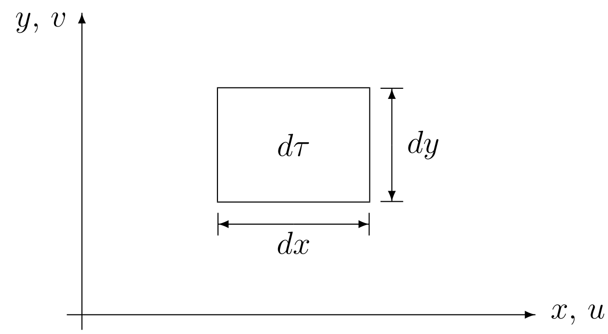
Computational Fluid Dynamics I

Exercise 1

1. Formulate the conservation of mass for a two-dimensional infinitesimal volume as shown in the sketch.



   (a) Formulate the conservation equation in integral form and derive its differential form.

   (b) Formulate the differential equation in a non-conservative form.

2. Reformulate the conservative form of the 2-D Euler equations in Cartesian coordinates into a form with the variables $\vec{V} = (\varrho, \vec{v}, E)^\top$ and the substantial derivative $\frac{D\vec{V}}{Dt}$.

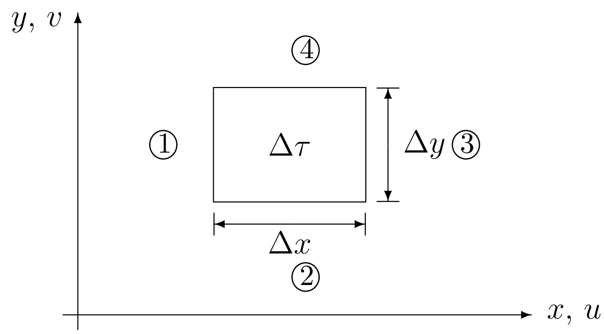
3. Derive the potential equation for compressible flow from the Euler equations under the assumption of steady, isoenergetic, and irrotational flow ($\vec{\zeta} = 0 \Rightarrow ds = 0$ (Crocco's theorem) $\Rightarrow \nabla p = \left.\frac{\partial p}{\partial \varrho}\right|_s \nabla \varrho \Rightarrow \nabla p = a^2 \nabla \varrho$).

Computational Fluid Dynamics I

Exercise 1 (solution)

1. (a) conservation of mass:

$$\int_{\tau} \frac{\partial U_1}{\partial t}\, d\tau + \oint_{A} \vec{H}_1 \cdot \vec{n} dA = 0 \quad , \qquad \begin{array}{l} U_1 = \varrho \\ \vec{H}_1 = \varrho \vec{v} = \varrho \binom{u}{v} \end{array}$$



① $\vec{H}_1 \cdot \vec{n} dA = \binom{\varrho u}{\varrho v} \cdot \binom{-dy}{0} = -\varrho u\, dy$

② $\vec{H}_2 \cdot \vec{n} dA = \binom{\varrho u}{\varrho v} \cdot \binom{0}{-dx} = -\varrho v\, dx$

③ $\vec{H}_3 \cdot \vec{n} dA = \binom{(\varrho u + \frac{\partial(\varrho u)}{\partial x}\Delta x) dy}{(\varrho v + \frac{\partial(\varrho v)}{\partial x}\Delta x) dy} \cdot \binom{dy}{0} = (\varrho u + \frac{\partial(\varrho u)}{\partial x}\Delta x) dy$

④ $\vec{H}_4 \cdot \vec{n} dA = \binom{(\varrho u + \frac{\partial(\varrho u)}{\partial y}\Delta y) dx}{(\varrho v + \frac{\partial(\varrho v)}{\partial y}\Delta y) dx} \cdot \binom{0}{dx} = (\varrho v + \frac{\partial(\varrho v)}{\partial y}\Delta y) dx$

$$\Rightarrow \int_{\Delta\tau} \frac{\partial \varrho}{\partial t}\, d\tau + \int_{\Delta x} -\varrho v\, dx + \int_{\Delta y} (\varrho u + \frac{\partial(\varrho u)}{\partial x}\Delta x) dy$$

$$+ \int_{\Delta x} (\varrho v + \frac{\partial(\varrho v)}{\partial y}\Delta y) dx + \int_{\Delta y} -\varrho u\, dy \;=\; 0$$

$$\lim_{\Delta\tau, \Delta x, \Delta y \to d\tau, dx, dy} \Rightarrow \quad \frac{\partial \varrho}{\partial t} d\tau - \varrho v\, dx + (\varrho u + \frac{\partial(\varrho u)}{\partial x}\Delta x) dy + (\varrho v + \frac{\partial(\varrho v)}{\partial y}\Delta y) dx - \varrho u\, dy \;=\; 0$$

$$\Leftrightarrow \quad \frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u)}{\partial x} + \frac{\partial(\varrho v)}{\partial y} \;=\; \frac{\partial \varrho}{\partial t} + \nabla \cdot \underbrace{\begin{pmatrix} \varrho u \\ \varrho v \end{pmatrix}}_{\varrho \vec{v}} \;=\; 0$$

Alternative solution: use Gauss theorem

$$\oint_A \varrho\vec{v}\cdot\vec{n}dA = \int_\tau \mathrm{div}(\varrho\vec{v})d\tau$$

$$\Rightarrow \int_\tau \frac{\partial\varrho}{\partial t}d\tau + \int_\tau \mathrm{div}(\varrho\vec{v})d\tau = 0$$

$$\lim_{\Delta\tau\to d\tau} \quad \Rightarrow \frac{\partial\varrho}{\partial t} + \nabla\cdot\varrho\vec{v} = 0$$

