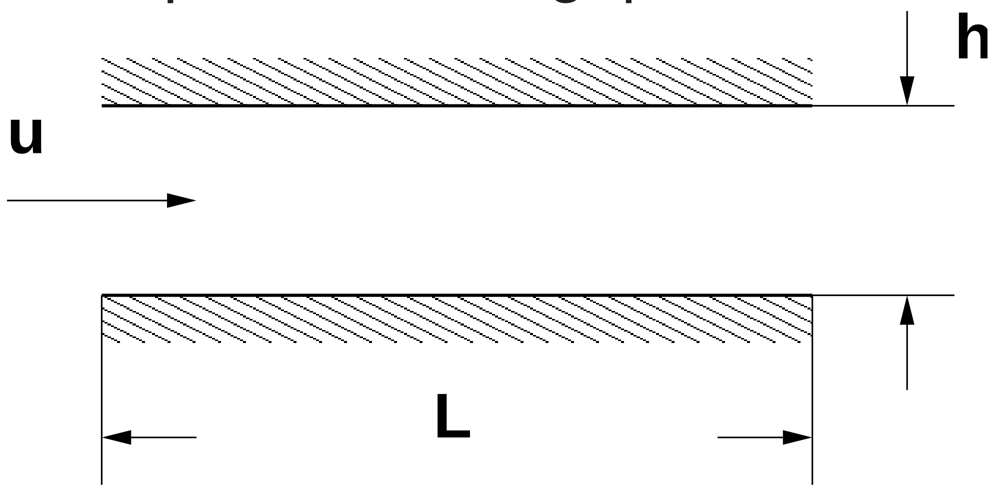
Theory of similarity

- Comparison of experimental results with real configuration
- Reduction of the number of physical quantities ($\rightarrow$ number of experiments)
- experimental results are independent on the scale (similarity parameters are dimensionsless)

Theory of similarity

similarity rules

- geometric similarity
  example: flow in a gap



  $\frac{h}{L} \triangleq$ parameter of the geometry

- dynamical similarity
  Flows are not necessarily similar, if only the flow quantities are scaled

Example

similarity concerning pressureDruckes

$$\text{Eu} = \frac{\text{pressure force}}{\text{inertia}} = \frac{\Delta p}{\rho u^2}$$

similarity concerning viscous stresses

$$\text{Re} = \frac{\text{inertia}}{\text{viscous forces}} = \frac{\rho u l}{\eta} = \left(\frac{\rho u^2 l}{u \eta}\right)$$

Re, Eu $\triangleq$ similarity parameters (dimensionless)

2 flows are similar, if they are geometrical and dynamical similar!

Beispiel

Methoden zur Bestimmung der Kennzahlen

- method of dimensional analysis
  (Buckingham‘s Π-Theorem)

Π-Theorem determines the maximum number of parameters to be considered

- number of physical quantities: k
- number of basical dimensions: r
  $[m], [s], [kg], [K]$

$\rightarrow$ number of dimensionless parameters

$$m = k - r$$

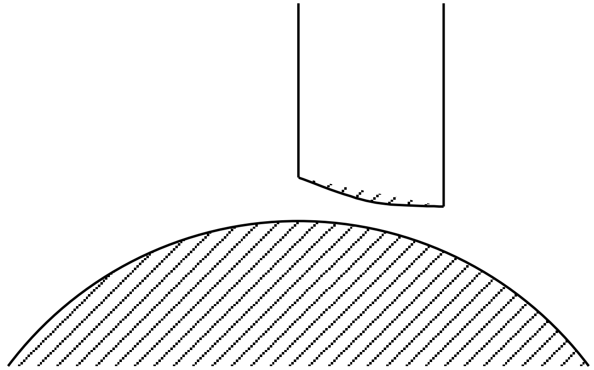
Procedure

1. Determine the number of physical quantitites (k)
   $G_1 = G_1(G_2, G_3, \ldots, G_k)$
2. Decompose and determine r
3. Determine m
4. Choose the recurring variables (constant values)
   (a) include all basical dimensions
   (b) linearly independent
   (c) don't choose the variables that are hard to measure
5. Determine the dimensionless parameters $\Pi_i = N\, G_1^{\alpha_1} \cdot M\, G_2^{\alpha_2} \cdot \ldots \cdot G_r^{\alpha_r}$
6. Check the dimensions
7. Formulate $\Pi_1 = \Pi_1(\Pi_2, \Pi_3, \ldots, \Pi_m)$

important numbers in Fluid Mechanics

Reynoldsnumber $\text{Re} = \frac{\rho u l}{\eta} \triangleq \frac{\text{inertia}}{\text{viscous forces}}$

