Computational Fluid Dynamics I

Exercise 7

1. Given is the PDE (convection-diffusion equation):

$$L(u) \; = \; u_t + a\,u_x - \nu u_{xx} \; = \; 0 \qquad \text{with} \quad a = const., \quad \nu = const. \geq 0$$

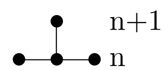
Check the convergence of the following generalised difference scheme with central differences:

$$L_\Delta(u) \; = \; \frac{u_i^{n+1} - u_i^n}{\Delta t} + (1-\Theta) Res_\Delta(u^n) + \Theta Res_\Delta(u^{n+1}) \; = \; 0$$

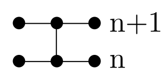
with $Res_\Delta(u) \; = \; \dfrac{a}{2\Delta x}\,(u_{i+1} - u_{i-1}) - \dfrac{\nu}{\Delta x^2}\,(u_{i+1} - 2u_i + u_{i-1})$

and the discretisation factor Θ:

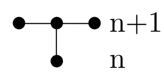
$\Theta = 0$ explicit scheme $\mathcal{O}\left(\Delta t, \Delta x^2\right)$



$\Theta = \frac{1}{2}$ implicit scheme $\mathcal{O}\left(\Delta t^2, \Delta x^2\right)$
(Crank-Nicholson)



$\Theta = 1$ implicit scheme $\mathcal{O}\left(\Delta t, \Delta x^2\right)$



Check with the general solution for $L_\Delta(u)$ the special cases
$\Theta = 0,\ \frac{1}{2},\ 1$ and the

convection-diffusion equation	:	$a \neq 0$	$\nu \neq 0$
convection equation	:	$a \neq 0$	$\nu = 0$
diffusion equation	:	$a = 0$	$\nu \neq 0$

Computational Fluid Dynamics I

Exercise 7 (solution)

1. (a) From Lax's theorem the convergence of a finite difference equation for an initial value problem requires consistency and stability.

consistency (see as well exercise 4):

separate checking time and space using Taylor series expansion in $x-$ and $t-$direction:

$$
\begin{aligned}
t: \quad & \frac{u^{n+1}-u^n}{\Delta t} &&= u_t|^n + \frac{\Delta t}{2}u_{tt}|^n + \frac{\Delta t^2}{6}u_{ttt}|^n + \dots \\
x: \quad & Res_\Delta(u) &&= au_x|_i - \nu u_{xx}|_i + a(\frac{\Delta x^2}{6}u_{xxx}|_i + \dots) - \nu(\frac{\Delta x^2}{12}u_{xxxx}|_i + \dots) \\
& &&= Res(u) + \mathcal{O}\left(\Delta x^2\right) \\
t: \quad & Res_\Delta(u^{n+1}) &&= Res_\Delta(u^n) + (Res_\Delta(u^n))_t|^n\Delta t + (Res_\Delta(u^n))_{tt}|^n\frac{\Delta t^2}{2} + \dots \\
& &&= Res(u^n) + (Res(u^n))_t|^n\Delta t + \mathcal{O}\left(\Delta t^2, \Delta x^2\right)
\end{aligned}
$$

apply to the difference scheme (with $u_t = -Res(u) \iff u_{tt} = -(Res(u))_t$):

$$u_t + Res(u) = (\Theta - \tfrac{1}{2})\Delta t u_{tt} + \mathcal{O}\left(\Delta t^2, \Delta x^2\right) \quad \Longrightarrow \text{consistent for } \Delta x, \Delta t \to 0$$

accuracy: $\mathcal{O}\left(\Delta t\ , \Delta x^2\right)$, if $\Theta \neq \frac{1}{2}$
$\mathcal{O}\left(\Delta t^2, \Delta x^2\right)$, if $\Theta = \frac{1}{2}$

stability: von Neumann analysis (approach see exercise 6):

