$$

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① $T_{ref} = T_R \; ; \; L_{ref} = \ell \; ; \; u_{ref} = u_R$

$$\Rightarrow \bar{T} = \frac{T}{T_R} \; ; \; \bar{y} = \frac{y}{\ell} \; ; \; \bar{u} = \frac{u}{u_R}$$

$$\Rightarrow \boxed{\lambda \cdot \frac{T_R}{\ell^2} \cdot \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \eta \cdot \frac{u_R^2}{\ell^2}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 = 0}$$

② $$\Rightarrow \boxed{\frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \frac{\eta \cdot u_R^2}{\lambda \cdot T_R}\left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2 = 0}$$

$$\Downarrow$$

$$K = \frac{\eta \, u_R^2}{\lambda \cdot T_R}$$

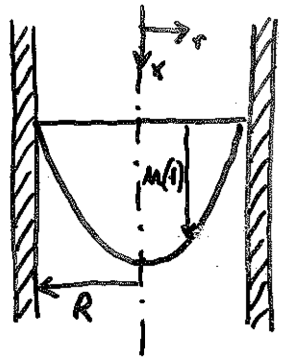
using $c_p = \frac{\gamma R}{\gamma - 1}$:

$$K = \frac{\eta \cdot c_p}{\lambda} \cdot \frac{u_R^2}{c_p \cdot T_R} = \underbrace{\frac{\eta \cdot c_p}{\lambda}}_{Pr} \cdot \underbrace{\frac{u_R^2}{\gamma R T_R}}_{Ma^2} \cdot (\gamma - 1)$$

$$Pr \cdot Ma^2 \cdot (\gamma-1)$$

example: laminar pipe flow

$$\boxed{-\frac{dp}{dx} + \varrho g + \frac{\eta}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)} \quad g \downarrow$$



Determine the dimensionless parameters of the flow-problem.

$$\bar{p} = \frac{p}{\Delta p} \; ; \; \bar{x} = \frac{x}{L} \; ; \; \bar{\varrho} = \frac{\varrho}{\varrho_{ref}} \; ; \; \bar{g} = \frac{g}{g_{ref}} \; ; \; \bar{\eta} = \frac{\eta}{\eta_{ref}} \; ; \; \bar{r} = \frac{r}{R} \; ; \; \bar{u} = \frac{u}{u_m}$$

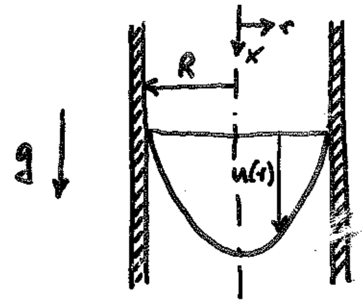
$$\Rightarrow -\frac{\Delta p}{L}\frac{d\bar{p}}{d\bar{x}} + \varrho_{ref}\, g_{ref} \cdot \bar{g}\,\bar{\varrho} + \frac{\eta_{ref}}{R^2} u_m \cdot \frac{\bar{\eta}}{\bar{r}} \cdot \frac{d}{d\bar{r}}\left(\bar{r}\frac{d\bar{u}}{d\bar{r}}\right) = 0$$

$$\Rightarrow -\underbrace{\frac{\Delta p \cdot R^2}{L \cdot \eta_{ref} \cdot u_m}}_{K_1}\frac{d\bar{p}}{d\bar{x}} + \underbrace{\frac{\varrho_{ref} \cdot g_{ref} \cdot R^2}{\eta_{ref}\, u_m}}_{K_2} \cdot \bar{g}\,\bar{\varrho} + \frac{\bar{\eta}}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\frac{d\bar{u}}{d\bar{r}}\right) = 0$$

$$K_1 = \frac{\Delta p}{\varrho_{ref} \cdot u_m^2} \cdot \frac{\varrho_{ref} \cdot u_m \cdot R}{\eta_{ref}} \cdot \frac{R}{L} = Eu \cdot Re \cdot \frac{R}{L}$$

$$K_2 = \frac{\varrho_{ref} \cdot u_m \cdot R}{\eta_{ref}} \cdot \frac{g_{ref} \cdot R}{u_m^2} = Re \cdot \frac{1}{Fr^2}$$

example: laminar pipe flow



Determine the characterizing parameters of this problem.

Hint: u_m, l, R

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① physical quantities: $\Delta p, \rho, u_m, \eta, l, R, g \Rightarrow k = 7$

② basic dimensions: $\left[\frac{kg}{s^2 m}\right], \left[\frac{kg}{m^3}\right], \left[\frac{m}{s}\right], \left[\frac{kg}{ms}\right], [m], [m], \left[\frac{m}{s^2}\right] \Rightarrow r = 3$

③ number of parameters: $m = k - r = 4$

④ recurring variables: $\rho\left[\frac{kg}{m^3}\right], u_m\left[\frac{m}{s}\right], R[m]$

⑤

$$K_1 = \Delta p \cdot \rho^{\alpha_1} \cdot u_m^{\beta_1} \cdot R^{\gamma_1}$$

$$K_2 = \eta \cdot \rho^{\alpha_2} \cdot u_m^{\beta_2} \cdot R^{\gamma_2}$$

$$K_3 = l \cdot \rho^{\alpha_3} \cdot u_m^{\beta_3} \cdot R^{\gamma_3}$$

$$K_4 = g \cdot \rho^{\alpha_4} \cdot u_m^{\beta_4} \cdot R^{\gamma_4}$$

⑥

$$K_1: \begin{array}{c|l} kg & 1 + \alpha_1 \\ m & -1 - 3\alpha_1 + \beta_1 + \gamma_1 \\ s & 2 \quad - \beta_1 \end{array} \Bigg\} \quad \alpha_1 = -1, \beta_1 = -2, \gamma_1 = 0 \quad \boxed{K_1 = \frac{\Delta p}{\rho u_m^2} = Eu}$$

$$K_2: \begin{array}{c|l} kg & 1 + \alpha_2 \\ m & -1 - 3\alpha_2 + \beta_2 + \gamma_2 \\ s & -1 \quad - \beta_2 \end{array} \Bigg\} \quad \alpha_2 = -1, \beta_2 = -1, \gamma_2 = -1 \quad \boxed{K_2 = \frac{\eta}{\rho u_m R} = \frac{1}{Re}}$$

$$K_3: \begin{array}{c|l} kg & \quad + \alpha_3 \\ m & 1 \quad - 3\alpha_3 + \beta_3 + \gamma_3 \\ s & \quad - \beta_3 \end{array} \Bigg\} \quad \alpha_3 = 0, \beta_3 = 0, \gamma_3 = -1 \quad \boxed{K_3 = \frac{l}{R}}$$

$$K_4: \begin{array}{c|l} kg & \quad + \alpha_4 \\ m & 1 \quad - 3\alpha_4 + \beta_4 + \gamma_4 \\ s & -2 \quad - \beta_4 \end{array} \Bigg\} \quad \alpha_4 = 0, \beta_4 = -2, \gamma_4 = 1 \quad \boxed{K_4 = \frac{g \cdot D}{u_m^2} = \frac{1}{Fr^2}}$$