- $\text{Re} \to 0$: creeping flow



- $\text{Re} \cdot h^2/l^2 \lesseqqgtr 1$: gap flow (bearing)

- $\text{Re} \to \infty$: frictionless (turbulent)

Eulernumber $\text{Eu} = \frac{\Delta p}{\rho u_\infty^2} \triangleq \frac{\text{pressure force}}{\text{inertia}}$

important numbers in Fluid Mechanics

Machnumber $\text{Ma} = \frac{u}{c} \triangleq \frac{\text{flow velocity}}{\text{speed of sound}}$

$$\text{Ma} < 0.3 \qquad \sim \text{incompressible}$$

$$\left.\begin{array}{lll} \text{Ma} & < 1. & \text{subsonic} \\ \text{Ma} & > 1. & \text{supersonic} \\ \text{Ma} & \gg 1. & \text{hypersonic} \end{array}\right\} \text{compressible flow}$$

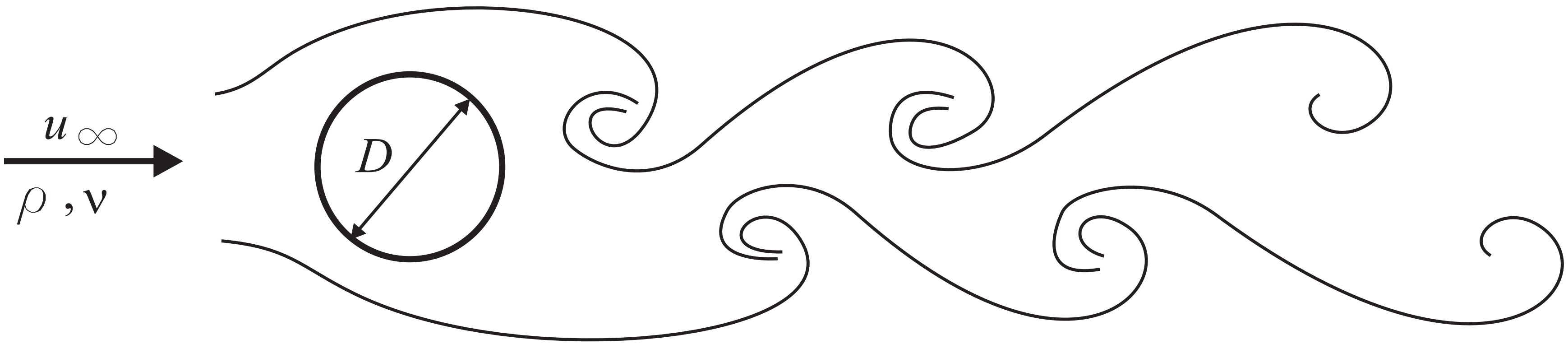
Froudenumber $\text{Fr} = \frac{u}{\sqrt{gh}} \triangleq \frac{\text{inertia}}{\text{gravitation}}$

Strouhalnumber $\text{Sr} = \frac{l}{ut} \triangleq$ ratio between characteristic times

Prandtlzahl $\text{Pr} = \eta \frac{c_p}{\lambda} = \frac{\mu}{a} \triangleq \frac{\text{kinematic viscosity}}{\text{temperature conductivity}}$

11.1

The wake of a long cylinder with the diameter D is anlyzed experimentally in a windtunnel. On certain circumstances a periodic vortex configuration is generated, the Kármán vortex street. The dimensionless parameters of the problem shall be determined. How many variations of parameters are necessary in this investigation to measure the frequency of the vortex street?



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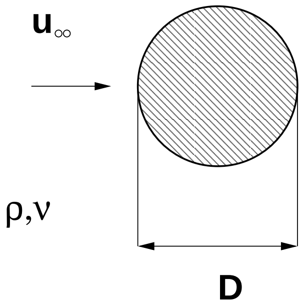


f=?

physical quantities

- incoming velocity: $u_\infty \left[\frac{m}{s}\right] \leftarrow$
- kinematic viscosity $\nu \left[\frac{m^2}{s}\right]$
- density $\rho \left[\frac{kg}{m^3}\right] \leftarrow$
- diameter of cylinder $D \; [m] \leftarrow$

