Vectors, Tensors, Operators

Tensors

Rank 0	Scalar	$p, \rho, T, \ldots$
Rank 1	Vector	$\vec{F}, \vec{v}, \vec{I}, \ldots$
Rank 2	Dyad	$\underbrace{\bar{\bar{\sigma}}}_{\text{stress tensor}}, \bar{\bar{\tau}}, \ldots$

Differential operators

Nabla-Operator in cartesian coordinates

$$\begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix}$$

Vektors, Tensors, Operators

Divergence (div): $\text{div}\,\vec{v} \;=\; \nabla \cdot \vec{v} \;=\; \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ (inner product)

Gradient (grad): $\text{grad}\,p \;=\; \nabla p \;=\; (\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z})^T$

Rotation (rot): $\text{rot}\,\vec{v} \;=\; \nabla \times \vec{v} \;=\; \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial y}{\partial y} \end{pmatrix}$ (outer product)

Vektors, Tensors, Operators

LAPLACE-Operator in cartesian coordinates

$$\Delta = \nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$\Delta p = \nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$

$$\Delta\vec{v} = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix}$$

Computational rules

1. Scalar - Vector $\longrightarrow$ Vector

$$a\,\vec{b} = a \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} a\,b_x \\ a\,b_y \\ a\,b_z \end{pmatrix} = \vec{c}$$

2. Vector - Vector $\longrightarrow$ Scalar (inner produkt)

$$\vec{a} \cdot \vec{b} = (a_x, a_y, a_z) \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = a_x b_x + a_y b_y + a_z b_z = c$$

3. Vector - Vector $\longrightarrow$ Vector (outer produkt)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{pmatrix} a_y b_z & - & a_z b_y \\ a_z b_x & - & a_x b_z \\ a_x b_y & - & a_y b_x \end{pmatrix} = \vec{c}$$

Computational rules

4. Vector - Vector $\longrightarrow$ Dyad (dyadic produkt)

$$\vec{a} \otimes \vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} (b_x, b_y, b_z) = \begin{pmatrix} a_x b_x & a_x b_y & a_x b_z \\ a_y b_x & a_y b_y & a_y b_z \\ a_z b_x & a_z b_y & a_z b_z \end{pmatrix} = \bar{\bar{c}}$$

5. Vector - Dyad $\longrightarrow$ Vector

$$\vec{a} \cdot \bar{\bar{b}} = (a_x, a_y, a_z) \cdot \begin{pmatrix} b_{xx} & b_{xy} & b_{xz} \\ b_{yx} & b_{yy} & b_{yz} \\ b_{zx} & b_{zy} & b_{zz} \end{pmatrix} = \begin{pmatrix} a_x b_{xx} + a_y b_{yx} + a_z b_{zx} \\ a_x b_{xy} + a_y b_{yy} + a_z b_{zy} \\ a_x b_{xz} + a_y b_{yz} + a_z b_{zz} \end{pmatrix}$$

Identities

$$\text{rot}\,(\text{grad}\,a) = \nabla \times (\nabla a) = \vec{0}$$

$$\text{div}\,(\text{rot}\,\vec{v}) = \nabla \cdot (\nabla \times \vec{a}) = 0$$

$$\vec{v} \times (\text{rot}\,\vec{v}) = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \times \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial y}{\partial y} \end{pmatrix} = \frac{1}{2}\nabla \vec{v}^2 - (\vec{v} \cdot \nabla)\vec{v}$$

Partial derivatives

Total derivative of a function $f(x, y, z)$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

the total derivative describes the increase of a function

$$\vec{v} = \vec{v}(t, x, y, z)$$

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t}dt + \frac{\partial \vec{v}}{\partial x}dx + \frac{\partial \vec{v}}{\partial y}dy + \frac{\partial \vec{v}}{\partial z}dz \bigg| : dt$$

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}\underbrace{\frac{dx}{dt}}_{u} + \frac{\partial \vec{v}}{\partial y}\underbrace{\frac{dy}{dt}}_{v} + \frac{\partial \vec{v}}{\partial z}\underbrace{\frac{dz}{dt}}_{w}$$

Substantial derivative

$$\underbrace{\frac{d\vec{v}}{dt}}_{\text{substantial}} = \overbrace{\frac{\partial \vec{v}}{\partial t}}^{\text{local}} + \underbrace{u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}}_{\text{convective acceleration}}$$

Summary of simplifications

steady: $\frac{\partial}{\partial t} = 0$ $\boxed{\text{not } \frac{d}{dt} = 0}$

incompressible: $\rho =$ const.

