Tutorenprogramm - Strömungsmechanik II

Potentialströmungen - Musterlösung

1. Aufgabe

1. Parallelströmung + Senke

   $F(z) = u_\infty z + \frac{E}{2\pi} \ln z$

2. $F(z) = u_\infty (x + iy) + \frac{E}{2\pi} \ln(re^{i\varphi})$

   $\rightarrow \phi = u_\infty x + \frac{E}{2\pi} \ln r, \quad r = \sqrt{x^2 + y^2}$

   $u = \frac{\partial \phi}{\partial x} = u_\infty + \frac{Ex}{2\pi(x^2 + y^2)}, \quad v = \frac{\partial \phi}{\partial y} = \frac{Ey}{2\pi(x^2 + y^2)}$

   Staupunkt: $u = v = 0; \qquad v = 0 \quad \rightarrow \quad y_s = 0, \quad u = 0 \quad \rightarrow \quad x_s = -\frac{E}{2\pi u_\infty}$

   mit $E = -\frac{2\dot{V}}{B} \quad \rightarrow x_s = \frac{\dot{V}/B}{\pi u_\infty}, \quad y_s = 0$

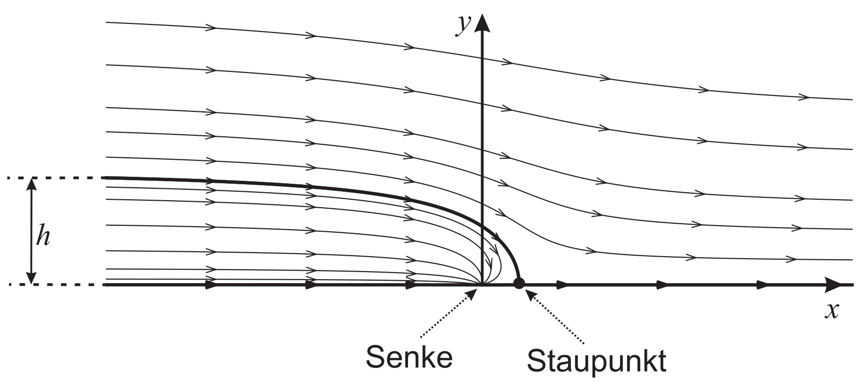
   Staupunktstromlinie:
   $\Psi = u_\infty y + \frac{E}{2\pi}\varphi; \quad \varphi_s = \arctan(y_s/x_s) = 0, \quad y_s = 0 \quad \Rightarrow \quad \Psi_s = 0$

   $0 = u_\infty r \sin\varphi + \frac{E}{2\pi}\varphi \quad \Rightarrow \quad r = \frac{\dot{V}}{\pi B u_\infty} \cdot \frac{\varphi}{\sin\varphi}$

3. Volumenstrombilanz: $\dot{V} < hBu_\infty \quad \Rightarrow \quad h > \frac{\dot{V}}{Bu_\infty}$

   oder $h = \lim_{\varphi \to \pi} y = \lim_{\varphi \to \pi} r \sin\varphi = \lim_{\varphi \to \pi} \frac{\dot{V}}{\pi B u_\infty}\varphi = \frac{\dot{V}}{Bu_\infty}$

4. Skizze:



Quelle: Herbst 2012

2. Aufgabe

1. Potentialfunktion:

   $$\Phi(r,\varphi) = u_\infty r\cos\varphi + v_\infty r\sin\varphi + \frac{E}{2\pi}\ln r + \frac{\Gamma}{2\pi}\varphi \text{ oder}$$

   $$\Psi(r,\varphi) = u_\infty r\sin\varphi - v_\infty r\cos\varphi + \frac{E}{2\pi}\varphi - \frac{\Gamma}{2\pi}\ln r$$

   $$\Rightarrow v_r(r,\varphi) = \frac{\partial\Phi}{\partial r} = u_\infty\cos\varphi + v_\infty\sin\varphi + \frac{E}{2\pi r}$$

   $$\Rightarrow v_\varphi(r,\varphi) = -\frac{\partial\Psi}{\partial r} = -u_\infty\sin\varphi + v_\infty\cos\varphi + \frac{\Gamma}{2\pi r}$$