- frequency $f \left[\frac{1}{s}\right]$

11.1

$$f = F(u_\infty, \nu, \rho, D) \Longleftrightarrow G(f, u_\infty, \nu, \rho, D) = 0$$

Π-Theorem: $k = 5$ (number of physical quantitites)

$r = 3$ (basical dimensions(m, s, kg))

$$\rightarrow m = k - r = 5 - 3 = 2$$

reference variables: u_∞, ρ, D

- All dimensions are included
- linearly independent

11.1

1. number

$$\Pi_1 = \underbrace{f}_{\text{nonrecurring}} \quad \underbrace{u_\infty^{\alpha_1} \rho^{\beta_1} D^{\gamma_1}}_{\text{recurring}}$$

Dimensional analysis

$$[-] = [\frac{1}{s}] \quad [\frac{m}{s}]^{\alpha_1} \; [\frac{kg}{m^3}]^{\beta_1} \; [m]^{\gamma_1}$$

comparison of exponents:

$$\begin{aligned} [kg] &: 0 = \beta_1 \\ [s] &: 0 = -1 - \alpha_1 && \rightarrow \alpha_1 = -1 \\ [m] &: 0 = \alpha_1 - 3\beta_1 + \gamma_1 && \rightarrow \gamma_1 = 1 \end{aligned}$$

$$\rightarrow \Pi_1 = \frac{f \; D}{u_\infty} = \text{Sr: Strouhalnumber}$$

11.1

$$\Pi_2 = \underbrace{\nu}_{\text{nonrecurring}} \underbrace{u_\infty^{\alpha_2} \rho^{\beta_2} D^{\gamma_2}}_{\text{recurring}}$$

$$[-] = [\frac{m^2}{s}] \quad [\frac{m}{s}]^{\alpha_2} [\frac{kg}{m^3}]^{\beta_2} [m]^{\gamma_2}$$

$$\begin{aligned}
[kg] &: 0 = \beta_2 \\
[s] &: 0 = -1 - \alpha_2 && \rightarrow \alpha_2 = -1 \\
[m] &: 0 = 2 + \alpha_2 - 3\beta_2 + \gamma_2 && \rightarrow \gamma_2 = -1
\end{aligned}$$

$$\rightarrow \Pi_2 = \frac{\nu}{u_\infty D} = \frac{1}{\text{Re}}$$

$\rightarrow Sr = \Phi(\text{Re}) \rightarrow$ variation of 1 parameter in experiment

11.10

In a gas flow the heat transfer is determined from the viscous effects and from heat conduction. The influencing quantities are the heat conductivity $\lambda \left[\frac{kgm}{s^3K}\right]$, the dynamic viscosity $\eta \left[\frac{kg}{ms}\right]$ and the reference values for the temperature, the velocity, and the length. The physical relationship can be described with the energy equation

$$\lambda \frac{\partial^2 T}{\partial y^2} + \eta \left(\frac{\partial u}{\partial y}\right)^2 = 0.$$

11.10

Deduce the dimensionless parameters of the problem
a) with the Π-theorem.
b) Expand the resulting parameter with the specific heat capacity c_p and formulate the new coefficient as a product of three different parameters.

Hint: The material quantities are constant. The fourth basic dimension is the temperature.

11.10

a)

$$K^* = \lambda^\alpha \, \eta^\beta \, T_R^\gamma \, u_R^\delta \, l^\varepsilon$$

choose $\beta = 1$

$$\begin{aligned} kg &: 0 = \alpha + 1 \\ m &: 0 = \alpha - 1 + \delta + \varepsilon \\ s &: 0 = -3\alpha - 1 - \delta \\ K &: 0 = -\alpha + \gamma \end{aligned}$$

11.10

$$\alpha = \gamma = -1$$
$$\delta = 2$$
$$\varepsilon = 0$$

$$K^* = \frac{\eta\, u_R^2}{\lambda_p\, T_R}$$

b)