Symmetry: Normal component of the velocity = 0

$\frac{\partial}{\partial \theta} = 0$

frictionless $\eta = 0$ ($\lambda = 0$ (heat conductivity)

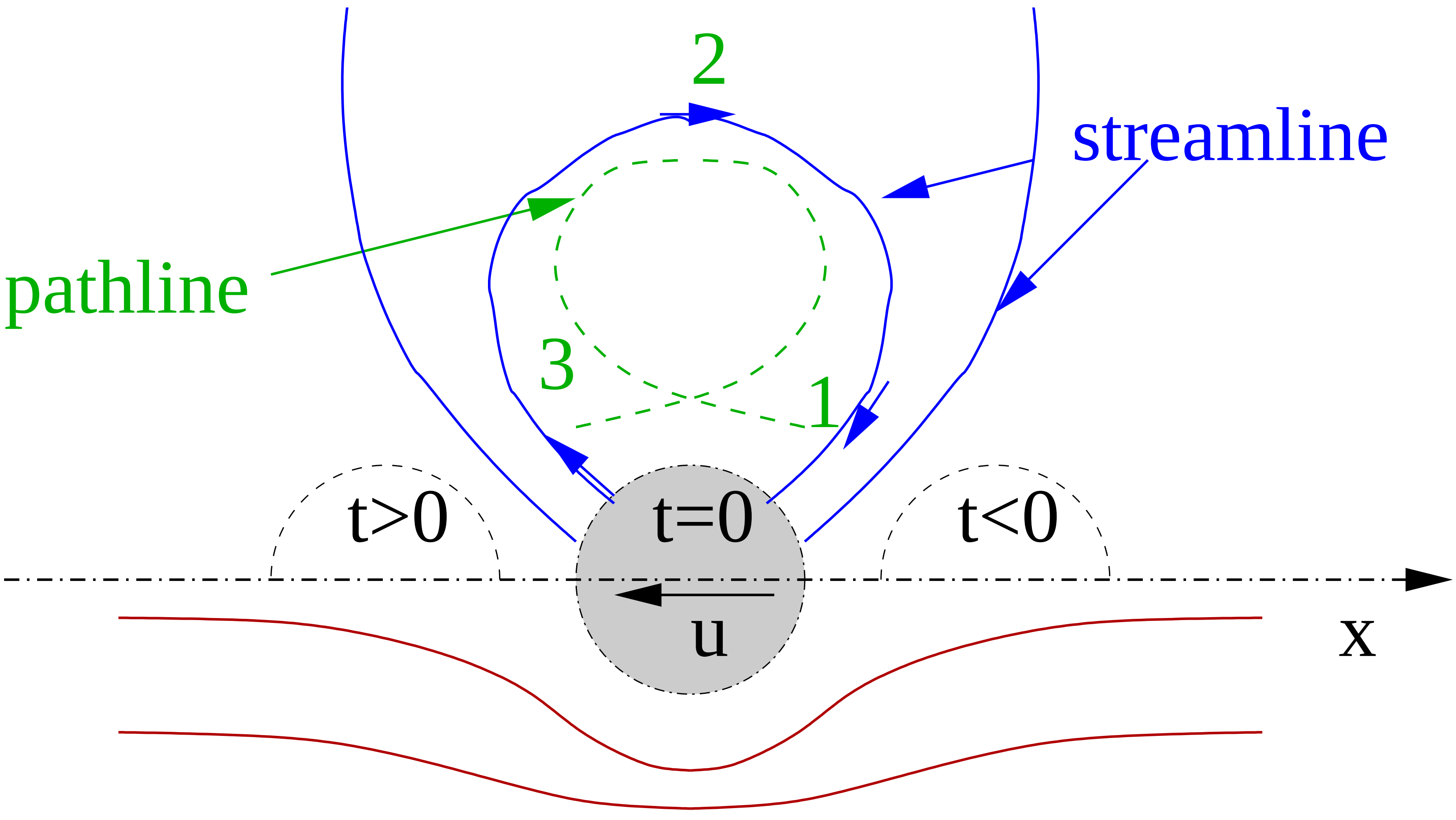
Summary of simplifications

2-dimensional reduction of the number of equations
reduction of the number of derivatives

fully developed flow $\frac{\partial}{\partial x} = 0$

$$\xrightarrow{continuity} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$\longrightarrow w = 0,$ since $w = 0$ athe wall

