Computational Fluid Dynamics I

Exercise 8

1. Formulate for the discretised Poisson equation

$$u_{i,j} - \Theta_x(u_{i-1,j} + u_{i+1,j}) - \Theta_y(u_{i,j-1} + u_{i,j+1}) \ = \ \delta^2 f_{i,j},$$

$$\Theta_x = \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}, \quad \Theta_y = \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}$$

   (a) the Jacobi–method
   (b) the method of Gauß–Seidel point iteration with overrelaxation
   (c) the method of Gauß–Seidel line iteration with overrelaxation

   Check the stability of these methods with the help of the von Neumann analysis.

Computational Fluid Dynamics I

Exercise 8 (solution)

1. (a) **Jacobi-method (ν is iteration counter):**

$$u_{i,j}^{\nu+1} = \Theta_x \left(u_{i-1,j}^{\nu} + u_{i+1,j}^{\nu}\right) + \Theta_y \left(u_{i,j-1}^{\nu} + u_{i,j+1}^{\nu}\right) + \delta^2 f_{i,j}$$

stability, approach: $u_{i,j}^{\nu} = u_{exact,i,j}^{\nu} + V^{\nu} e^{I\alpha i + I\beta j}$, where $u_{exact,i,j}$ is the exact solution of this equation, therefore

$$\begin{aligned} u_{exact,i,j}^{\nu+1} + V^{\nu+1} e^{I\alpha i + I\beta j} = \; & \Theta_x \; \left(u_{exact,i-1,j}^{\nu} + V^{\nu} e^{I\alpha(i-1)+I\beta j} + u_{exact,i+1,j}^{\nu} + V^{\nu} e^{I\alpha(i+1)+I\beta j}\right) + \\ & \Theta_y \; \left(u_{exact,i,j-1}^{\nu} + V^{\nu} e^{I\alpha i + I\beta(j-1)} + u_{exact,i,j+1}^{\nu} + V^{\nu} e^{I\alpha i + I\beta(j+1)}\right) + \delta^2 f_{i,j} \end{aligned}$$

where for the given definitions of Θ_x and Θ_y the terms $u_{exact,i,j}^{\nu}$ and $\delta^2 f_{i,j}$ fulfill the original FDE and thus falls out, dividing by $V^{\nu} e^{I\alpha i + I\beta j}$ then yields:

$$G = \frac{V^{\nu+1}}{V^{\nu}} = \Theta_x \left(e^{-I\alpha} + e^{I\alpha}\right) + \Theta_y \left(e^{-I\beta} + e^{I\beta}\right) = 2\left(\Theta_x \cos(\alpha) + \Theta_y \cos(\beta)\right)$$

with $\Theta_x = \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}$, $\Theta_y = \frac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}$ and $-\pi \leq \alpha \leq \pi, \; -\pi \leq \beta \leq \pi$
consider two cases:

$$\begin{aligned} 2\left(\Theta_x \cos(\alpha) + \Theta_y \cos(\beta)\right) \leq 2\left(\Theta_x + \Theta_y\right) &= 2\left(\frac{\Delta x^2 + \Delta y^2}{2\left(\Delta x^2 + \Delta y^2\right)}\right) = 1 \\ 2\left(\Theta_x \cos(\alpha) + \Theta_y \cos(\beta)\right) \geq 2\left(-\Theta_x - \Theta_y\right) &= 2\left(\frac{-\Delta x^2 - \Delta y^2}{2\left(\Delta x^2 + \Delta y^2\right)}\right) = -1 \end{aligned}$$

$$-1 \leq G \leq 1$$

Thus the Jacobi–method is stable.

(b) **Gauß–Seidel point iteration with overrelaxation**
($\tilde{u}$ is intermediate value):

$$\tilde{u}_{i,j} - \Theta_x \left(u_{i-1,j}^{\nu+1} + u_{i+1,j}^{\nu}\right) - \Theta_y \left(u_{i,j-1}^{\nu+1} + u_{i,j+1}^{\nu}\right) = \delta^2 f_{i,j}$$

$$u_{i,j}^{\nu+1} = u_{i,j}^{\nu} + \omega\left(\tilde{u}_{i,j} - u_{i,j}^{\nu}\right)$$

or

$$u_{i,j}^{\nu+1} = u_{i,j}^{\nu} + \omega\left(\Theta_x \left(u_{i-1,j}^{\nu+1} + u_{i+1,j}^{\nu}\right) + \Theta_y \left(u_{i,j-1}^{\nu+1} + u_{i,j+1}^{\nu}\right) + \delta^2 f_{i,j} - u_{i,j}^{\nu}\right)$$

with $\omega > 0$ and the order of calculation $i = 1, \cdots, im$ and $j = 1, \cdots, jm$ for $u_{i,j}^{\nu+1}$

stability, approach see above:

$$\frac{V^{\nu+1}}{V^{\nu}} = 1 + \omega\left(\Theta_x\left(\frac{V^{\nu+1}}{V^{\nu}}e^{-I\alpha} + e^{I\alpha}\right) + \Theta_y\left(\frac{V^{\nu+1}}{V^{\nu}}e^{-I\beta} + e^{I\beta}\right) - 1\right)$$

with $c = \Theta_x \cos(\alpha) + \Theta_y \cos(\beta)$ and $s = \Theta_x \sin(\alpha) + \Theta_y \sin(\beta)$

$$\Leftrightarrow G = \frac{V^{\nu+1}}{V^{\nu}} = \frac{\frac{1}{\omega} - 1 + c + I\ s}{\frac{1}{\omega} - c + I\ s}$$