2. Staupunkt:

   $$v_r = 0 \quad\Leftrightarrow\quad r = \frac{-E}{2\pi(u_\infty\cos\varphi + v_\infty\sin\varphi)}$$

   $$v_\varphi = 0 \quad\Leftrightarrow\quad r = \frac{-\Gamma}{2\pi(v_\infty\cos\varphi - u_\infty\sin\varphi)}$$

   $$\Rightarrow -E\,(v_\infty\cos\varphi - u_\infty\sin\varphi) = -\Gamma\,(u_\infty\cos\varphi + v_\infty\sin\varphi)$$

   $$-E\,(v_\infty - u_\infty\tan\varphi) = -\Gamma\,(u_\infty + v_\infty\tan\varphi)$$

   $$\Rightarrow \varphi = \arctan\left(\frac{Ev_\infty - \Gamma u_\infty}{\Gamma v_\infty + Eu_\infty}\right)$$

   Berechne v_∞ so, dass $\varphi_s = 45° = \frac{\pi}{4} \quad\Leftrightarrow\quad \frac{Ev_\infty - \Gamma u_\infty}{\Gamma v_\infty + Eu_\infty} = 1$

   $$v_\infty(E-\Gamma) = u_\infty(E+\Gamma) \quad\Rightarrow\quad v_\infty = u_\infty\frac{E+\Gamma}{E-\Gamma}$$

   Koordinaten des Staupunktes: $\varphi_s = \frac{\pi}{4}$

   $$r_s = \frac{-E}{2\pi\left(u_\infty\frac{1}{\sqrt{2}} + u_\infty\frac{E+\Gamma}{E-\Gamma}\frac{1}{\sqrt{2}}\right)}$$

   $$= -\frac{(E-\Gamma)}{2\sqrt{2}\pi u_\infty}$$

3. Stromfunktion:

   $$\Psi(r,\varphi) = u_\infty r\sin\varphi - v_\infty r\cos\varphi + \frac{E}{2\pi}\varphi - \frac{\Gamma}{2\pi}\ln r$$

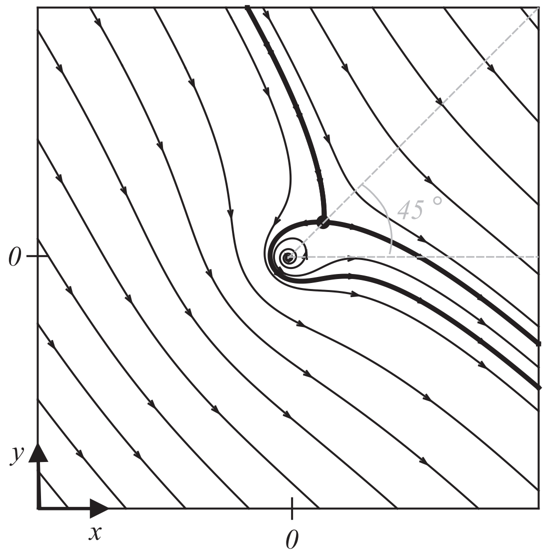
   Punkt *London* hat Koordinaten $r = 1$, $\varphi = -\frac{\pi}{4} \quad\Rightarrow \Psi_L = -\frac{u_\infty}{\sqrt{2}} - \frac{v_\infty}{\sqrt{2}} - \frac{E}{2\pi}\frac{\pi}{4} - \frac{\Gamma}{2\pi}\ln 1$

   $$= -\frac{u_\infty}{\sqrt{2}}\left(1 + \frac{E+\Gamma}{E-\Gamma}\right) - \frac{E}{8} = -\frac{\sqrt{2}u_\infty E}{E-\Gamma} - \frac{E}{8}$$

   Stromfunktion im Ursprung: $\quad\Psi(0,0) \to \infty$

4. Berechne erst den Wert der Stromfunktion auf der Staupunktstromlinie Ψ_s. Der Ursprung liegt sicher in der Aschewolke (Quelle der Aschewolke liegt im Ursprung) und besitzt ein Maximum der Stromfunktion. Daher entscheide, ob $\Psi_L \in [\Psi_s, \Psi(0,0)]$. In diesem Fall befindet sich London unter der Aschewolke. Andernfalls befindet sich saubere Luft über London.

5. Skizze:



Quelle: Herbst 2010

3. Aufgabe

1.
$$F(z) = \frac{a}{n} \cdot r^n \cdot e^{i\varphi n} = \frac{a}{n} r^n \cdot (\cos(n\varphi) + i \sin(n\varphi)) = \Phi + i\Psi$$
$$\rightarrow \Phi = \frac{a}{n} \cdot r^n \cdot \cos(n\varphi), \Psi = \frac{a}{n} \cdot r^n \cdot \sin(n\varphi)$$

2.
$$\bar{w} = \frac{dF}{dz} = u - iv = a \cdot z^{n-1} = a \cdot r^{n-1} \cdot \cos((n-1)\varphi) + i \cdot a \cdot r^{n-1} \sin((n-1)\varphi)$$
$$\rightarrow u = a \cdot r^{n-1} \cdot \cos((n-1)\varphi), v = \cdot a \cdot r^{n-1} \sin((n-1)\varphi)$$