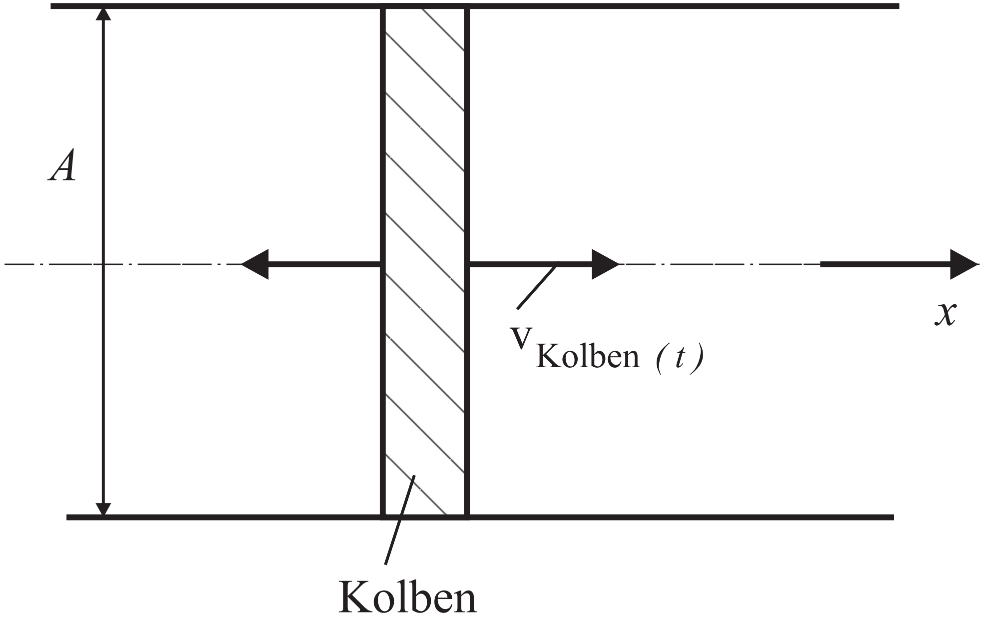

2
streamline
pathline
3
1
t>0
t=0
t<0
u
x

motionless observer: pathline $\neq$ streamline
moving observer: pathline $=$ streamline

motionless environment
constant velocity $\longrightarrow$ unsteady flow for fixed observer
steady flow for moving observer