$$\Rightarrow |G|^2 = \frac{\left(\frac{1}{\omega} - 1 + c\right)^2 + s^2}{\left(\frac{1}{\omega} - c\right)^2 + s^2} = \frac{\frac{1}{\omega^2} - \frac{2}{\omega} + 1 + \frac{2}{\omega}c - 2c + c^2 + s^2}{\frac{1}{\omega^2} - \frac{2}{\omega}c + c^2 + s^2}$$

with the condition $|G|^2 \leq 1$:

$$\Rightarrow 1 - \frac{2}{\omega} + \frac{4}{\omega}c - 2c \leq 0 \quad \Leftrightarrow \quad \omega - 2 - 2c(\omega - 2) \leq 0$$

$$\Leftrightarrow (1 - 2c)(\omega - 2) \leq 0$$

because of $\Theta_x + \Theta_y = \frac{1}{2}$ the value of c is between $-\frac{1}{2} \leq c \leq \frac{1}{2}$, therefore the expression in the first bracket is $0 \leq 1 - 2c \leq 2$, consider the adverse case $(1 - 2c) = 2$, then

$$\Rightarrow \omega \leq 2$$

Thus the Gauß–Seidel point iteration with overrelaxation is stable for $0 < \omega \leq 2$.

(c) **Gauß–Seidel line iteration with overrelaxation:**

$$-\Theta_x \tilde{u}_{i-1,j} + \tilde{u}_{i,j} - \Theta_x \tilde{u}_{i+1,j} = \Theta_y \left(u^{\nu+1}_{i,j-1} + u^{\nu}_{i,j+1} \right) + \delta^2 f_{i,j}$$

$$u^{\nu+1}_{i,j} = u^{\nu}_{i,j} + \omega \left(\tilde{u}_{i,j} - u^{\nu}_{i,j} \right)$$

with $\omega > 0$ and a line iteration in i-direction and the order of calculation $j = 1, \cdots, jm$ for $u^{\nu+1}_{i,j}$.

stability, approach for $u^{\nu}_{i,j}$ see above, $\tilde{u}_{i,j} = u_{exact} + \tilde{V} e^{I\alpha i + I\beta j}$:

$$\Rightarrow -\Theta_x \tilde{V} e^{-I\alpha} + \tilde{V} - \Theta_x \tilde{V} e^{I\alpha} = \Theta_y \left(V^{\nu+1} e^{-I\beta} + V^{\nu} e^{I\beta} \right)$$

$$\frac{V^{\nu+1}}{V^{\nu}} = 1 + \omega \left(\frac{\tilde{V}}{V^{\nu}} - 1 \right)$$

$$\Leftrightarrow \frac{\tilde{V}}{V^{\nu}} \left(1 - \Theta_x \left(e^{-I\alpha} + e^{I\alpha} \right) \right) = \Theta_y \left(\frac{V^{\nu+1}}{V^{\nu}} e^{-I\beta} + e^{I\beta} \right)$$

$$\frac{\tilde{V}}{V^{\nu}} = \frac{1}{\omega} \left(\frac{V^{\nu+1}}{V^{\nu}} - 1 \right) + 1$$

$$\Leftrightarrow G = \frac{V^{\nu+1}}{V^{\nu}} = \frac{\left(\frac{1}{\omega} - 1 \right) \left(1 - 2\Theta_x \cos(\alpha) \right) + \Theta_y \cos(\beta) + I\Theta_y \sin(\beta)}{\frac{1}{\omega} \left(1 - 2\Theta_x \cos(\alpha) \right) - \Theta_y \cos(\beta) + I\Theta_y \sin(\beta)}$$

$$\Rightarrow |G|^2 = \frac{\left(\left(\frac{1}{\omega} - 1 \right) \left(1 - 2\Theta_x \cos(\alpha) \right) + \Theta_y \cos(\beta) \right)^2 + \Theta_y^2 \sin^2(\beta)}{\left(\frac{1}{\omega} \left(1 - 2\Theta_x \cos(\alpha) \right) - \Theta_y \cos(\beta) \right)^2 + \Theta_y^2 \sin^2(\beta)}$$

with the condition $|G|^2 \leq 1$ it follows:

$$\Rightarrow \left(\frac{1}{\omega^2} - \frac{2}{\omega} + 1 \right) \left(1 - 2\Theta_x \cos(\alpha) \right)^2 + \left(\frac{2}{\omega} - 2 \right) \left(1 - 2\Theta_x \cos(\alpha) \right) \Theta_y \cos(\beta)$$

$$\leq \frac{1}{\omega^2} \left(1 - 2\Theta_x \cos(\alpha) \right)^2 - \frac{2}{\omega} \left(1 - 2\Theta_x \cos(\alpha) \right) \Theta_y \cos(\beta)$$

with $c = \Theta_x \cos(\alpha) + \Theta_y \cos(\beta)$

$$\Rightarrow \left(\frac{2}{\omega} - 1 \right) \underbrace{\left(\underbrace{2\Theta_x}_{\leq 1} \cos(\alpha) - 1 \right)}_{\leq 0} \underbrace{(1 - 2c)}_{\geq 0} \leq 0$$

With $2\Theta_x \cos(\alpha) - 1 \leq 0$ and $0 \leq 1 - 2c \leq 2$, the expression in the first bracket has to be $\frac{2}{\omega} - 1 \geq 0$

Thus the Gauß–Seidel line iteration with overrelaxation is stable for $0 < \omega \leq 2$.