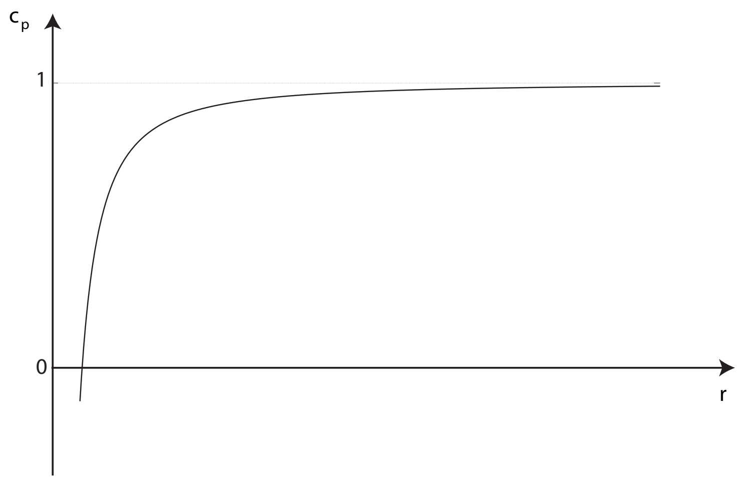
3. Wand ist Stromlinie $\Rightarrow \Psi = konst. = C = \frac{a}{n} r^n \cdot \sin(n\varphi)$

$$\begin{array}{llll} \text{für} & y = 0, x \geq 0: & r = x, \varphi = 0 & \Rightarrow C = \frac{a}{n} \cdot x^n \cdot \sin(0°) = 0 \\ & y \leq 0, x = 0: & r = y, \varphi = \frac{3}{2}\pi & \Rightarrow C = \frac{a}{n} \cdot y^n \cdot \sin(n \cdot \frac{3}{2}\pi) = 0 \end{array}$$
$$\Rightarrow \sin(n \cdot \frac{3}{2}\pi) = 0 \Rightarrow n = \frac{2}{3}, \quad \text{da} \quad 0 < n < 1 \quad \text{(siehe Skizze)}$$

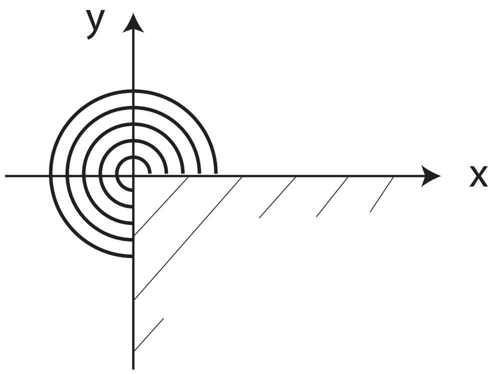
alternativ: konvexe Ecke, Umlenkwinkel $\Delta\varphi = \frac{3}{2}\pi$, Ergänzungswinkel: $\Theta = \pi - \Delta\varphi = (n-1)\frac{\pi}{2} = -\frac{\pi}{2} \Rightarrow n = \frac{2}{3}$
$$u(r = L, \Theta = 0°) = u_L \Rightarrow a = u_L \cdot L^{\frac{1}{3}}$$

4.
$$c_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty u_L^2} = 1 - \frac{V^2}{u_L^2}$$

mit
$$V = \sqrt{u^2 + v^2}$$
$$\Rightarrow c_p = 1 - \frac{a^2 \cdot r^{-\frac{2}{3}}}{u_L^2} = 1 - \left(\frac{L}{r}\right)^{\frac{2}{3}}$$



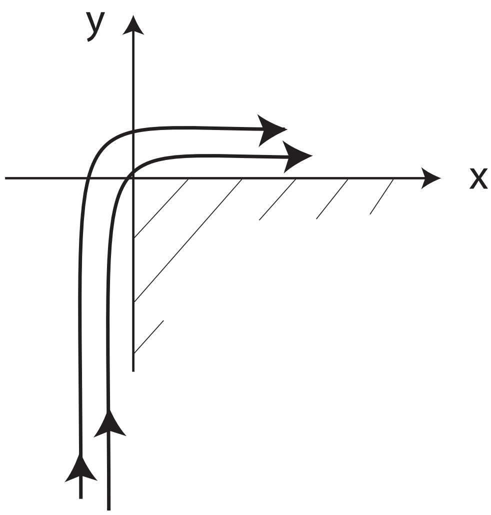
5. $c_p = konst :\Rightarrow r = konst.$

