Computational Fluid Dynamics I

Exercise 3

1. Consider the non-linear, hyperbolic partial differential equation

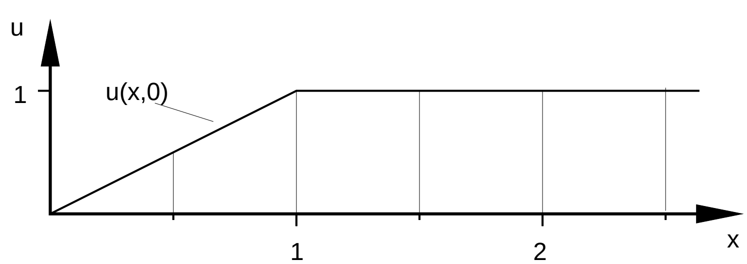
$$u_t + uu_x = 0$$

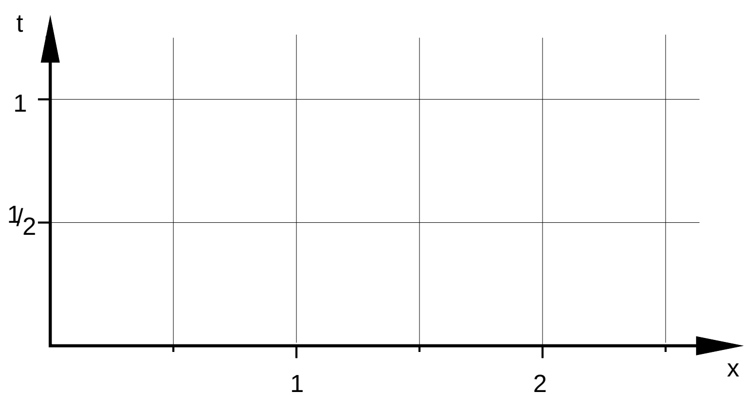
   (a) Determine the characteristic line and the characteristic solution.

   (b) The following initial condition is given:

$$u(x, t = 0) = \begin{cases} 0 & \text{for} \quad x \leq 0 \\ x & \text{for} \quad 0 < x \leq 1 \\ 1 & \text{for} \quad x > 1 \end{cases}$$

   Determine the solution for the time levels $t=\frac{1}{2}$ and t=1 in the (x,t)-diagram





Computational Fluid Dynamics I

Exercise 3 (solution)

1. (a) PDE: $u_t + uu_x = 0$:
   Use coordinate transformation

$$\begin{aligned} u_x &= u_\Omega \Omega_x + u_S S_x \\ u_t &= u_\Omega \Omega_t + u_S S_t \end{aligned}$$

   and fill into PDE:

$$\begin{aligned} u_\Omega \Omega_t + u_S S_t + u(u_\Omega \Omega_x u_S S_x) &= 0 \\ \Rightarrow \underbrace{(\Omega_t + u\Omega_x)}_{=Q} u_\Omega + (S_t + uS_x) u_S &= 0 \end{aligned}$$

   Set $Q = 0$ for the cross-wise derivative u_Ω to be undetermined and determine the slope of the characteristic line:

$$Q = \Omega_t + u\Omega_x = 0 \Leftrightarrow \frac{-\Omega_t}{\Omega_x} = \frac{dx}{dt} = u$$

   When assuming $u = const.$ for the slope of the characteristic line, integration from x_0, t_0 yields the characteristic line C:

$$C: \quad x - x_0 = u\,(t - t_0) \quad \text{or} \quad t = t_0 + \frac{x - x_0}{u}$$

   The characteristic solution is obtained by transforming the PDE from (x, y) to (τ, ξ): $d\tau = dt$, $d\xi = dx - udt$:

$$\begin{aligned} u_t &= \xi_t u_\xi + \tau_t u_\tau = -uu_\xi + u_\tau \\ u_x &= \xi_x u_\xi + \tau_x u_\tau = u_\xi \end{aligned}$$

   apply to PDE $\Rightarrow$

$$-uu_\xi + u_\tau + uu_\xi = u_\tau = 0$$

   That is, the value of u will be constant in time **on** each individual characteristic curve due to the absence of any source/sink terms in the original PDE. Integration over τ with $\xi = x - ut = const.$ $\Rightarrow$