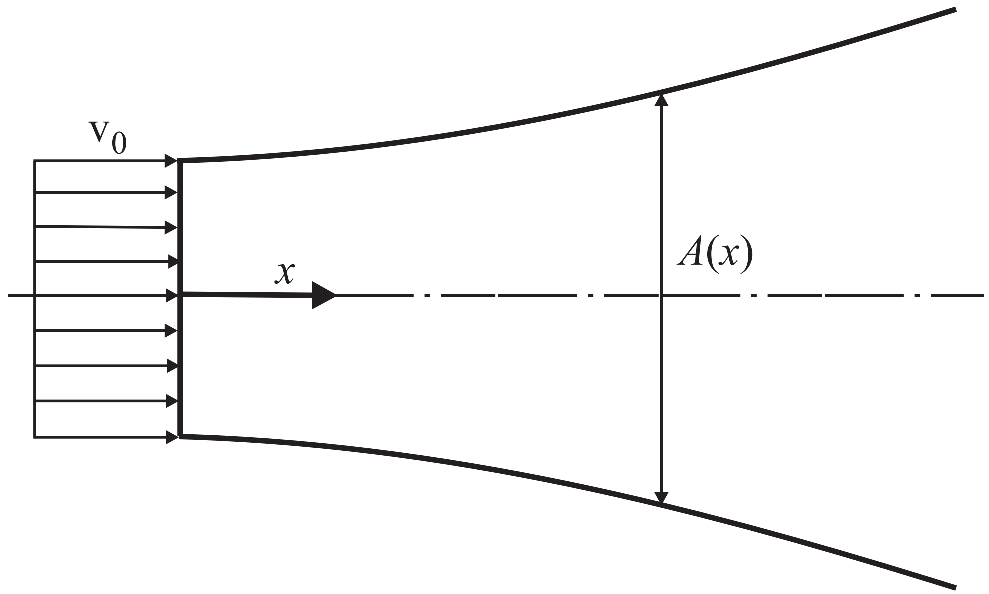
4.2

a) A piston is moving in a tube of infinite length and with constant cross section A with the velocity $v_{\text{piston}}(t)$. The density of the fluid is constant.



Determine the substantial acceleration in the tube.

b) A fluid of constant density flows into a diffusor with the constant velocity $v = v_0$. The cross section of the diffusor is $A(x)$. Determine the substantial acceleration of the fluid along the axis x.



4.2

a) substantial derivative:

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$$

continuity: $A \cdot v =$ const. mit $A =$ const.

$$\longrightarrow \frac{\partial v}{\partial x} = 0$$

$$\text{but } \frac{\partial v}{\partial t} = \frac{\partial v_{piston}}{\partial t} \longrightarrow \frac{dv}{dt} = \frac{\partial v_{piston}}{\partial t}$$

only local acceleration

4.2

b) again: $\frac{dv}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$

constant inflow velocity: $v = v_0 \longrightarrow \frac{\partial v}{\partial t} = 0$

continuity: $A(x) \cdot v(x) =$ const.

1st derivative: $\frac{\partial A(x)}{\partial x} \cdot v(x) + A(x) \cdot \frac{\partial v(x)}{\partial x} = 0$

$$\frac{\partial v(x)}{\partial x} = -\frac{v(x)}{A(x)} \cdot \frac{\partial A(x)}{\partial x}$$

$$\longrightarrow \frac{dv}{dt} = -\frac{v^2(x)}{A(x)}\frac{\partial A(x)}{\partial x}$$

since: $\dot{V} = v_0 \cdot A_0 = v(x)A(x)$

$$\longrightarrow v(x) = \frac{v_0 A_0}{A(x)} \qquad \boxed{\mathbf{\frac{dv}{dt} = -\frac{v_0^2 A_0^2}{A^3(x)}\frac{\partial A}{\partial x}}}$$

4.3

The following continuity equation is formulated in cartesian coordinates.

$$\frac{d\rho}{dt} + \rho \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \quad \text{(Eq. 1)}$$

with $v = \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$

Transform the equation into
a) cylindrical coordinates

Hint:

$$\begin{aligned} x &= r \cdot \cos\phi & \qquad r &= \sqrt{x^2 + y^2} \\ y &= r \cdot \sin\phi & \qquad \phi &= \arctan\left(\frac{y}{x}\right) \\ z &= z & \qquad z &= z \end{aligned}$$

sperical coordinates

Hint:

$$
\begin{aligned}
x &= r \cdot \sin\Theta \cdot \cos\phi \qquad & r &= +\sqrt{x^2 + y^2 + z^2} \\
y &= r \cdot \sin\Theta \cdot \sin\phi \qquad & \Theta &= \arctan\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \\
z &= r \cdot \cos\Theta \qquad & \phi &= \arctan\left(\frac{y}{x}\right)
\end{aligned}
$$

2-dimensional cylinder coordinates

$$x = r\cos\phi : \frac{\partial x}{\partial r} = \cos\phi \quad \frac{\partial x}{\partial \phi} = -r\sin\phi$$

$$y = r\sin\phi : \frac{\partial y}{\partial r} = \sin\phi \quad \frac{\partial y}{\partial \phi} = r\cos\phi$$

$$r = \sqrt{x^2+y^2} : \frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\phi$$

