Computational Fluid Dynamics I

Exercise 10

1. The Laplace equation

$$\nabla \cdot \vec{f} = 0 \quad , \qquad \text{with} \quad \vec{f} = \nabla u$$

   is to be solved on a curvilinear structured grid.

   (a) Transform the equation for $\vec{f}$ into curvilinear coordinates $(x, y) \to (\xi, \eta)$ (conservative form) and discretize the equation for an equidistant grid in curvilinear space.

   (b) Formulate a discretization based on a finite volume method for the solution of the equation for $\vec{f}$. Reformulate the equation as a surface integral, define a meaningful control volume and discretize the equation.

   (c) Show that the formulation obtained with the transformation in cuvilinear coordinates is identical to the finite volume formulation.

Computational Fluid Dynamics I

Exercise 10 (solution)

1. (a)

$$\nabla \cdot \vec{f} = 0 \qquad \vec{f} = \nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

$$(x, y) \Rightarrow (\xi, \eta) \qquad \nabla \cdot \vec{f} = g_x + h_y = 0$$

with

$$\begin{aligned} g_x &= \xi_x g_\xi + \eta_x g_\eta \\ h_y &= \xi_y h_\xi + \eta_y h_\eta \end{aligned}$$

follows for the terms in the square brackets

$$\begin{aligned} \xi_x g_\xi + \eta_x g_\eta + \xi_y h_\xi + \eta_y h_\eta &= 0 \qquad | \cdot J \\ J\xi_x g_\xi + J\eta_x g_\eta + J\xi_y h_\xi + J\eta_y h_\eta &= 0 \end{aligned}$$

product rule

$$\frac{\partial}{\partial \xi}\left(J\xi_x g + J\xi_y h\right) + \frac{\partial}{\partial \eta}\left(J\eta_x g + J\eta_y h\right) - g\left[\frac{\partial}{\partial \xi}(J\xi_x) + \frac{\partial}{\partial \eta}(J\eta_x)\right] - h\left[\frac{\partial}{\partial \xi}(J\xi_y) + \frac{\partial}{\partial \eta}(J\eta_y)\right] = 0$$

with metric terms

$$\xi_x = \frac{y_\eta}{J} \qquad \xi_y = -\frac{x_\eta}{J} \qquad \eta_x = -\frac{y_\xi}{J} \qquad \eta_y = \frac{x_\xi}{J}$$

follows

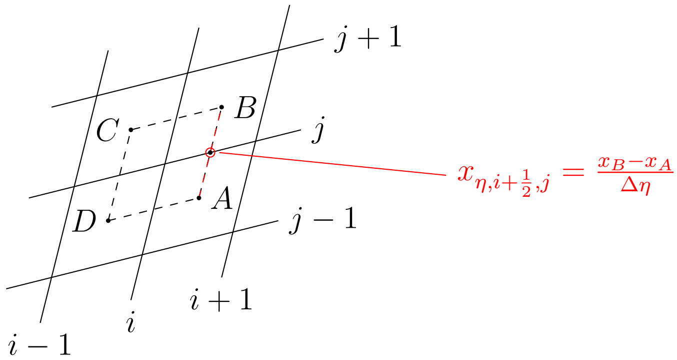
$$\begin{aligned} \frac{\partial}{\partial \xi}(J\xi_x) + \frac{\partial}{\partial \eta}(J\eta_x) &= +\frac{\partial}{\partial \xi} y_\eta - \frac{\partial}{\partial \eta} y_\xi = 0 \\ \frac{\partial}{\partial \xi}(J\xi_y) + \frac{\partial}{\partial \eta}(J\eta_y) &= -\frac{\partial}{\partial \xi} x_\eta + \frac{\partial}{\partial \eta} x_\xi = 0 \end{aligned}$$

final formulation in curvilinear coordinates

$$[J(\xi_x g + \xi_y h)]_\xi + [J(\eta_x g + \eta_y h)]_\eta = (y_\eta g - x_\eta h)_\xi + (-y_\xi g + x_\xi h)_\eta = 0$$

