Tutorenprogramm - Strömungsmechanik II

Laminare Grenzschichten - Musterlösung

1. Aufgabe

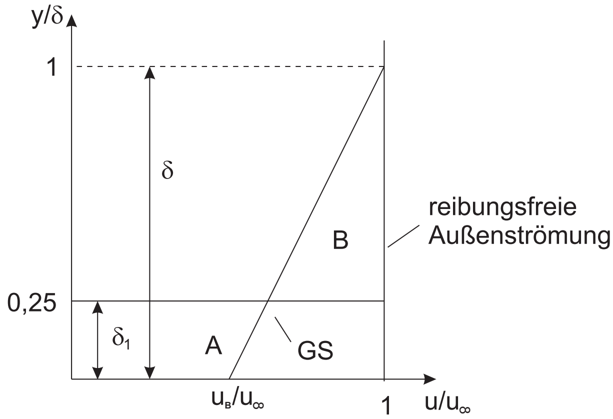
a) $\frac{u_1(y)}{u_\infty} = a_0 + a_1(\frac{y}{\delta})$

1. RB.: $y = 0 : u = u_B \rightarrow a_0 = \frac{u_B}{u_\infty} = K$

2. RB.: $y = \delta : u = u_\infty \rightarrow a_0 + a_1 = 1 \rightarrow a_1 = 1 - K$

$\rightarrow \frac{u(\frac{y}{\delta})}{u_\infty} = K + (1-K)\frac{y}{\delta}$

b) $K = 0.5$



Fläche A entspricht Fläche B

$\frac{\delta_1}{\delta} = \int_0^1 (1 - \frac{u}{u_\infty})d(\frac{y}{\delta}) = \frac{1}{2}\int_0^1 (1 - \frac{y}{\delta})d(\frac{y}{\delta}) = 0.25$

c) Ebene Grenzschichtsrömung: $\frac{\partial p}{\partial x} = 0$

mit Euler-Gleichung $u_a\frac{du_a}{dx} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \rightarrow \frac{du_a}{dx} = 0$

$\rightarrow u_a = u_\infty = \text{const}$

v. Karman Int.-bez.: $\frac{d\delta_2}{dx} + \cancelto{0}{\frac{1}{u_a}\frac{du_a}{dx}(2\delta_2 + \delta_1)} - \frac{\tau_w}{\rho u_a^2} = 0$ (*)

$\tau_w = -\tau(y=0) = \eta\frac{\partial u}{\partial y}|_{y=0} = \eta\frac{u_\infty}{\delta}\frac{\partial(\frac{u}{u_\infty})}{\partial(\frac{y}{\delta})}|_{\frac{y}{\delta}=0} = \eta\frac{u_\infty}{\delta}(1-K)$

$\frac{\delta_2}{\delta} = \int_0^1 [\frac{u}{u_\infty}(1 - \frac{u}{u_\infty})]d(\frac{y}{\delta})$

$= \int_0^1 [K - K^2 + (2K^2 - 3K + 1)\frac{y}{\delta} - (1-K)^2(\frac{y}{\delta})^2]d(\frac{y}{\delta})$

$\rightarrow \frac{\delta_2}{\delta} = \frac{1}{6} + \frac{1}{6}K - \frac{1}{3}K^2 =: C$

einsetzen in (*) : $C\frac{d\delta}{dx} - \eta\frac{u_\infty}{\delta}\frac{(1-K)}{\rho u_\infty^2} = 0$

$\rightarrow \delta d\delta = \eta\frac{(1-K)}{C\rho u_\infty}dx$

Integration: $\frac{1}{2}\delta^2(x) = \eta\frac{(1-K)}{C\rho u_\infty}x + C_1$ mit RB: $x \rightarrow 0$: $\delta \rightarrow 0 \rightarrow C_1 = 0$