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2+y^2}} = \frac{x}{r} = \sin\phi$$

$$\phi = \arctan(\tfrac{y}{x}) : \frac{\partial \phi}{\partial x} = \frac{-y}{x^2(1+\frac{y^2}{x^2})} = -\frac{y}{r^2} = -\frac{\sin\phi}{r}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{x(1+\frac{y^2}{x^2})} = \frac{x}{r^2} = \frac{\cos\phi}{r}$$

4.3

continuity: $\dfrac{d\rho}{dt} + \rho(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}) = 0$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial x}; \; \frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial y}$$

$\dfrac{d}{dt} = \dfrac{d}{dr}\dfrac{dr}{dt} + \dfrac{d}{d\phi}\dfrac{d\phi}{dt}$ with $v_r = \dfrac{dr}{dt}$ and $v_\phi = r\dfrac{d\phi}{dt}$

$$\begin{aligned} u &= \frac{dx}{dt} = v_r \cos\phi - v_\phi \sin\phi \\ v &= \frac{dy}{dt} = v_r \sin\phi + v_\phi \cos\phi \end{aligned}$$

4.3 continued

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \phi}\frac{\partial \phi}{\partial x} = (\underline{\frac{\partial v_r}{\partial r}\cos\phi} - \underline{\frac{\partial v_\phi}{\partial r}\sin\phi})\cos\phi -$$

$$-\frac{\sin\phi}{r}(\underline{\frac{\partial v_r}{\partial \phi}\cos\phi} - \underline{v_r\sin\phi} - \underline{\frac{\partial v_\phi}{\partial \phi}\sin\phi} - \underline{v_\phi\cos\phi})$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \phi}\frac{\partial \phi}{\partial x} = (\underline{\frac{\partial v_r}{\partial r}\sin\phi} + \underline{\frac{\partial v_\phi}{\partial r}\cos\phi})\sin\phi +$$

$$+\frac{\cos\phi}{r}(\underline{\frac{\partial v_r}{\partial \phi}\sin\phi} + \underline{v_r\cos\phi} + \underline{\frac{\partial v_\phi}{\partial \phi}\cos\phi} - \underline{v_\phi\sin\phi})$$

$$\begin{aligned}(\sin\phi^2 + \cos\phi^2) &= 1 \\ (\ldots) &= 0\end{aligned}$$

4.3 continued

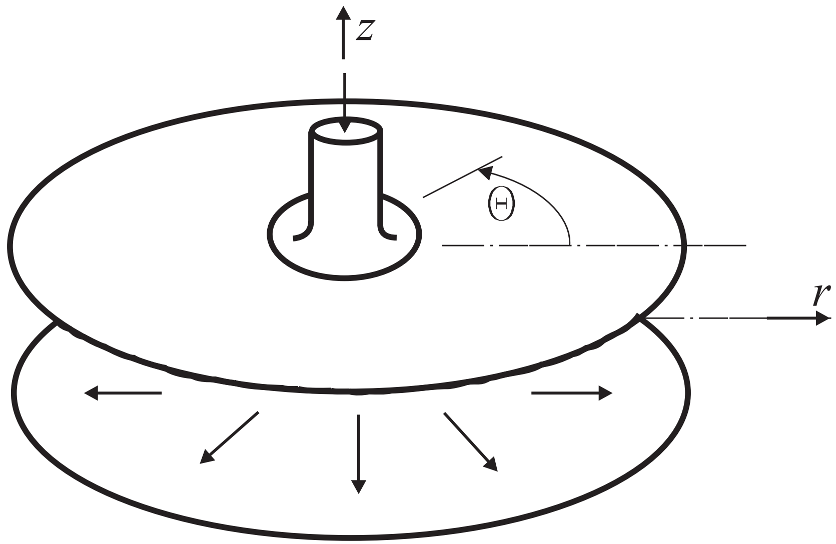
Final result:

$$\frac{d\rho}{dt} + \rho\frac{\partial v_r}{\partial r} + \rho\frac{v_r}{r} + \rho\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} = 0$$

$$\longrightarrow \boxed{\frac{\mathbf{d}\rho}{\mathbf{dt}} + \rho\frac{\mathbf{1}}{\mathbf{r}}\frac{\partial}{\partial \mathbf{r}}(\mathbf{r}\mathbf{v_r}) + \rho\frac{\mathbf{1}}{\mathbf{r}}\frac{\partial \mathbf{v}_\phi}{\partial \phi} = \mathbf{0}}$$

4.4

An incompressible fluid with the viscosity η is flowing laminar and steady between two parallel plates



The flow is radial from inside to outside.