(b) conservative form:

$$\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\vec{v}) = 0$$

$$\Rightarrow \underbrace{\frac{\partial\varrho}{\partial t} + \vec{v}\cdot\nabla\varrho}_{\text{Substantial/material derivative} \quad \frac{D\varrho}{Dt}} + \varrho\nabla\cdot\vec{v} = 0 \quad \Rightarrow$$

non-conservative form:

$$\frac{D\varrho}{Dt} + \varrho\nabla\cdot\vec{v} = 0$$

2.

$$\begin{aligned}
\varrho_t + (\varrho u)_x + (\varrho v)_y &= 0 \quad \text{(mass)}\\
(\varrho u)_t + (\varrho u^2 + p)_x + (\varrho uv)_y &= 0 \quad \text{(x} - \text{momentum)}\\
(\varrho v)_t + (\varrho uv)_x + (\varrho v^2 + p)_y &= 0 \quad \text{(y} - \text{momentum)}\\
(\varrho E)_t + (\varrho uE + up)_x + (\varrho vE + vp)_y &= 0 \quad \text{(energy)}
\end{aligned}$$

conservation of mass:

$$\varrho_t + u\varrho_x + v\varrho_y + \varrho(u_x + v_y) \;=\; 0 \Leftrightarrow \frac{D\varrho}{Dt} + \varrho\nabla\cdot\vec{v} \;=\; 0$$

x-momentum eq.. :

$$\begin{aligned}
\varrho u_t + \varrho u u_x + \varrho v u_y + u\varrho_t + u(\varrho u)_x + u(\varrho v)_y + p_x &= 0\\
\varrho u_t + \varrho u u_x + \varrho v u_y + u \underbrace{\Big(\varrho_t + (\varrho u)_x + (\varrho v)_y\Big)}_{=0 \quad \text{(mass-conservation eq.)}} + p_x &= 0\\
\varrho\frac{Du}{Dt} + p_x &= 0
\end{aligned}$$

$$\Leftrightarrow \qquad \frac{Du}{Dt} + \frac{1}{\varrho}p_x \;=\; 0$$

energy equation:

$$\begin{aligned}
&\varrho E_t + \varrho u E_x + \varrho v E_y +\\
&\quad \underbrace{E\varrho_t + E(\varrho u)_x + E(\varrho v)_y}_{=0 \quad \text{(mass-conservation eq.)}} + (up)_x + (vp)_y \;=\; 0
\end{aligned}$$

$$\Leftrightarrow \qquad \frac{DE}{Dt} + \frac{1}{\varrho}\Big((up)_x + (vp)_y\Big) \;=\; 0$$

3. Derivative of pressure can be transformed to derivative of density:

$$\nabla p = \begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \frac{\partial\varrho}{\partial\varrho}\frac{\partial p}{\partial x} \\ \frac{\partial\varrho}{\partial\varrho}\frac{\partial p}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial p}{\partial\varrho}\frac{\partial\varrho}{\partial x} \\ \frac{\partial p}{\partial\varrho}\frac{\partial\varrho}{\partial y} \end{pmatrix} = \frac{\partial p}{\partial\varrho}\nabla\varrho = a^2\nabla\varrho \qquad a \text{ is speed of sound}$$

Introduce potential Φ:

$$\vec{v} = \nabla\Phi \qquad u = \Phi_x \quad , \quad v = \Phi_y \qquad dp = a^2 d\varrho$$

Euler equations (2–D, steady for Cartesian coordinates) :

$$\begin{aligned}
(\varrho u)_x + (\varrho v)_y &= 0\\
(\varrho u^2 + p)_x + (\varrho uv)_y &= 0\\
(\varrho uv)_x + (\varrho v^2 + p)_y &= 0
\end{aligned}$$

$$\begin{aligned}
\varrho u u_x + \underbrace{u(\varrho u)_x + u(\varrho v)_y}_{=0} + \varrho v u_y + p_x &= 0 \qquad | \cdot u\\
\varrho u v_x + \underbrace{v(\varrho u)_x + v(\varrho v)_y}_{=0} + \varrho v v_y + p_y &= 0 \qquad | \cdot v
\end{aligned}$$

Replace u and v by potential Φ:

$$\begin{array}{rcl} \varrho u^2\Phi_{xx} + \varrho uv\Phi_{xy} + up_x &=& 0 \\ \varrho uv\Phi_{xy} + \varrho v^2\Phi_{yy} + vp_y &=& 0 \end{array} \Bigg| +$$

$$u^2\Phi_{xx} + 2uv\Phi_{xy} + v^2\Phi_{yy} + \frac{1}{\varrho}(\underbrace{up_x + vp_y}_{\binom{u}{v}\cdot\nabla p \,=\binom{u}{v}\cdot a^2\nabla\varrho}) = 0$$

$$\Rightarrow \qquad u^2\Phi_{xx} + 2uv\Phi_{xy} + v^2\Phi_{yy} + ua^2\frac{1}{\varrho}\varrho_x + va^2\frac{1}{\varrho}\varrho_y = 0$$

with the conservation of mass $\quad u\varrho_x + v\varrho_y = -\varrho u_x - \varrho v_y = -\varrho\Phi_{xx} - \varrho\Phi_{yy}$

potential equation: $\quad (u^2 - a^2)\Phi_{xx} + 2uv\Phi_{xy} + (v^2 - a^2)\Phi_{yy} = 0$