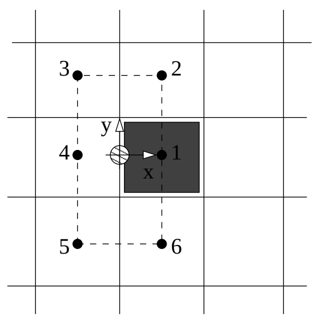
Computational Fluid Dynamics I

Exercise 11

1. The Laplace equation $\nabla^2 u = 0$ is discretized on a Cartesian grid, where the variables are stored at the cell centers. The discretization is carried out with a finite volume method, the values on the surface of the cell are reconstructed with the assumption of a linear function, i.e., applying a first-order Taylor-series expansion around the surface centroid located at $(0,0)$:

$$\begin{aligned} u(x,y) &= u(0,0) + u_x(0,0)x + u_y(0,0)y \\ &= a_0 + a_1 x + a_2 y \end{aligned}$$

For the reconstruction on the cell surface the cell centered values of points 1-6, see Figure, are used.



This yields a overdetermined, 6×3 linear equation system

$$\underline{A} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ \vdots & \vdots & \vdots \\ 1 & x_6 & y_6 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{pmatrix}$$

The constants a_0, a_1, a_2 can be determined by a least-squares approach. Thereby, the constants are chosen such that the sum of squared errors, $\sum_i (a_0 + a_1 x_i + a_2 y_i - u_i)^2$, is minimal. This is achieved by solving

$$\underline{A}^T \underline{A} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \underline{A}^T \begin{pmatrix} u_1 \\ \vdots \\ u_6 \end{pmatrix}$$

which yields the 3×3 system

$$\begin{pmatrix} n & \sum x_i & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i \\ \sum y_i & \sum x_i y_i & \sum y_i^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum u_i \\ \sum u_i \cdot x_i \\ \sum u_i \cdot y_i \end{pmatrix}$$

Determine the truncation error of the finite volume method.

Computational Fluid Dynamics I

Exercise 11 (solution)

1. From

$$\begin{pmatrix} n & \sum x_i & \sum y_i \\ \sum x_i & \sum x_i^2 & \sum x_i y_i \\ \sum y_i & \sum x_i y_i & \sum y_i^2 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum u_i \\ \sum u_i \cdot x_i \\ \sum u_i \cdot y_i \end{pmatrix}$$

For an equidistant grid (n = number of points):

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & \frac{3}{2}\Delta x^2 & 0 \\ 0 & 0 & 4\Delta y^2 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum u_i \\ \sum u_i \cdot x_i \\ \sum u_i \cdot y_i \end{pmatrix}$$

$$\begin{aligned} \Rightarrow \quad a_0 &= \frac{1}{6}(u_1 + u_2 + u_3 + u_4 + u_5 + u_6) \\ a_1 &= \frac{2}{3\Delta x^2} \cdot \frac{\Delta x}{2}(u_2 + u_1 + u_6 - u_3 - u_4 - u_5) = \frac{1}{3\Delta x}(u_2 + u_1 + u_6 - u_3 - u_4 - u_5) \\ a_2 &= \frac{1}{4\Delta y^2} \cdot \Delta y(u_3 + u_2 - u_5 - u_6) = \frac{1}{4\Delta y}(u_3 + u_2 - u_5 - u_6) \end{aligned}$$

From $u(x, y) = a_0 + a_1 x + a_2 y$

$$\nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Using Gauß' theorem for $\nabla^2 u = 0$

$$\int_\tau \nabla \cdot \vec{f}\, d\tau = \oint_A \vec{f} \cdot \vec{n}\, dA = \oint_A \begin{pmatrix} u_x \\ u_y \end{pmatrix} \cdot \vec{n}\, dA = 0 \qquad \text{with} \qquad \vec{f} = \nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