4.4

The determining differential equations in cylindrical coordinates are

$$\frac{1}{r}\frac{\partial(\rho\, r\, v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho\, v_\Theta)}{\partial \Theta} + \frac{\partial(\rho\, v_z)}{\partial z} = 0$$

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\Theta}{r}\frac{\partial v_r}{\partial \Theta} - \frac{v_\Theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \eta\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r\, v_r)}{\partial r}\right)\right.$$
$$\left. + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \Theta^2} - \frac{2}{r^2}\frac{\partial v_\Theta}{\partial \Theta} + \frac{\partial^2 v_r}{\partial z^2}\right)$$

Simplify the equations for the flow problem described.

$$\frac{1}{r}\,\frac{\partial(\rho\; r\; v_r)}{\partial r} + \frac{1}{r}\,\frac{\partial(\rho\; v_\Theta)}{\partial\Theta} + \frac{\partial(\rho\; v_z)}{\partial z} = 0$$

$\rho = const.$ radial symmetry radial flow parallel plates

$$\longrightarrow \boxed{\frac{\partial(r\; v_r)}{\partial\; r} = 0}$$ Continuity

radial radial parallel continuity

$$\rho\left(v_r\frac{\partial v_r}{\partial r} + \frac{v_\Theta}{r}\frac{\partial v_r}{\partial \Theta} - \frac{v_\Theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \eta\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r\ v_r)}{\partial r}\right)\right.$$

$$\left. + \frac{1}{r^2}\frac{\partial^2\ v_r}{\partial \Theta^2} - \frac{2}{r^2}\frac{\partial v_\Theta}{\partial \Theta} + \frac{\partial^2\ v_r}{\partial z^2}\right)$$

symmetry

symmetry symmetry

$$\longrightarrow \boxed{\rho \mathbf{v_r}\frac{\partial \mathbf{v_r}}{\partial \mathbf{r}} = -\frac{\partial \mathbf{p}}{\partial \mathbf{r}} + \eta\frac{\partial^{\mathbf{2}}\ \mathbf{v_r}}{\partial \mathbf{z^2}}}$$ radial momentum equation

4.6

The Navier-Stokes equations for unsteady, incompressible flows in graviational field read:

$$\nabla \cdot \vec{v} = 0$$

$$\rho \frac{d\vec{v}}{dt} = -\nabla p + \eta \nabla^2 \vec{v} + \rho \vec{g}$$

Formulate the equations for a steady, frictionless, twodimensional flow in a cartesian coordinate system $(x,\ y)$.

continuity: $\nabla \cdot \vec{v} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

4.6

momentum: frictionless: $\eta = 0$

x-direction: $$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \rho \mathsf{g}_x$$

$$\rho \left(\frac{\partial u}{\partial t}\Big|_{=0} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \rho \mathsf{g}_x$$

y-direction: $$\rho \frac{dv}{dt} = -\frac{\partial p}{\partial y} + \rho \mathsf{g}_y$$

$$\rho \left(\frac{\partial v}{\partial t}\Big|_{=0} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \rho \mathsf{g}_y$$

example

The continuity equation and the Navier-Stokes equations are given:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\nabla \cdot \vec{v}) = 0$$

$$\frac{\partial \rho \vec{v}}{\partial t} + \nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla p - \nabla \,\overline{\overline{\tau}} + \rho \vec{g}$$

with $$\overline{\overline{\tau}} = \begin{pmatrix} -2\eta\frac{\partial u}{\partial x} + \frac{3}{2}\eta(\nabla \cdot \vec{v}) & -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & -2\eta\frac{\partial v}{\partial y} + \frac{3}{2}\eta(\nabla \cdot \vec{v}) \end{pmatrix}$$

The equations are to be simplified for:
1.) steady flows $(\frac{\partial}{\partial t} = 0)$
2.) like 1.) and inkompressible flows $(\rho =$ const$)$
3.) like 2.) and with constant viscosity $(\eta =$ const$)$
4.) like 3.) and frictionless flows $(\eta = 0)$

example

continuity equation

1. $\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\nabla \cdot \vec{v}) = 0 \Rightarrow \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} = 0$