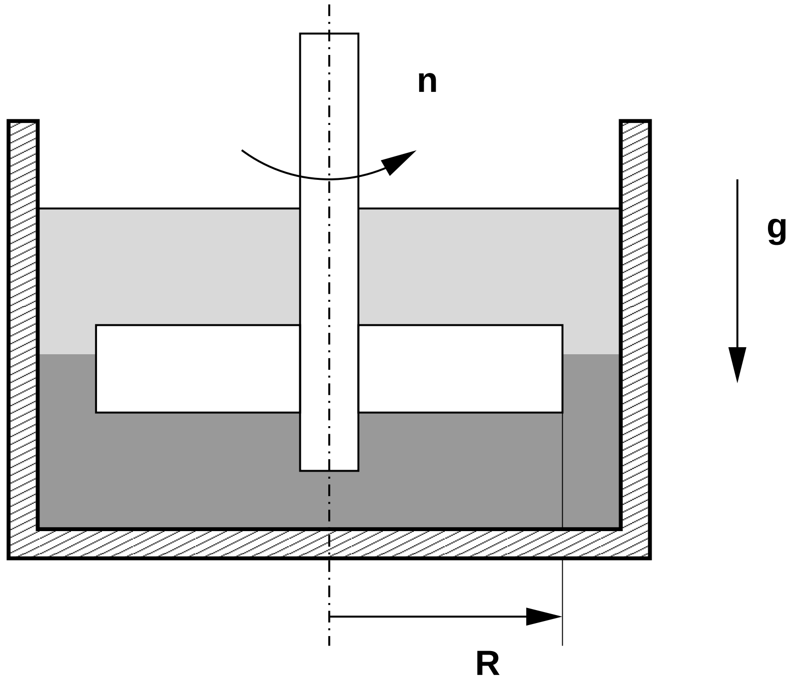
$$K^* = \frac{\eta\, c_p}{\lambda}\, \frac{u_R^2}{c_p\, T_R} = \frac{\eta\, c_p}{\lambda}\, \frac{u_R^2}{\gamma\, R\, T_R}(\gamma - 1)$$

$$c_p = \frac{\gamma\, R}{\gamma - 1}$$

$$K^* = Pr \cdot Ma^2(\gamma - 1))$$

Exam

In a cylindrical container two fluids (ρ_1, ν_1 and ρ_2, ν_2) shall be mixed using a blade agitator. For the design of the agitator a 1;3 subscaled model is built. In the following the model quantities are denoted by (').



Exam

Determine

1. the rotational frequency n' and the kinematic viscosities ν_1' and ν_2' of the model,
2. the torque M from the experimentally determined momentum M'.

Given: $R, n, \nu_1, \nu_2, \rho_1, \rho_2, \rho_1'$

Exam

1. $n', \nu_1', \nu_2' = ?$

   $Fr = Fr', Re = Re'$ mit $u = \omega\ R = 2\pi n\ R$

   $$Fr = \frac{u}{\sqrt{gR}} = \frac{2\pi nR}{\sqrt{gR}} = \frac{2\pi n' R'}{\sqrt{gR'}} \Rightarrow n' = n \cdot \sqrt{\frac{R}{R'}} = \sqrt{3}\ n$$

   $$Re = \frac{u \cdot R}{\nu_1} = \frac{2\pi nR \cdot R}{\nu_1} = \frac{2\pi n' R'^2}{\nu_1'}$$

   $$\Rightarrow \nu_1' = \nu_1 (\frac{R'}{R})^2\ \frac{n'}{n} = \nu_1 (\frac{R'}{R})^{\frac{3}{2}} = 0.19 \cdot \nu_1$$

   und wegen $\frac{\nu_1}{\nu_2} = \frac{\nu_1'}{\nu_2'} \Rightarrow \nu_2' = 0.19 \cdot \nu_2$

2. $c_M = c_M' \Rightarrow M = c_M \cdot \frac{1}{2}\rho_1 u^2 R \cdot \pi R^2$

   $$\Rightarrow \frac{M}{M'} = \frac{\rho_1 (R\ n)^2 R^3}{\rho_1' (R' n')^2 R'^3} = (\frac{R}{R'})^4 \cdot \frac{\rho_1}{\rho_1'}$$

   $$\Rightarrow M = 81\ \frac{\rho_1}{\rho_1'} M'$$

3. $P = M \cdot \omega = M \; 2\pi \; n = 508.94 \cdot \frac{\rho_1}{\rho_1'} \; n \; M'$

11.15

The laminar boundary layer flow on a flat plate, neglecting the viscous heat, can be described with the continuity, the momentum, and the energy equation in the following form:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \right) = \eta \frac{\partial^2 u}{\partial y^2}$$

$$\rho c_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = \lambda \frac{\partial^2 T}{\partial y^2}$$

11.15

a) Determine the dimensionless parameters of the problem.

b) Reformulate the resulting parameters by using well known parameters of fluid mechanics.