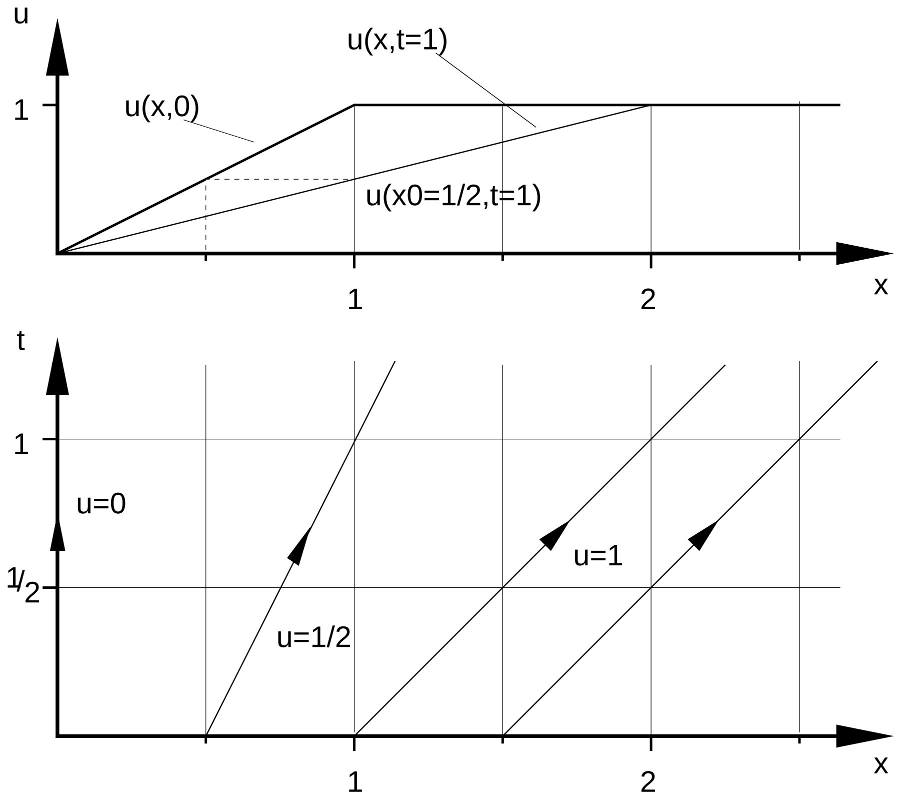
$$u(\tau, \xi) = c(\xi)$$

   With the initial condition $u_0(x_0, t_0)$ on the characteristic line $\xi = \xi_0 = x_0 - ut_0$:

$$u(x, t) = u_0(x_0, t_0)$$

   i.e. the solution remains constant on the characteristic line.
   The general transformation $(x, y) \rightarrow (\xi, \tau)$ yields an result independent in u, therefore the assumption $u = const.$ for the first integration of the slope of the characteristic line is valid, as long as it does not cross other characteristic lines. In this case the nonlinearity would result in a new slope.

(b) graphical solution, e.g. for time level $t = 1$:



$$C: \quad x - x_0 = u\,(t - t_0) \quad \text{or} \quad t = t_0 + \frac{x - x_0}{u}$$

Computational Fluid Dynamics I

Exercise 3 Appendix

Given is the PDE $\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u}{\partial x} = 1$.

1. Determine the characteristic slope and the characteristic solution.
2. Determine the solution at time level $t = 1$ for the initial condition $u(t = 0, x) = sin(\pi x)$.

a)

$$u_t + \frac{1}{2}u_x = 1$$

$$\Rightarrow \Omega_t + \frac{1}{2}\Omega_x = 0$$
$$\Rightarrow -\frac{\Omega_t}{\Omega_x} = \frac{dx}{dt}|_c = \frac{1}{2}$$

$$d\xi = 2dx - dt \qquad d\tau = dt$$
$$u_x = u_\xi \cdot 2 \qquad u_t = u_\xi \cdot (-1) + u_\tau$$

$$\Rightarrow -u_\xi + u_\tau + \frac{1}{2} \cdot 2u_\xi = 1$$
$$\Rightarrow u_\tau = 1$$
$$\Rightarrow u(\xi, \tau) = \tau + c(\xi)$$

b)

$$u(t = 0, x) = \sin(\pi x)$$

$t = \tau$:

$$\Rightarrow u(\tau = 0, \xi) = \sin\left(\frac{\pi}{2}\xi\right) \qquad \xi = 2x - t$$
$$= c(\xi)$$
$$\Rightarrow u(\xi, \tau) = \tau + \sin\left(\frac{\pi}{2}\xi\right)$$
$$\Rightarrow u(x, t) = t + \sin\left(\pi\left(x - \frac{1}{2}t\right)\right)$$

t=1

$$u(x, t = 1) = 1 + \sin\left(\pi\left(x - \frac{1}{2}\right)\right)$$