   $\Rightarrow \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \;\leftrightarrow\; \nabla \cdot (\rho\vec{v}) = 0$

2. (=3. =4.) $\nabla \cdot (\rho\vec{v}) = 0 \;\Rightarrow\; \rho\nabla \cdot \vec{v} = 0 \Leftrightarrow \nabla \cdot \vec{v} = 0 \leftrightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

example

Navier-Stokes / momentum equations

1. $\nabla \cdot (\rho \vec{v}\vec{v}) = -\nabla p - \nabla \overline{\overline{\tau}} + \rho \vec{g}$

2. $\nabla \cdot (\rho \vec{v}\vec{v}) = \rho \nabla \cdot (\vec{v}\vec{v}) = \rho \nabla \cdot \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix} = \rho \begin{pmatrix} \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \\ \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} \end{pmatrix}$

$$=\rho \begin{pmatrix} u(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ v(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{pmatrix} = \rho(\vec{v} \cdot \nabla) \cdot \vec{v}$$

with $\overline{\overline{\tau}} = \begin{pmatrix} -2\eta\frac{\partial u}{\partial x} & -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ -\eta(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) & -2\eta\frac{\partial v}{\partial y} \end{pmatrix}$

example

3. $-\nabla \cdot \overline{\overline{\tau}} = \eta \begin{pmatrix} 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ \frac{\partial}{\partial x}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + 2\frac{\partial^2 v}{\partial y^2} \end{pmatrix} = \eta \begin{pmatrix} 2\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} \end{pmatrix}$

$$= \eta \begin{pmatrix} \frac{\partial}{\partial x}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\ \frac{\partial}{\partial y}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \end{pmatrix} = \eta \nabla^2 \cdot \vec{v} \quad (= \eta \Delta \cdot \vec{v})$$

$\Rightarrow$ Navier-Stokes equations: $\rho(\vec{v} \cdot \nabla) \cdot \vec{v} = -\nabla p + \eta \nabla^2 \cdot \vec{v} + \rho \vec{g}$

4. $\eta = 0 \Rightarrow$ Euler equations: $\rho(\vec{v} \cdot \nabla) \cdot \vec{v} = -\nabla p + \rho \vec{g}$

Basic equations, operators

The Navier-Stokes equations for incompressible fluids are given:

$$\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\,\vec{v}\right) = \rho\vec{g} - \nabla p + \eta\nabla^2\vec{v} \quad (1)$$

1. What is the physical meaning of the terms? What is the difference between the terms of the left hand side?
2. Is

$$\rho\frac{d\vec{v}}{dt} = \rho\vec{g} - \nabla p + \eta\nabla^2\vec{v} \quad (2)$$

equivalent zu (1)? Give reasons for your answer.

Basic equations, operators

3. Write the following terms in components for the 3d cartesian space:

(a) ∇p

(b) $(\vec{v} \cdot \nabla)\, \vec{v}$

(c) $\nabla \cdot (\vec{v}\vec{v})$

(d) $\nabla^2 \vec{v}$

(e) $\nabla \vec{v}$

4. Proof that the following expression is equivalent to (1):

$$\rho \left(\frac{\partial \vec{v}}{\partial t} + \nabla \cdot (\vec{v}\vec{v}) \right) = \rho \vec{g} - \nabla p + \eta \nabla^2 \vec{v} \qquad (3)$$

5. Proof the following identitites:

(a) rot(grad $\phi) = \vec{0}$

(b) div(rot $\vec{\phi}) = 0$

(c) $(\vec{v} \cdot \nabla)\, \vec{v} = \vec{v} \cdot (\nabla \vec{v})$

Basic equations, operators

1.)

$\varrho\frac{\partial\vec{v}}{\partial t}$: local acceleration in an unsteady flow

$\varrho(\vec{v}\cdot\nabla)\vec{v}$: convective acceleration

$\varrho\vec{g}$: gravitation, volume forces

$-\nabla p$: pressure force

$\eta\nabla^2\vec{v}$: friction term in an incompressible flow with constant η

2.)

