Computational Fluid Dynamics I

Exercise 5

1. The heat conduction equation is given:

$$T_t = \alpha T_{xx}, \quad \alpha = const. > 0$$

The equation is discretised with a 3-time level scheme (Dufort-Frankel scheme):

$$L_\Delta(T) = \frac{T_i^{n+1} - T_i^{n-1}}{2\Delta t} - \alpha \frac{T_{i+1}^n - (T_i^{n+1} + T_i^{n-1}) + T_{i-1}^n}{\Delta x^2} = 0$$

Check the consistency of this scheme.

2. Discretise the above equation with an explicit scheme. Check the stability of this scheme with

   (a) the discrete perturbation theory.

   (b) the help of a periodical test function,

$$T(x,t) = V(t)\, cos(kx) \quad \text{resp.} \quad T_i^n = V^n\, cos(\Theta i)$$

   with $t = n\Delta t$, $x = i\Delta x$, $\Theta = k\Delta x$, by analysing, whether the amplitude $V(t)$ is in- or decreasing with the time level.

   advice: $cos(\alpha \pm \beta) = cos\alpha\, cos\beta \mp sin\alpha\, sin\beta$

Computational Fluid Dynamics I

Exercise 5 (solution)

1. We are given a finite difference equation without the truncation error, it can be determined by developing Taylor series for time level "n" and location "i"

$$T^{n\pm1} = T^n \pm T_t^n \Delta t + T_{tt}^n \frac{\Delta t^2}{2} \pm T_{ttt}^n \frac{\Delta t^3}{6} + \textbf{t}\text{erms of }\textbf{h}\text{igher }\textbf{o}\text{rder}$$

$$\Longrightarrow \qquad \frac{T^{n+1} - T^{n-1}}{2\Delta t} = T_t^n + T_{ttt}^n \frac{\Delta t^2}{6} + \text{tho}$$

$$T^{n+1} + T^{n-1} = 2\,T^n + 2\,T_{tt}^n \frac{\Delta t^2}{2} + \text{tho}$$

apply in $L_\Delta(T)$:

$$T_{t}{}_i^n + T_{ttt}{}_i^n \frac{\Delta t^2}{6} + \ldots - \alpha \Bigg(\underbrace{\frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}}_{= T_{xxi} + T_{xxxxi}\frac{\Delta x^2}{12} + \ldots} - T_{tt}{}_i^n \frac{\Delta t^2}{\Delta x^2} + \ldots \Bigg) = 0$$

The truncation error of the spatial discretization $T_{xxi} + T_{xxxxi}\frac{\Delta x^2}{12} + \ldots$ can be either determined by knowledge (second-order accurate approximation of second-order derivative, see script pp. 3-3) or also via spatial Taylor series expansions. Finally, the **original PDE** (left hand side) and the **truncation error** (right hand side) is

recovered $\quad \Longrightarrow \quad (T_t - \alpha T_{xx})_i^n = \underbrace{-T_{ttt}\frac{\Delta t^2}{6} + \alpha T_{xxxx}\frac{\Delta x^2}{12} - \alpha T_{tt}\frac{\Delta t^2}{\Delta x^2} + \text{tho}}_{\tau}$

which together is the **modified PDE**.

<u>consistency:</u>

$\lim\limits_{\Delta x, \Delta t \to 0} \tau = 0$? $\qquad$ only fulfilled, if $\lim\limits_{\Delta x, \Delta t \to 0} \frac{\Delta t^2}{\Delta x^2} = 0$

i.e. Δt has to vanish faster than Δx

- for finite Δx, Δt choose: $\frac{\Delta t}{\Delta x} \ll 1$, e.g. $\frac{\Delta t}{\Delta x} = O(\Delta x)$
- irrelevant for steady solution, since in this case $T_t = T_{tt} = \ldots = 0$

2. discretisation:

$$\delta_t T = \frac{T_i^{n+1} - T_i^n}{\Delta t} \qquad \delta_{xx} T = \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2}$$

explicit scheme:

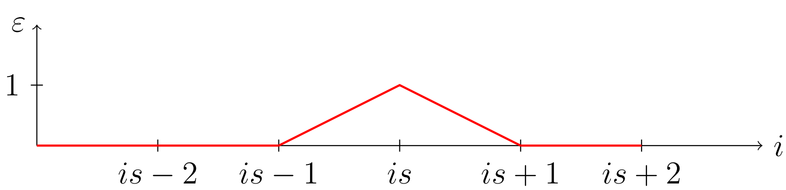
$$T_i^{n+1} = T_i^n + \sigma \left(T_{i+1}^n - 2T_i^n + T_{i-1}^n \right) \qquad \text{with} \quad \sigma = \frac{\alpha \Delta t}{\Delta x^2}$$

(a) **Discrete perturbation theory:** For linear equations a perturbation ε (error) satisfies the same difference equation as the solution, therefore

$$\varepsilon_i^{n+1} = \sigma \varepsilon_{i-1}^n + (1-2\sigma)\varepsilon_i^n + \sigma \varepsilon_{i+1}^n \qquad \text{mit} \quad \sigma = \nu \frac{\Delta t}{\Delta x^2} \qquad (*)$$

The analysis of the error behaviour yields the following results:

- Initial condition $n = 0$
  $\varepsilon_i^0 = \varepsilon$ für $i = is$ $\qquad \varepsilon_i^0 = 0$ für $i \neq is$



- Time step $n = 1$ compute solution of equation * with values from $n = 0$:

$$\begin{aligned} \varepsilon_{is}^1 &= \sigma \varepsilon_{is-1}^0 + (1-2\sigma)\varepsilon_{is}^0 + \sigma \varepsilon_{is+1}^0 = (1-2\sigma)\varepsilon \\ \varepsilon_{is+1}^1 &= \sigma \varepsilon_{is}^0 + (1-2\sigma)\varepsilon_{is+1}^0 + \sigma \varepsilon_{is+2}^0 = \sigma\varepsilon \\ \varepsilon_{is-1}^1 &= \sigma \varepsilon_{is-2}^0 + (1-2\sigma)\varepsilon_{is-1}^0 + \sigma \varepsilon_{is}^0 = \sigma\varepsilon \end{aligned}$$

  solution at all other points $i < is - 1$ and $i > is + 1$ is zero.
  from $\dfrac{\max|\varepsilon^1|}{\max|\varepsilon^0|} \leq 1$ folgt $|\sigma| \leq 1$ bzw. $|1 - 2\sigma| \leq 1$
  $\rightarrow \quad 0 < \sigma \leq 1$

Repeat procedure for following time steps (see script, p. 3-8ff), for $n \rightarrow \infty$ the asymptotical stability limit is $\underline{0 < \sigma \leq 1/2}$.

(a) **von Neumann stability analysis:**

A periodic error function

$$\begin{aligned} T_{i,j}^n &= V^n \cdot e^{Ik_x x} \\ &= V^n \cdot e^{Ik_x i\Delta x} \\ &= V^n \cdot e^{I\Theta i} \end{aligned}$$

is applied to the original PDE

$$T_i^{n+1} = T_i^n + \sigma\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)$$

such that

$$V^{n+1}e^{I\Theta i} = V^n e^{I\Theta i} + \sigma\left(V^n e^{I\Theta(i+1)} - 2V^n e^{I\Theta i} + V^n e^{I\Theta(i-1)}\right)$$

divide by $V^n e^{I\Theta i}$

$$\frac{V^{n+1}}{V^n} = 1 + \sigma\left(e^{I\Theta} - 2 + e^{-I\Theta}\right)$$

use $e^{\pm I\Theta} = \cos(\Theta) \pm I\sin(\Theta)$ and $G = \frac{V^{n+1}}{V^n}$

$$\begin{aligned} G &= 1 + \sigma\left(\cos(\Theta) + I\sin(\Theta) - 2 + \cos(\Theta) - I\sin(\Theta)\right) \\ G &= 1 - 2\sigma\left(1 - \cos(\Theta)\right) \end{aligned}$$

stable, if $\quad |G| \leq 1 \quad \rightarrow \quad -1 \leq G \leq 1 \quad \text{for} \quad -\pi \leq \Theta \leq \pi$

$$\Longrightarrow \qquad \sigma \leq \frac{1}{2} \qquad \text{resp.} \qquad \Delta t \leq \frac{\Delta x^2}{2\alpha}$$