Computational Fluid Dynamics I

Exercise 6

1. For the convection equation

$$u_t + a\,u_x \;=\; 0, \quad a = const. \neq 0$$

the following general scheme will be used:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + a\,(1-\Theta)\,\delta_x u_i^n + a\,\Theta\,\delta_x u_i^{n+1} \;=\; 0$$

where $\delta_x u_i = \dfrac{u_{i+1} - u_{i-1}}{2\,\Delta x}$ and $0 \leq \Theta \leq 1$ ($\Theta = 0$: explicit scheme, $\Theta = 1$: implicit scheme).

(a) Show with the help of the analysis of Hirt for which values of the parameter Θ the scheme above will be stable.

(b) Check the result with the von Neumann analysis.

Computational Fluid Dynamics I

Exercise 6 (solution)

1. (a) **Hirt's analysis:**

$$\frac{u_i^{n+1}-u_i^n}{\Delta t} + a\,(1-\Theta)\,\delta_x u_i^n + a\,\Theta\,\delta_x u_i^{n+1} \;=\; 0$$

with

$$\delta_x u_i = \frac{u_{i+1}-u_{i-1}}{2\,\Delta x}$$

becomes

$$\Rightarrow \frac{u_i^{n+1}-u_i^n}{\Delta t} + a\,(1-\Theta)\,\frac{u_{i+1}^n-u_{i-1}^n}{2\,\Delta x} + a\,\Theta\,\frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2\,\Delta x} \;=\; 0$$

To recover the truncation error formulate a Taylor series expansion for the variable u:

$$\begin{aligned} u_i^{n+1} &= u_i^n + u_t|_i^n\,\Delta t + u_{tt}|_i^n\,\frac{\Delta t^2}{2} + u_{ttt}|_i^n\,\frac{\Delta t^3}{6} + \ldots \\ u_{i\pm1}^n &= u_i^n \pm u_x|_i^n\,\Delta x + u_{xx}|_i^n\,\frac{\Delta x^2}{2} + u_{xxx}|_i^n\,\frac{\Delta x^3}{6} + \ldots \end{aligned}$$

and rearrange to get expressions for finite difference expressions:

$$\begin{aligned} \frac{u^{n+1}-u^n}{\Delta t} &= u_t^n + \frac{\Delta t}{2}u_{tt}^n + \frac{\Delta t^2}{6}u_{ttt}^n + \cdots \\ \frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x} &= u_x^n + \frac{\Delta x^2}{6}u_{xxx}^n + \cdots \\ \frac{u_{i+1}^{n+1}-u_{i-1}^{n+1}}{2\Delta x} &= u_x^{n+1} + \frac{\Delta x^2}{6}u_{xxx}^{n+1} + \cdots \\ &= \underbrace{u_x^n + \Delta t u_{xt}^n + \frac{\Delta t^2}{2}u_{xtt}^n}_{\text{temporal Taylor series expansion for } u_x^{n+1}} + \cdots + O(\Delta x^2) \end{aligned}$$

follows

$$\begin{aligned} u_t^n + \frac{\Delta t}{2}u_{tt}^n + a\left((1-\Theta)\,u_x^n + \Theta\,(u_x^n + \Delta t u_{xt}^n)\right) + O(\Delta x^2,\Delta t^2) &= 0 \\ \Longrightarrow u_t^n + a u_x + \frac{\Delta t}{2}u_{tt}^n + a\Theta\Delta t u_{xt}^n + O(\Delta x^2,\Delta t^2) &= 0 \end{aligned}$$

Index n for the time layer will be omitted:

$$\Longrightarrow u_t + a u_x = -\frac{\Delta t}{2}u_{tt} - a\Theta\Delta t u_{xt} + O(\Delta x^2,\Delta t^2),$$

which is the **modified PDE**. Using the original PDE $u_t + au_x = 0$:

$$\Longrightarrow u_{tt} = -au_{xt}, \qquad u_{tx} = -au_{xx}, \qquad u_{tt} = a^2 u_{xx}$$

we can transform temporal derivatives to spatial ones:

$$\Longleftrightarrow u_t + au_x = \underbrace{a^2 \Delta t (\Theta - \frac{1}{2})}_{\text{numerical viscosity}} u_{xx} + O(\Delta x^2, \Delta t^2)$$

from the condition, that only a positive (numerical) viscosity has a damping or stabilizing effect, it follows:

$$\frac{1}{2} \leq \Theta \leq 1$$

(b) **von Neumann analysis:**

Approach for the error function ϵ:

$$\epsilon_i^n = \sum_{\phi=-\pi}^{\phi=\pi} V^n(\Phi)e^{i\Phi I} \ , \quad \Phi = \frac{2\pi\Delta x}{\lambda} \ , \quad t = n\Delta t \ , \quad I = \sqrt{-1}$$

Inserting the approach into the finite difference equation, omitting the sums and requiring the equation is satisfied for every discrete wave angle Φ (see script, p. 3-10 to 3-12) yields:

$$\frac{V^{n+1}e^{i\Phi I} - V^n e^{i\Phi I}}{\Delta t} + a(1-\Theta)\frac{V^n\left(e^{(i+1)\Phi I} - e^{(i-1)\Phi I}\right)}{2\Delta x} + a\Theta\frac{V^{n+1}\left(e^{(i+1)\Phi I} - e^{(i-1)\Phi I}\right)}{2\Delta x} = 0$$

$$\Longleftrightarrow \frac{\frac{V^{n+1}}{V^n} - 1}{\Delta t} + a(1-\Theta)\frac{\left(e^{\Phi I} - e^{-\Phi I}\right)}{2\Delta x} + a\Theta\frac{V^{n+1}}{V^n}\frac{\left(e^{\Phi I} - e^{-\Phi I}\right)}{2\Delta x} = 0$$

with $e^{\Phi I} = cos(\Phi) + Isin(\Phi)$ one receives a term for the amplification factor G:

$$\Longleftrightarrow G = \frac{V^{n+1}}{V^n} = \frac{1 - (1-\Theta)a\frac{\Delta t}{\Delta x}Isin(\Phi)}{1 + \Theta a\frac{\Delta t}{\Delta x}Isin(\Phi)}$$

The absolute value of a complex number is $\left|\frac{a+b*I}{c+d*I}\right| = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$:

$$\Longrightarrow |G|^2 = \frac{1 + (1 - 2\Theta + \Theta^2)\left(a\frac{\Delta t}{\Delta x}sin(\Phi)\right)^2}{1 + \left(\Theta a\frac{\Delta t}{\Delta x}sin(\Phi)\right)^2}$$

for a stable difference scheme it is required that: $|G|^2 \leq 1$ for $-\pi \leq \Phi \leq \pi$:

$$(1 - 2\Theta)sin^2(\Phi) \leq 0$$

according to the problem is $\Theta \leq 1$:

$$\Longrightarrow \frac{1}{2} \leq \Theta \leq 1$$