$$
\begin{aligned}
&\frac{V^{n+1}e^{i\Phi I} - V^n e^{i\Phi I}}{\Delta t} \\
&\quad +(1-\Theta)V^n\left(\frac{a}{2\Delta x}\left(e^{(i+1)\Phi I} - e^{(i-1)\Phi I}\right) - \frac{\nu}{\Delta x^2}\left(e^{(i+1)\Phi I} - 2e^{i\Phi I} + e^{(i-1)\Phi I}\right)\right) \\
&\quad\quad +\Theta V^{n+1}\left(\frac{a}{2\Delta x}\left(e^{(i+1)\Phi I} - e^{(i-1)\Phi I}\right) - \frac{\nu}{\Delta x^2}\left(e^{(i+1)\Phi I} - 2e^{i\Phi I} + e^{(i-1)\Phi I}\right)\right) = 0
\end{aligned}
$$

with $c = \frac{a\Delta t}{\Delta x}$ and $\sigma = \frac{\nu \Delta t}{\Delta x^2}$ follows:

$$G = \frac{V^{n+1}}{V^n} = \frac{1 - (1-\Theta)(2\sigma(1-\cos(\Phi)) + cI\sin(\Phi))}{1 + \Theta(2\sigma(1-\cos(\Phi)) + cI\sin(\Phi))}$$

stability condition:

$$\Longrightarrow |G|^2 = \frac{\left(1-(1-\Theta)2\sigma(1-\cos(\Phi))\right)^2 + \left((1-\Theta)c\sin(\Phi)\right)^2}{\left(1+\Theta 2\sigma(1-\cos(\Phi))\right)^2 + \left(\Theta c\sin(\Phi)\right)^2} \leq 1$$

$$\iff (1-2\Theta)\underbrace{\left(4\sigma^2(1-\cos(\Phi))^2 + c^2\sin^2(\Phi)\right)}_{\geq 0} - \underbrace{4\sigma(1-\cos(\Phi))}_{\geq 0} \leq 0$$

$\Longrightarrow$ scheme is unconditionally stable for $\Theta \geq \frac{1}{2}$

analysis for $0 \leq \Theta < \frac{1}{2}$:

with $\sin^2(\Phi) = 1 - \cos^2(\Phi) = (1 + \cos(\Phi))(1 - \cos(\Phi))$:

$$\Longrightarrow \underbrace{(1-2\Theta)(c^2+4\sigma^2)}_{\geq 0} + \underbrace{(1-2\Theta)}_{>0}(c^2-4\sigma^2)\cos(\Phi) - \underbrace{4\sigma}_{\geq 0} \leq 0$$

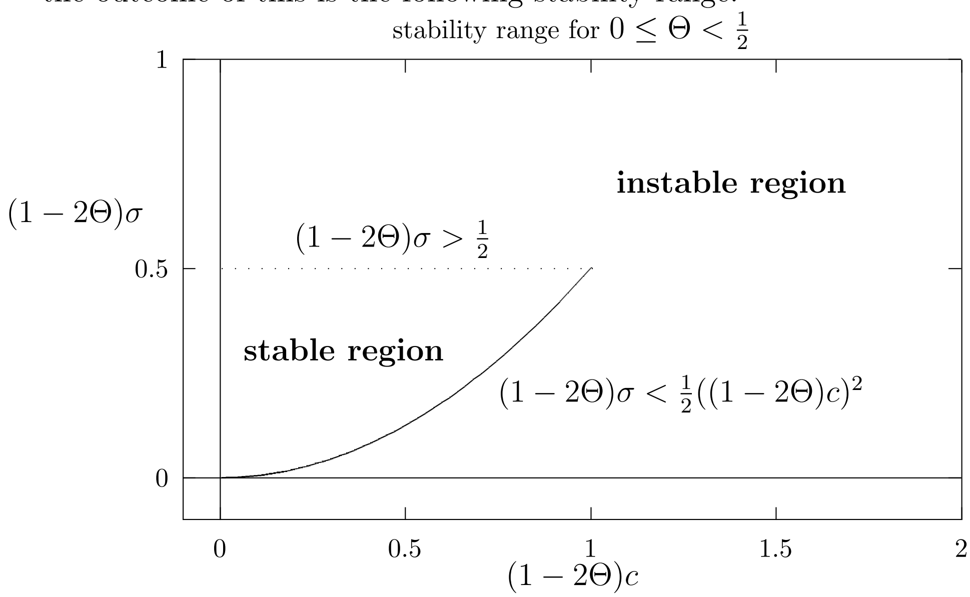
for $c^2 - 4\sigma^2 > 0$ is $\cos(\Phi) = 1$ the adverse case:

$$\Longrightarrow (1-2\Theta)c^2 \leq 2\sigma$$

for $c^2 - 4\sigma^2 \leq 0$ is $\cos(\Phi) = -1$ the adverse case:

$$\Longrightarrow (1-2\Theta)\sigma \leq \frac{1}{2}$$

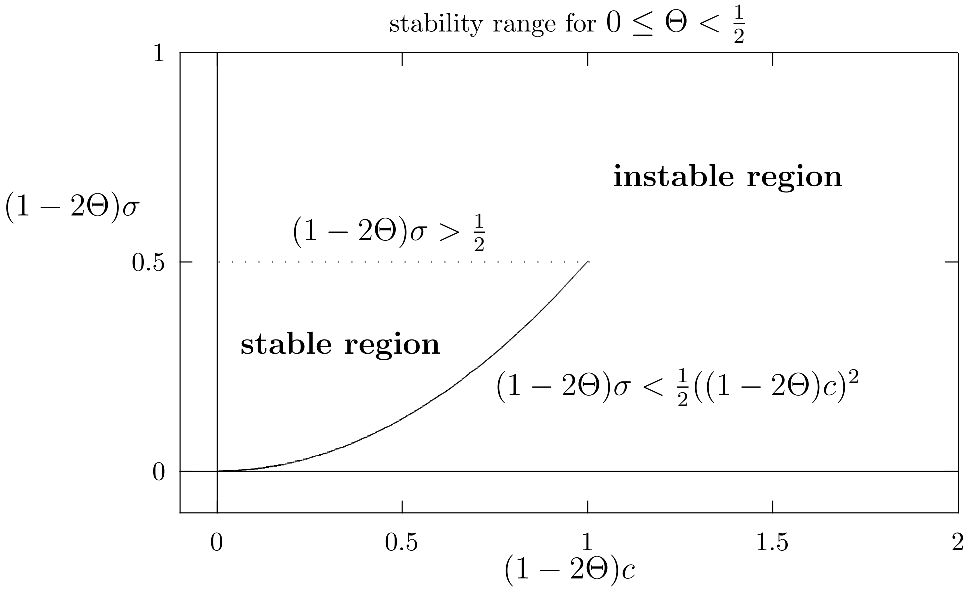
the outcome of this is the following stability range:



summary:

From consistency and stability follows convergence (theorem of Lax).

- consistency of the difference approximation is obtained for all parameters (Θ, ν, a)
- stability is obtained
  - ⋆ $\frac{1}{2} \leq \Theta \leq 1$ for all values of (ν, a)
  - ⋆ $0 \leq \Theta < \frac{1}{2}$ see diagram
- instability is obtained for
  - ⋆ $0 \leq \Theta < \frac{1}{2}$ for the pure convection equation ($\nu = 0$ or $\sigma = 0$)
  - ⋆ $0 \leq \Theta < \frac{1}{2}$ and $\sigma > \frac{1}{2(1-2\Theta)}$ for the pure diffusion equation ($a = 0$ or $c = 0$)



stability range for $0 \leq \Theta < \frac{1}{2}$
1
0.5
0
0
0.5
1
1.5
2
$(1-2\Theta)\sigma$
$(1-2\Theta)c$
instable region
$(1-2\Theta)\sigma > \frac{1}{2}$
stable region
$(1-2\Theta)\sigma < \frac{1}{2}((1-2\Theta)c)^2$