discretisation

$$(y_\eta g - x_\eta h)_{i+\frac{1}{2},j} - (y_\eta g - x_\eta h)_{i-\frac{1}{2},j} + (-y_\xi g + x_\xi h)_{i,j+\frac{1}{2}} - (-y_\xi g + x_\xi h)_{i,j-\frac{1}{2}} = 0$$



procedure for the computation (example) for an element

$$y_\eta g = y_\eta u_x \qquad \rightarrow \qquad (y_\eta u_x)_{i+\frac{1}{2},j} = (y_\eta)_{i+\frac{1}{2},j} \cdot (\xi_x u_\xi + \eta_x u_\eta)_{i+\frac{1}{2},j}$$

For this we need the metric terms at the point $i+\frac{1}{2}, j$, we can compute these for example by second-order accurate central differences (other formulations possible)

$$y_{\eta,i+\frac{1}{2},j} = \frac{y_B - y_A}{\Delta\eta} = \frac{y_B - y_A}{1}$$

where y_A and y_B are the averages of the surrounding 4 grid points

$$y_A = \frac{1}{4}\left(y_{i,j} + y_{i+1,j} + y_{i,j-1} + y_{i+1,j-1}\right)$$
$$y_B = \frac{1}{4}\left(y_{i,j} + y_{i+1,j} + y_{i,j+1} + y_{i+1,j+1}\right)$$

The other metric terms, e.g., ξ_x,η_x, etc, can also be first transformed to the inverse metric terms and then be discretized at $i+\frac{1}{2}, j$ in a similar manner. The terms u_ξ and u_η can be computed as simple central differences on the computational mesh, e.g.

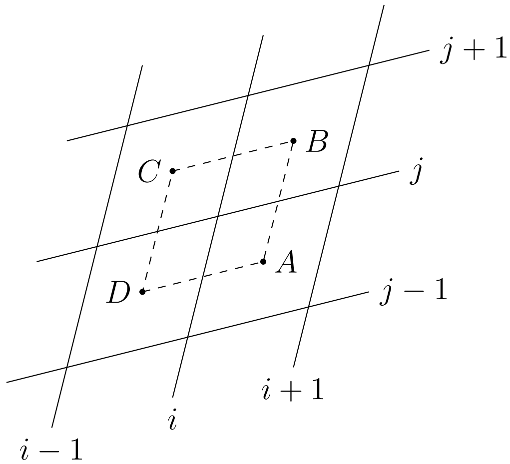
$$u_{\xi,i+\frac{1}{2},j} = \frac{u_{i+1,j} - u_{i,j}}{1}$$

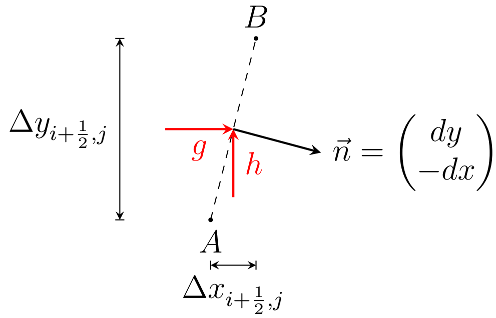
(b) finite volume formulation

$$\int_\tau \nabla \cdot \vec{f} d\tau = \oint_A \vec{f} \cdot \vec{n} dA \qquad \vec{f} = \begin{pmatrix} g \\ h \end{pmatrix} \qquad \vec{n} dA = \begin{pmatrix} dy \\ -dx \end{pmatrix}$$

$$\rightarrow \qquad \oint_A g dy - h dx = 0$$

Possible discretization with node-centered formulation (for mathematical positive direction)