Yes, since

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t} + (\vec{v}\cdot\nabla)\Phi$$

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial x}\frac{dx}{dt} + \frac{\partial\Phi}{\partial y}\frac{dy}{dt} + \frac{\partial\Phi}{\partial z}\frac{dz}{dt} \qquad \frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}$$

Basic equations, operators

3.)

a)

$$\nabla p = \text{grad} p = \begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \\ \partial p/\partial z \end{pmatrix}$$

b)

$$(\vec{v}\cdot\nabla)\vec{v} = (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z})\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$= \begin{pmatrix} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{pmatrix}$$

∇ is an operator, not a vector: $(\vec{v}\cdot\nabla) \neq \nabla\cdot\vec{v}$

Basic equations, operators

c)

$$\nabla \cdot (\vec{v}\vec{v}) = \text{div}(\vec{v}\vec{v}^T)$$
$$= \nabla \cdot \begin{pmatrix} u^2 & uv & uw \\ uv & v^2 & vw \\ uw & vw & w^2 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} \\ \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} \\ \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} \end{pmatrix}$$

Basic equations, operators

d)

$$\nabla^2 \vec{v} = \Delta \vec{v} \qquad \text{LaPlace-operator}$$

onedimensional (scalar)

$$\nabla^2 \Phi = \Delta \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

vector

$$\nabla^2 \vec{v} = \begin{pmatrix} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \end{pmatrix}$$

e)

$$\nabla \vec{v} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{pmatrix}$$

Basic equations, operators

4) (1-3): $(\vec{v}\cdot\nabla)\vec{v} - \nabla\cdot(\vec{v}\vec{v}) = 0$

$$\begin{pmatrix} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{pmatrix} - \begin{pmatrix} \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} \\ \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} \\ \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + u\frac{\partial w}{\partial z} \\ v\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + v\frac{\partial w}{\partial z} \\ w\frac{\partial u}{\partial x} + w\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z} \end{pmatrix} = \vec{v}(\nabla\cdot\vec{v}) = 0$$

conti

Basic equations, operators

5)

$$\text{rot}(\text{grad}(\Phi)) = \nabla \times (\nabla\Phi) = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} \partial\Phi/\partial x \\ \partial\Phi/\partial y \\ \partial\Phi/\partial z \end{pmatrix} = \begin{pmatrix} \frac{\partial^2\Phi}{\partial y\partial z} - \frac{\partial^2\Phi}{\partial z\partial y} \\ \frac{\partial^2\Phi}{\partial x\partial z} - \frac{\partial^2\Phi}{\partial z\partial x} \\ \frac{\partial^2\Phi}{\partial y\partial x} - \frac{\partial^2\Phi}{\partial x\partial y} \end{pmatrix} = \vec{0}$$

$$\text{div}(\text{rot}\vec{\Phi}) = \nabla \cdot (\nabla \times \vec{\Phi}) = \nabla \cdot \begin{pmatrix} \frac{\partial\Phi_z}{\partial y} - \frac{\partial\Phi_y}{\partial z} \\ \frac{\partial\Phi_x}{\partial z} - \frac{\partial\Phi_z}{\partial x} \\ \frac{\partial\Phi_y}{\partial x} - \frac{\partial\Phi_x}{\partial y} \end{pmatrix}$$

$$= \frac{\partial}{\partial x}(\frac{\partial\Phi_z}{\partial y} - \frac{\partial\Phi_y}{\partial z}) + \frac{\partial}{\partial y}(\frac{\partial\Phi_x}{\partial z} - \frac{\partial\Phi_z}{\partial x}) + \frac{\partial}{\partial z}(\frac{\partial\Phi_y}{\partial x} - \frac{\partial\Phi_x}{\partial y}) = \vec{0}$$

Basic equations, operators

5c)

$$(\vec{v}\cdot\nabla)\,\vec{v} = \vec{v}\cdot(\nabla\vec{v}) = \left((u\,v\,w)\cdot\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}\end{pmatrix}\right)^T =$$

$$\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}\end{pmatrix}\cdot(u\,v\,w) = \begin{pmatrix}u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\\ u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\\ u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\end{pmatrix} = \vec{v}\cdot(\nabla\vec{v})$$