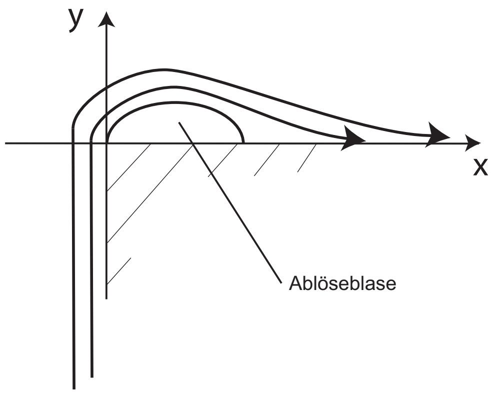


6. Stromlinie:

$$\Psi = \frac{a}{\frac{2}{3}} \cdot r^{\frac{2}{3}} \cdot \sin(\frac{2}{3}\varphi) = konst.$$

$$\Rightarrow r = \left(\frac{\frac{2}{3}K}{u_L \cdot L^{\frac{1}{3}} \sin(\frac{2}{3}\Theta)} \right)^{\frac{3}{2}}$$





Quelle: Herbst 2013

4. Aufgabe

1.
$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \eta \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad \Big| \cdot \frac{\partial}{\partial y} \qquad (1)$$
$$\rho \frac{dv}{dt} = -\frac{\partial p}{\partial y} + \eta \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad \Big| \cdot \frac{\partial}{\partial x} \qquad (2)$$

$$\Rightarrow \rho \frac{\partial}{\partial y} \frac{du}{dt} = -\frac{\partial}{\partial y} \left(\frac{\partial p}{\partial x} \right) + \eta \frac{\partial}{\partial y} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad (1)$$
$$\Rightarrow \rho \frac{\partial}{\partial x} \frac{dv}{dt} = -\frac{\partial}{\partial x} \left(\frac{\partial p}{\partial y} \right) + \eta \frac{\partial}{\partial x} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \qquad (2)$$

(1) - (2)

$$\Rightarrow \rho \left[\frac{\partial}{\partial y} \left(\frac{du}{dt} \right) - \frac{\partial}{\partial x} \left(\frac{dv}{dt} \right) \right] = \eta \left[\frac{\partial}{\partial y} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial}{\partial x} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right]$$

$$\Rightarrow \rho \left[\frac{\partial}{\partial y} \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v \right) - \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} u + \frac{\partial v}{\partial y} v \right) \right]$$
$$= \eta \left[\frac{\partial}{\partial y} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \frac{\partial}{\partial x} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \right] = -\eta \nabla^2 2\omega_z$$

mit $\omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$

$$\Rightarrow \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial t} \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} u \right) + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} v \right)$$
$$-\frac{\partial}{\partial x} \left(\frac{\partial v}{\partial t} \right) - \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial x} u \right) - \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial y} v \right) = -\frac{\eta}{\rho} \nabla^2 2\omega_z$$

$$\Rightarrow \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial t} \right) - \frac{\partial}{\partial x} \left(\frac{\partial v}{\partial t} \right) + \frac{\partial^2 u}{\partial x \partial y} u + \frac{\partial^2 u}{\partial y^2} v - \frac{\partial^2 v}{\partial x \partial y} v - \frac{\partial^2 v}{\partial x^2} u$$
$$+ \underbrace{\frac{\partial u}{\partial y} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) - \frac{\partial v}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)}_{=0 \quad \text{(Konti)}} = -\frac{\eta}{\rho} \nabla^2 2\omega_z$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + u \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) + v \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) = -\frac{\eta}{\rho} \nabla^2 2\omega_z$$

$$\Rightarrow -2 \frac{\partial \omega_z}{\partial t} - 2u \frac{\partial \omega_z}{\partial x} - 2v \frac{\partial \omega_z}{\partial y} = -\frac{\eta}{\rho} \nabla^2 2\omega_z$$

$$\Rightarrow \frac{\partial \omega_z}{\partial t} + u \frac{\partial \omega_z}{\partial x} + v \frac{\partial \omega_z}{\partial y} = \frac{\eta}{\rho} \nabla^2 \omega_z$$

$$\Rightarrow \frac{d\omega_z}{dt} = \frac{\eta}{\rho} \nabla^2 \omega_z$$

2. Für die obige Transportgleichung ist $\omega_z =$ konst. eine Lösung.
   Für reibungsfreie Strömung gilt zusätzlich: $\frac{d\omega_z}{dt} = 0$, sodass $\omega_z = 0$ eine Lösung ist.

Quelle: Herbst 2013