Assuming constant material quantities the flow field is independent of the temperature field. Both distributions can be computed separately.

c) Specify the assumptions to determine the temperature distribution in the boundary layer directly from the velocity distribution.

Hint for c): Compare the differential equations and assume that the velocity distribution $\vec{v}(x,\ y)$ is already known.

11.15

a) conti: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

momentum: $$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \eta \frac{\partial^2 u}{\partial y^2}$$

qnergy: $$\rho\, c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \lambda \frac{\partial^2 T}{\partial y^2}$$

dimensionsless parameters:

$$\bar{u} = \frac{u}{u_\infty}; \quad \bar{v} = \frac{v}{u_\infty}; \quad \bar{\rho} = \frac{\rho}{\rho_\infty}; \quad \bar{x} = \frac{x}{L}; \quad \bar{y} = \frac{y}{L};$$

$$\bar{\eta} = \frac{\eta}{\eta_\infty}; \quad \bar{c_p} = \frac{c_p}{c_{p\infty}}; \quad \bar{T} = \frac{T}{T_\infty}; \quad \bar{\lambda} = \frac{\lambda}{\lambda_\infty}$$

11.15

b) conti: $$\frac{u_\infty}{L}\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}}\right) = 0 \qquad \Longrightarrow \text{ no parameter}$$

momentum: $$\rho_\infty \frac{{u_\infty}^2}{L}\,\bar{\rho}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \eta_\infty\,\bar{\eta}\,\frac{u_\infty}{L^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\bar{\rho}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \bar{\eta}\,\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\underbrace{\left(\frac{\eta_\infty\, u_\infty\, L}{L^2\,\rho_\infty\, u_\infty^2}\right)}_{K_1}$$

$$K_1 = \frac{\eta_\infty}{L\,\rho_\infty\,u_\infty} = \frac{1}{Re}$$

11.15

energy: $$\frac{\rho_\infty\, c_{p_\infty}\, u_\infty\, T_\infty}{L}\, \bar{\rho}\, \bar{c}_p \left(\bar{u}\, \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\, \frac{\partial \bar{T}}{\partial \bar{y}} \right) = \frac{\lambda_\infty\, T_\infty}{L^2}\, \bar{\lambda}\, \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}$$

$$\bar{\rho}\, \bar{c}_p \left(\bar{u}\, \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\, \frac{\partial \bar{T}}{\partial \bar{y}} \right) = \underbrace{\frac{\lambda_\infty\, T_\infty\, L}{L^2\, \rho_\infty\, c_{p_\infty}\, u_\infty\, T_\infty}}_{K_2} \left(\bar{\lambda}\, \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} \right)$$

$$K_2 = \frac{\lambda_\infty}{L\, \rho_\infty\, c_{p_\infty}\, u_\infty}$$

11.15

c) dimensionless PDE.:

conti: $$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

momentum: $$\bar{\rho}\left(\bar{u}\,\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \frac{1}{Re}\,\bar{\eta}\left(\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right)$$

energy: $$\bar{\rho}\,\bar{c}_p\left(\bar{u}\,\frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v}\,\frac{\partial \bar{T}}{\partial \bar{y}}\right) = \frac{1}{Pr}\cdot\frac{1}{Re}\,\bar{\lambda}\left(\frac{\partial^2 \bar{T}}{\partial \bar{y}^2}\right)$$

11.13

constant fluid properties: $\bar{\rho} = \bar{c_p} = \bar{\lambda} = \bar{\eta} = 1$

Comparison between momentum and energy equation:
By replacing T with u and proposing

$$\boxed{\text{Pr} = 1}$$

the energy and the momentum equation are the same

i.e. $$\frac{\eta_\infty \, c_{p_\infty}}{\lambda_\infty} = 1$$

11.13

The energy equation for steady, compressible flows with constant material quantities is

$$\rho u \frac{\partial}{\partial x}\left(c_p T + \frac{u^2}{2}\right) + \rho v \frac{\partial}{\partial y}\left(c_p T + \frac{u^2}{2}\right) = u\eta \frac{\partial^2 u}{\partial y^2} + \eta \left(\frac{\partial u}{\partial y}\right)^2 + \lambda \frac{\partial^2 T}{\partial y^2}$$

Determine with the method of differential equations

a) the dimensionless form of the differential equation,
b) the dimensionless parameters of the problem.
c) Determine the isentropic coefficient γ , if the equation is independent of the Mach-number $M_\infty = \dfrac{u_\infty}{c_\infty}$.