Compute $\int \begin{pmatrix} u_x \\ u_y \end{pmatrix} \cdot \vec{n}\, dA$:

normal vector $\vec{n}$ for side 1 to 4:

side 1: $\vec{n} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ side 2: $\vec{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ side 3: $\vec{n} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ side 4: $\vec{n} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{aligned} \int \begin{pmatrix} u_x \\ u_y \end{pmatrix} \cdot \vec{n}\, dA &= -a_1(\text{side } 1) \cdot \Delta y + a_1(\text{side } 2) \cdot \Delta y - a_2(\text{side } 3) \cdot \Delta x + a_2(\text{side } 4) \cdot \Delta x = 0 \\ &= (u_{xx} + u_{yy})\, \Delta x \Delta y \end{aligned}$$

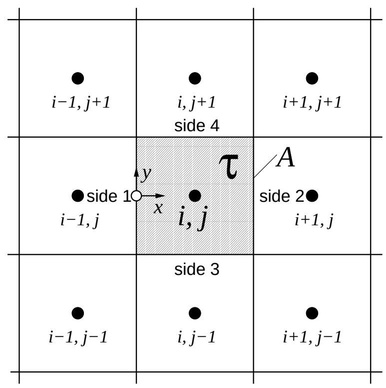
Use u_{xx} to evaluate the truncation error (same procedere for u_{yy}):

$$u_{xx} = \frac{a_1(\text{side } 2) - a_1(\text{side } 1)}{\Delta x}$$

$$a_1(\text{side } 1) = \frac{1}{3\Delta x}(u_{i,j+1} + u_{i,j} + u_{i,j-1} - u_{i-1,j+1} - u_{i-1,j} - u_{i-1,j-1})$$

$$a_1(\text{side } 2) = \frac{1}{3\Delta x}(u_{i+1,j+1} + u_{i+1,j} + u_{i+1,j-1} - u_{i,j+1} - u_{i,j} - u_{i,j-1})$$

$$= \frac{1}{3\Delta x^2}(u_{i+1,j+1} + u_{i+1,j} + u_{i+1,j-1} - 2u_{i,j+1} - 2u_{i,j} - 2u_{i,j-1} + \\ + u_{i-1,j+1} + u_{i-1,j} + u_{i-1,j-1}) \ (*)$$



Taylor series (multidimensional):

$$f(x,y) = \sum_{\substack{s=0\\t=0}}^{\infty} \frac{1}{s! \cdot t!} \cdot \frac{\partial^{s+t} f}{\partial x^s \, \partial y^t} \, (x - x_0)^s \, (y - y_0)^t$$

Here for $u_{i+1,j+1}$ (similar for the remaining terms...):

$$u(x + \Delta x, y + \Delta y) = u(x,y) + \Delta x \cdot u_x + \Delta y \cdot u_y + \Delta x \Delta y \cdot u_{xy} + \frac{\Delta x^2}{2} \cdot u_{xx} + \frac{\Delta y^2}{2} \cdot u_{yy} \\ + \frac{\Delta x^2 \Delta y}{2} \cdot u_{xxy} + \frac{\Delta y^2 \Delta x}{2} \cdot u_{xyy} + \frac{\Delta x^3}{6} \cdot u_{xxx} + \frac{\Delta y^3}{6} \cdot u_{yyy} \\ + \frac{\Delta x^3 \Delta y}{6} \cdot u_{xxxy} + \frac{\Delta y^3 \Delta x}{6} \cdot u_{xyyy} + \frac{\Delta x^2 \Delta y^2}{4} \cdot u_{xxyy} \\ + \frac{\Delta x^4}{24} \cdot u_{xxxx} + \frac{\Delta y^4}{24} \cdot u_{yyyy} + ...$$

Inserting Taylor series in $(*)$ yields:

$$u_{xx} = \frac{1}{3\Delta x^2}\left(3\Delta x^2 \cdot u_{xx} + \Delta x^2 \Delta y^2 \cdot u_{xxyy} + \frac{\Delta x^4}{4} \cdot u_{xxxx}\right)$$

$$= \left(u_{xx} + \frac{\Delta y^2}{3} \cdot u_{xxyy} + \frac{\Delta x^2}{12} \cdot u_{xxxx}\right)$$

$\Rightarrow$ Truncation error for u_{xx}: $\tau = \mathcal{O}(\Delta x^2, \Delta y^2)$

Similar for u_{yy}, therefore truncation error $\tau = \mathcal{O}(\Delta x^2, \Delta y^2)$