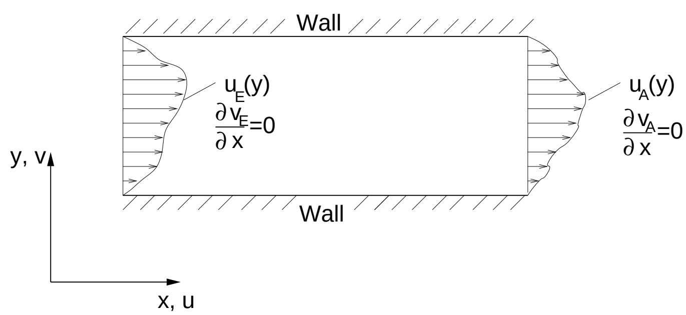
Computational Fluid Dynamics I

Exercise 2

1. (a) Derive the vorticity transport equation and the Poisson equation for the stream function Ψ for a two dimensional incompressible and viscous flow.

   (b) Formulate the boundary conditions for the stream function and the vorticity component at the boundaries of the channel flow domain shown in the sketch.



2. Formulate for incompressible flows (without taking into account the energy equation)

   (a) the Euler equations

   - with the velocity vector $\vec{v}$ and the pressure p
   - with stream function Ψ and vorticity component ω

   (b) the potential equation

   - with the velocity components u, v (Cauchy–Riemann differential equation)
   - with Φ
   - with Ψ

   Determine for a two-dimensional and steady flow the characteristic lines and the type of the equations.

Computational Fluid Dynamics I

Exercise 2 (solution)

1. (a) Navier-Stokes equations 2D, incompressible flow ($\rho = const \Rightarrow \rho_t = 0$):

$$\begin{aligned} u_x + v_y &= 0 \\ u_t + uu_x + vu_y + \frac{1}{\rho}p_x &= \nu\nabla^2 u \\ v_t + uv_x + vv_y + \frac{1}{\rho}p_y &= \nu\nabla^2 v \end{aligned}$$

The vorticity transport equation is obtained by taking the curl ($\nabla \times \vec{f}$) of the momentum equations: $\frac{\partial}{\partial x}$(y-momentum equation) - $\frac{\partial}{\partial y}$(x-momentum equation)

$$\begin{array}{rcl} v_{xt} + u_x v_x + uv_{xx} + v_x v_y + vv_{xy} + \frac{1}{\rho}p_{xy} & & \nabla^2(v_x \\ -u_{yt} - u_y u_x - uu_{xy} - v_y u_y - vu_{yy} - \frac{1}{\rho}p_{xy} & = \nu & -u_y) \end{array}$$

where the pressure terms fall out:

$$(v_x - u_y)_t + \underbrace{u_x(v_x - u_y) + v_y(v_x - u_y)}_{=0 \quad \text{(mass-conserv. eq.)}} + v(v_x - u_y)_y + u(v_x - u_y)_x = \nu\nabla^2(v_x - u_y)$$

With the vorticity component $\omega = v_x - u_y$:

$$\begin{aligned} \omega_t + \underbrace{u\omega_x + v\omega_y}_{\text{convection of vorticity}} &= \underbrace{\nu\nabla^2\omega}_{\text{diffusion of vorticity}} \\ \Rightarrow \frac{D\omega}{Dt} &= \nu\nabla^2\omega \end{aligned}$$

which is the vorticity- (or eddy-) transport equation. The Poisson equation for the stream function Ψ is obtained with $u = \Psi_y, v = -\Psi_x$:

$$-\omega = -v_x + u_y = \Psi_{xx} + \Psi_{yy} = \nabla^2\Psi$$

Finally, we have two coupled partial differential equations for the two variables ω and Ψ, the velocities u and v in the vorticity-transport equation can be replaced by $u = \Psi_y$ and $v = -\Psi_x$.

(b) We have boundary conditions given for u and v, but we need them for ω and Ψ:

- **In- and outflow boundary:**
  The velocity profile $u(y)$ is given and we know that $\Psi_y = \frac{d\Psi}{dy} = u$, therefore integration of $d\Psi = u(y)dy$ yields:

  $$\Psi_E(y) = \int_{y_{\text{wall}}}^{y} u_E(y')dy' + \Psi(y_{\text{wall}})$$

  where the value at the wall $\Psi(y_{\text{wall}})$ can be chosen arbitrary as our PDE contains only derivatives of Ψ. For the vorticity boundary condition we compute the derivatives of u and v:

  $$\omega_E = v_x - u_y = -\frac{\partial u_E(y)}{\partial y}$$

- **Solid wall:**
  The no slip condition $u = v = 0$ holds, therefore $v = \Psi_x = 0 \Rightarrow \Psi_{\text{wall}} = const.$
  From the Poisson function for the stream function $\Psi_{xx} + \Psi_{yy} = -\omega$ with $\Psi_{xx} = 0$:
  $\Rightarrow -\omega_{\text{wall}} = u_y = \Psi_{yy}$ and $u_{\text{wall}} = 0 = \Psi_{y,\text{wall}}$

  Therefore we use a Taylor series expansion for y_{wall}:

  $$\begin{aligned}
  \Psi(y_{\text{wall}} + \Delta y) &= \Psi(y_{\text{wall}}) + \underbrace{\Psi_y(y_{\text{wall}})}_{=0}\Delta y + \Psi_{yy}(y_{\text{wall}})\frac{\Delta y^2}{2} + \dots \\
  \Rightarrow \Psi_{yy}(y_{\text{wall}}) &= 2\frac{\Psi(y_{\text{wall}} + \Delta y) - \Psi(y_{\text{wall}})}{\Delta y^2} \\
  \Rightarrow \omega(y_{\text{wall}}) &= -\Psi_{yy}(y_{\text{wall}}) = -2\frac{\Psi(y_{\text{wall}} + \Delta y) - \Psi(y_{\text{wall}})}{\Delta y^2}
  \end{aligned}$$