$$
\begin{aligned}
(g\Delta y)_{i+\frac{1}{2},j} &- (h\Delta x)_{i+\frac{1}{2},j} + (g\Delta y)_{i,j+\frac{1}{2}} - (h\Delta x)_{i,j+\frac{1}{2}} \\
+(g\Delta y)_{i-\frac{1}{2},j} &- (h\Delta x)_{i-\frac{1}{2},j} + (g\Delta y)_{i,j-\frac{1}{2}} - (h\Delta x)_{i,j-\frac{1}{2}} = 0
\end{aligned}
$$

where the corresponding signs (+ for flux entering the volume, − for flux leaving the volume) are contained in the Δ terms:

$$
\begin{aligned}
\Delta x_{i+\frac{1}{2},j} &= x_B - x_A & \Delta y_{i+\frac{1}{2},j} &= y_B - y_A \\
\Delta x_{i-\frac{1}{2},j} &= x_D - x_C & \Delta y_{i-\frac{1}{2},j} &= y_D - y_C \\
\Delta x_{i,j+\frac{1}{2}} &= x_C - x_B & \Delta y_{i,j+\frac{1}{2}} &= y_C - y_B \\
\Delta x_{i,j-\frac{1}{2}} &= x_A - x_D & \Delta y_{i,j-\frac{1}{2}} &= y_A - y_D
\end{aligned}
$$

give the surface over which the flux is integrated and the correct sign. The coordinates at points A,B,C, and D are computed by averages of the surrouding four grid points, as shown before.

(c) curvilinear form

$$
\begin{aligned}
&(y_\eta \cdot g)_{i+\frac{1}{2},j} \quad -(x_\eta \cdot h)_{i+\frac{1}{2},j} \quad -(y_\eta \cdot g)_{i-\frac{1}{2},j} \quad +(x_\eta \cdot h)_{i-\frac{1}{2},j} \\
&-(y_\xi \cdot g)_{i,j+\frac{1}{2}} \quad +(x_\xi \cdot h)_{i,j+\frac{1}{2}} \quad +(y_\xi \cdot g)_{i,j-\frac{1}{2}} \quad -(x_\xi \cdot h)_{i,j-\frac{1}{2}} \quad = 0
\end{aligned} \tag{1}
$$

finite volume formulation

$$
\begin{aligned}
&(\Delta y \cdot g)_{i+\frac{1}{2},j} \quad -(\Delta x \cdot h)_{i+\frac{1}{2},j} \quad +(\Delta y \cdot g)_{i-\frac{1}{2},j} \quad -(\Delta x \cdot h)_{i-\frac{1}{2},j} \\
&+(\Delta y \cdot g)_{i,j+\frac{1}{2}} \quad -(\Delta x \cdot h)_{i,j+\frac{1}{2}} \quad +(\Delta y \cdot g)_{i,j-\frac{1}{2}} \quad -(\Delta x \cdot h)_{i,j-\frac{1}{2}} \quad = 0
\end{aligned} \tag{2}
$$

the metric coefficients, e.g., x_η, y_ξ, etc, are then equal to the lengths from the finite volume approach Δx and Δy. For example for surface $i+\frac{1}{2}, j$ we have the metric terms

$$x_{\eta,i+\frac{1}{2},j} = \frac{x_B - x_A}{\Delta\eta} = \frac{\Delta x_{i+\frac{1}{2},j}}{1}$$

$$y_{\eta,i+\frac{1}{2},j} = \frac{y_B - y_A}{\Delta\eta} = \frac{\Delta y_{i+\frac{1}{2},j}}{1}$$

The opposite signs in eqs. 1 and 2 are caused by opposite signs in metric terms in comparison with the lengths, for example

$$-(y_\eta \cdot g)_{i-\frac{1}{2},j} = -\frac{y_C - y_D}{\Delta\eta}\,(g)_{i-\frac{1}{2},j} = \frac{y_D - y_C}{\Delta\eta}\,(g)_{i-\frac{1}{2},j} = \Delta y_{i-\frac{1}{2},j}(g)_{i-\frac{1}{2},j}$$

as we compute the metric terms going into positive ξ and η direction, but for the lengths in the finite volume approach we follow the surface in positive rotation direction, here counterclockwise.