Computational Fluid Dynamics I

Exercise 4

1. The vorticity transport equation for unsteady one-dimensional flow is given:

$$\omega_t + u\omega_x = \nu\omega_{xx}$$

The viscosity ν $(\nu > 0)$ and the velocity $u = u\,(x,t)$ are assumed to be known. The equation should discretised for constant time and spatial steps $\Delta t, \Delta x$:

$$x_i = i\Delta x, \quad t^n = n\Delta t, \quad \omega(x_i, t^n) = \omega_i^n$$

(a) Determine with the help of Taylor series:
- ω_t for t^n, resp. t^{n+1} (forward, resp. backward difference)
- ω_x and ω_{xx} around x_i (central differences)

(b) Formulate an explicit and an implicit solution scheme for the PDE and check the consistency.

Computational Fluid Dynamics I

Exercise 4 (solution)

1. (a) **Discretisation of the time derivative:**
Formulate Taylor series expansion for $\omega^{\mathbf{n+1}}$ around $\omega^{\mathbf{n}}$:

$$\omega_i^{n+1} = \omega_i^n + \omega_t|_i^n \, \Delta t + \omega_{tt}|_i^n \, \frac{\Delta t^2}{2!} \; + \; \ldots$$

and reformulate to get **forward difference:**

$$\omega_t|_i^n = \frac{\omega_i^{n+1} - \omega_i^n}{\Delta t} \; - \; \omega_{tt}|_i^n \, \frac{\Delta t}{2} \; + \; \ldots$$

Formulate Taylor series expansion for $\omega^{\mathbf{n}}$ around $\omega^{\mathbf{n+1}}$:

$$\omega_i^n = \omega_i^{n+1} - \omega_t|_i^{n+1} \, \Delta t + \omega_{tt}|_i^{n+1} \, \frac{\Delta t^2}{2!} \; + \; \ldots$$

and reformulate to get **backward difference:**

$$\omega_t|_i^{n+1} = \frac{\omega_i^{n+1} - \omega_i^n}{\Delta t} \; + \; \omega_{tt}|_i^{n+1} \, \frac{\Delta t}{2} \; + \; \ldots$$

Discretisation of the spatial derivative:
Formulate Taylor series expansion for ω_{i+1} and ω_{i-1} around ω_i:

$$\omega_{i\pm1}^n = \omega_i^n \pm \omega_x|_i^n \, \Delta x + \omega_{xx}|_i^n \, \frac{\Delta x^2}{2!} \pm \omega_{xxx}|_i^n \, \frac{\Delta x^3}{3!} + \omega_{xxxx}|_i^n \, \frac{\Delta x^4}{4!} \; + \; \ldots$$

Subtract ω_{i-1}^n from ω_{i+1}^n to get finite difference expression for ω_x:

$$\omega_x|_i^n = \frac{\omega_{i+1}^n - \omega_{i-1}^n}{2\,\Delta x} \; - \; \omega_{xxx}|_i^n \, \frac{\Delta x^2}{6} \; + \; \ldots$$

Add ω_{i+1}^n and ω_{i-1}^n to get finite difference expression for ω_{xx}:

$$\omega_{xx}|_i^n = \frac{\omega_{i+1}^n - 2\omega_i^n + \omega_{i-1}^n}{\Delta x^2} \; - \; \omega_{xxxx}|_i^n \, \frac{\Delta x^2}{12} \; + \; \ldots$$

(b) • **Explicit solution scheme:**

$$\underbrace{\frac{\omega_i^{n+1} - \omega_i^n}{\Delta t}}_{\text{forward} \quad \omega_t} + u_i^n \underbrace{\frac{\omega_{i+1}^n - \omega_{i-1}^n}{2\,\Delta x}}_{\text{central} \quad \omega_x} - \nu \underbrace{\frac{\omega_{i+1}^n - 2\omega_i^n + \omega_{i-1}^n}{\Delta x^2}}_{\text{central} \quad \omega_{xx}} = 0$$

Explicit, since only one term ω_i^{n+1} is defined at the highest time level $(n+1)$, thus the equation can be explicitly solved.

$$\Rightarrow \quad \omega_i^{n+1} \; = \; f_i(\omega^n, u^n, \nu, \Delta t, \Delta x)$$

truncation error

$$\begin{aligned}
\tau \;&=\; L(\omega) \,-\, L_\Delta(\omega) \\
&=\; -\,\omega_{tt}|_i^n \,\frac{\Delta t}{2} \,-\, u\,\omega_{xxx}|_i^n \,\frac{\Delta x^2}{6} \,+\, \nu\,\omega_{xxxx}|_i^n \,\frac{\Delta x^2}{12} \,+\, \text{terms of higher order} \\
&=\; \mathcal{O}(\Delta t, \Delta x^2) \qquad \text{consistent, since } \lim_{\Delta t, \Delta x \to 0} \tau = 0
\end{aligned}$$

- **Implicit solution scheme:**

$$\underbrace{\frac{\omega_i^{n+1} - \omega_i^n}{\Delta t}}_{\text{backward} \quad \omega_t} + u_i^{n+1} \underbrace{\frac{\omega_{i+1}^{n+1} - \omega_{i-1}^{n+1}}{2\,\Delta x}}_{\text{central} \quad \omega_x} - \nu \underbrace{\frac{\omega_{i+1}^{n+1} - 2\omega_i^{n+1} + \omega_{i-1}^{n+1}}{\Delta x^2}}_{\text{central} \quad \omega_{xx}} = 0$$

Several terms ω_i^{n+1}, ω_{i-1}^{n+1}, and ω_{i+1}^{n+1} are defined at the highest time level $(n+1)$, therefore the equation can not be explicityly solved. The unknowns at the highest time level are implicityly coupled and build a tridiagonal system of equations:

$$\Rightarrow \; \text{tridiagonal system of equations} \qquad a_i\omega_{i-1}^{n+1} + b_i\omega_i^{n+1} + c_i\omega_{i+1}^{n+1} \; = \; f_i(\omega^n, u^n, \nu, \Delta t, \Delta x)$$

truncation error

$$\begin{aligned}
\tau \;&=\; \omega_{tt}|_i^{n+1} \,\frac{\Delta t}{2} \,-\, u\,\omega_{xxx}|_i^{n+1} \,\frac{\Delta x^2}{6} \,+\, \nu\,\omega_{xxxx}|_i^{n+1} \,\frac{\Delta x^2}{12} \,+\, \text{terms of higher order} \\
&=\; \mathcal{O}(\Delta t, \Delta x^2) \qquad \text{consistent, since } \lim_{\Delta t, \Delta x \to 0} \tau = 0
\end{aligned}$$