2. (a) Euler equations for incompressible flow $(\vec{v}, p)$:

$$\nabla \cdot \vec{v} = 0$$
$$\frac{D\vec{v}}{Dt} + \frac{1}{\rho}\nabla p = 0$$

characteristic lines (steady 2D flow):

$$\begin{aligned} u_x + v_y &= 0 \\ uu_x + vu_y + 1/\rho p_x &= 0 \Leftrightarrow \\ uv_x + vv_y + 1/\rho p_y &= 0 \end{aligned} \quad \begin{pmatrix} \partial_x & \partial_y & 0 \\ u\partial_x + v\partial_y & 0 & \frac{1}{\rho}\partial_x \\ 0 & u\partial_x + v\partial_y & \frac{1}{\rho}\partial_y \end{pmatrix} \begin{pmatrix} u \\ v \\ p \end{pmatrix} = 0$$

Use chain rule of PDE ($u_x = u_\Omega \Omega_x + u_S S_x$) to transform PDE to

$$\underbrace{\begin{pmatrix} \Omega_x & \Omega_y & 0 \\ u\Omega_x + v\Omega_y & 0 & \frac{1}{\rho}\Omega_x \\ 0 & u\Omega_x + v\Omega_y & \frac{1}{\rho}\Omega_y \end{pmatrix} \begin{pmatrix} u_\Omega \\ v_\Omega \\ p_\Omega \end{pmatrix}}_{\text{crosswise derivative}} + \begin{pmatrix} S_x & S_y & 0 \\ uS_x + vS_y & 0 & \frac{1}{\rho}S_x \\ 0 & uS_x + vS_y & \frac{1}{\rho}S_y \end{pmatrix} \begin{pmatrix} u_S \\ v_S \\ p_S \end{pmatrix} = 0$$

We need the determinant of the coefficients matrix of the crosswise derivatives to be zero:

$$\begin{vmatrix} \Omega_x & \Omega_y & 0 \\ u\Omega_x + v\Omega_y & 0 & \frac{1}{\rho}\Omega_x \\ 0 & u\Omega_x + v\Omega_y & \frac{1}{\rho}\Omega_y \end{vmatrix} = 0 = -(u\Omega_x + v\Omega_y)\frac{1}{\rho}\Omega_x^2 - (u\Omega_x + v\Omega_y)\frac{1}{\rho}\Omega_y^2$$

$$\Leftrightarrow (u\frac{\Omega_x}{\Omega_y} + v)(\frac{\Omega_x^2}{\Omega_y^2} + 1) = 0 \Rightarrow \frac{dy}{dx} = -\frac{\Omega_x}{\Omega_y} = \frac{v}{u} \quad \text{or} \quad \frac{\Omega_x}{\Omega_y} = \pm\sqrt{-1}$$

i. e. 1 real, 2 imaginary characteristic lines $\Rightarrow$ mixed hyperbolic elliptic type

Euler equations (2D) Ψ, ω:

$$\begin{aligned} \nabla^2\Psi &= -\omega \\ \frac{D\omega}{Dt} &= 0 \end{aligned}$$

characteristic lines (steady flow):

$$\begin{aligned} \Psi_{xx} + \Psi_{yy} &= -\omega \\ u\omega_x + v\omega_y &= 0 \end{aligned} \Leftrightarrow \begin{pmatrix} \partial_{xx} + \partial_{yy} & 1 \\ 0 & u\partial_x + v\partial_y \end{pmatrix} \begin{pmatrix} \Psi \\ \omega \end{pmatrix} = 0$$

to solve:

$$\begin{vmatrix} \Omega_x^2 + \Omega_y^2 & 0 \\ 0 & u\Omega_x + v\Omega_y \end{vmatrix} = 0 = (u\Omega_x + v\Omega_y)(\Omega_x^2 + \Omega_y^2)$$

$\Rightarrow$ see Euler equations $(\vec{v}, p)$

(b) Euler equations (incompressible, 2D, irrotational: $\omega = 0$):
($\Psi_y = u, \Psi_x = -v, \Phi_x = u, \Phi_y = v$)

$$\begin{aligned} \nabla^2\Phi &= \Phi_{xx} + \Phi_{yy} = 0 \qquad \text{Potential formulation} \quad \vec{v} = \nabla\Phi \\ \nabla^2\Psi &= \Psi_{xx} + \Psi_{yy} = 0 \qquad \text{Stream function formulation} \end{aligned}$$

for which the characteristic slopes are computed by

$$Q = \Omega_x^2 + \Omega_y^2 = 0 \Rightarrow \frac{dy}{dx} = \frac{-\Omega_x}{\Omega_y} = \pm\sqrt{-1}$$

which results in two imaginary lines $\Rightarrow$ the PDE is of **elliptic type**.

Either of the above second-order PDEs can be transformed to a system of two first-order PDEs:

$$\begin{aligned} u_x + v_y &= 0 \\ v_x - u_y &= 0 \end{aligned}$$

which in this case are known as the Cauchy-Riemann differential equation, to compute the characteristic lines solve:

$$\begin{vmatrix} \Omega_x & \Omega_y \\ -\Omega_y & \Omega_x \end{vmatrix} = 0 \Rightarrow \Omega_x^2 + \Omega_y^2 = 0 \Leftrightarrow \frac{dy}{dx} = -\frac{\Omega_x}{\Omega_y} = \pm\sqrt{-1}$$

i. e. 2 imaginary characteristic lines $\Rightarrow$ **elliptic type** (same results as above)