Hint: $c = \sqrt{\gamma R T}$ $\qquad c_p = \dfrac{\gamma R}{\gamma - 1}$

11.13

a)

$$c_p\left(\rho\, u\,\frac{\partial T}{\partial x} + \rho\, v\,\frac{\partial T}{\partial y}\right) + \left(\rho\, u\,\frac{\partial \frac{u^2}{2}}{\partial x} + \rho\, v\,\frac{\partial \frac{u^2}{2}}{\partial y}\right)$$

$$= \eta\, u\,\frac{\partial^2 u}{\partial\, y^2} + \eta\left(\frac{\partial u}{\partial y}\right)^2 + \lambda\,\frac{\partial^2 T}{\partial\, y^2}$$

reference values: $\rho_\infty,\ u_\infty,\ T_\infty,\ L,\ \eta_\infty,\ c_{p\,\infty},\ \lambda_\infty$

$$\Longrightarrow\ \bar{u} = \frac{u}{u_\infty};\ \ \bar{v} = \frac{v}{u_\infty};\ \ \bar{T} = \frac{T}{T_\infty};\ \ \bar{\rho} = \frac{\rho}{\rho_\infty};\ \ \bar{x} = \frac{x}{L};$$

$$\bar{y} = \frac{y}{L};\ \ \bar{\eta} = \frac{\eta}{\eta_\infty};\ \ \bar{c_p} = \frac{c_p}{c_{p\infty}};\ \ \bar{\lambda} = \frac{\lambda}{\lambda_\infty}$$

introduce:

$$\Longrightarrow \; a_1 \, \bar{c_p} \left(\bar{\rho} \, \bar{u} \, \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{\rho} \, \bar{v} \, \frac{\partial \bar{T}}{\partial \bar{y}} \right) + a_2 \left(\bar{\rho} \, \bar{u} \, \frac{\partial \frac{\bar{u}^2}{2}}{\partial \bar{x}} + \bar{\rho} \, \bar{v} \, \frac{\partial \frac{\bar{u}^2}{2}}{\partial \bar{y}} \right)$$

$$= a_3 \, \bar{\eta} \, \bar{u} \, \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + a_4 \, \bar{\eta} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 + a_5 \, \bar{\lambda} \, \frac{\partial^2 \bar{T}}{\partial \bar{y}^2}$$

with

$$a_1 = \frac{c_{p\infty} \, \rho_\infty \, u_\infty \, T_\infty}{L}; \quad a_2 = \frac{\rho_\infty \, u_\infty^3}{L}; \quad a_3 = a_4 = \frac{\eta_\infty \, u_\infty^2}{L^2}; \quad a_5 = \frac{\lambda_\infty \, T_\infty}{L^2}$$

dimensionsless form via division of the equation with e. g. a_1:

11.13

b) $\Longrightarrow$ dimensionless parameters

$$(i) \quad K_1 = \frac{a_2}{a_1} = \frac{\rho_\infty \, u_\infty^3}{L} \, \frac{L}{c_{p\infty} \, \rho_\infty \, u_\infty \, T_\infty} = \frac{u_\infty^2 \, \gamma \, R}{\gamma \, R \, c_\infty^2} (\gamma - 1) = (\gamma - 1) \, M_\infty^2$$

$$(ii) \quad K_2 = \frac{a_3}{a_1} = \frac{a_4}{a_1} = \frac{\eta_\infty \, u_\infty^2}{L^2} \, \frac{L}{c_{p\infty} \, \rho_\infty \, u_\infty \, T_\infty} =$$

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$$\frac{\eta_\infty}{u_\infty\ \rho_\infty\ L}\ (\gamma - 1)\ \frac{u_\infty^2}{c_\infty^2}\ =\ \frac{1}{Re}\ (\gamma - 1)\ M_\infty^2$$

$$(iii)\ \ K_3\ =\ \frac{a_5}{a_1}\ =\ \frac{\lambda_\infty\ T_\infty}{L^2}\ \frac{L}{c_{p\infty}\ \rho_\infty\ u_\infty\ T_\infty}\ =\ \frac{\eta_\infty}{u_\infty\ \rho_\infty\ L}\ \frac{\lambda_\infty}{\eta_\infty\ c_{p\infty}}\ =\ \frac{1}{Re}\ \frac{1}{Pr}$$

c) $$K_1\ =\ K_2\ =\ 0\ \Longrightarrow\ (\gamma - 1)\ =\ 0\ \Longrightarrow\ \gamma\ =\ 1$$