$\rightarrow \delta(x) = \sqrt{\frac{2\eta(1-K)}{C\rho u_\infty}x}$

d) $P = Fu_B = \int_0^L \tau_W B dx u_B$

$\rightarrow \frac{P}{B} = u_B \int_0^L \eta \frac{u_\infty}{\delta(x)}(1-K)dx$

$= \eta u_B u_\infty (1-K) \int_0^L (\frac{2\eta(1-K)}{C\rho u_\infty} x)^{-\frac{1}{2}} dx$

$= \frac{\eta K u_\infty^2 (1-K)}{\sqrt{\frac{2\eta(1-K)}{C\rho u_\infty}}} \int_0^L x^{-\frac{1}{2}} dx$

$= K u_\infty^2 \sqrt{\frac{\eta(1-K)C\rho u_\infty}{2}} 2\sqrt{L}$

$= K u_\infty^2 \sqrt{2\eta(1-K)C\rho u_\infty L}$

Quelle: Herbst 2008

2. Aufgabe

1.

$$\begin{aligned} &\text{Haftbedingung:} && \frac{y}{\delta} = 0 \to \frac{u}{u_a} = 0 && \Rightarrow && a_0 = 0 \\ &\text{Grenzschichtrand:} && \frac{y}{\delta} = 1 \to \frac{u}{u_a} = 1 && \Rightarrow && a_1 = 1 \\ &\Rightarrow && \frac{u}{u_a} = \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) \end{aligned}$$

2. $p(x=0) = p_0 \quad \Rightarrow \quad$ Bernoulli:

$$p(x) + \frac{\rho}{2}u_a^2 = p_0 - C\frac{x^2}{2} + \frac{\rho}{2}u_a^2 = p_0$$

$$u_a = x\sqrt{\frac{C}{\rho}}$$

3. $$\delta_1 = \delta\int_0^1 \left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \delta\int_0^1 \left(1 - \sin\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right) d\left(\frac{y}{\delta}\right) = \delta\left[\frac{y}{\delta} + \frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right]_0^1$$

$$\Rightarrow \delta_1 = \left(1 - \frac{2}{\pi}\right)\delta = k_1\delta$$

$$\delta_2 = \delta\int_0^1 \frac{u}{u_a}\left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \delta\int_0^1 \left(\sin\left(\frac{\pi}{2}\frac{y}{\delta}\right) - \sin^2\left(\frac{\pi}{2}\frac{y}{\delta}\right)\right) d\left(\frac{y}{\delta}\right)$$

$$= \delta\left[-\frac{2}{\pi}\cos\left(\frac{\pi}{2}\frac{y}{\delta}\right) - \frac{1}{2}\frac{y}{\delta} + \frac{1}{2\pi}\sin\left(\frac{\pi y}{\delta}\right)\right]_0^1$$

$$\Rightarrow \delta_2 = \left(\frac{2}{\pi} - \frac{1}{2}\right)\delta = k_2\delta$$

$$\tau(y=0) = -\eta\left.\frac{du}{dy}\right|_{y=0} = -\frac{\eta\, u_a}{\delta}\left.\frac{d\left(\frac{u}{u_a}\right)}{d\left(\frac{y}{\delta}\right)}\right|_{\frac{y}{\delta}=0} = -\frac{\eta\, u_a}{\delta}\frac{\pi}{2}$$

4. Einsetzen in die von Kármánsche Integralbeziehung:

$$k_2\frac{d\delta}{dx} + \sqrt{\frac{\rho}{C}}\frac{1}{x}\sqrt{\frac{C}{\rho}}\,(2k_2 + k_1)\,\delta - \frac{\eta\, x\, \pi\, \rho}{2\,\delta\,\rho\, x^2\, C}\sqrt{\frac{C}{\rho}} = 0$$

Umformen:

$$\frac{d\delta}{dx} + \underbrace{\frac{2\,k_2 + k_1}{k_2}}_{\Gamma}\frac{\delta}{x} - \underbrace{\frac{\eta\,\pi}{2k_2\sqrt{C\,\rho}}}_{\Omega}\frac{1}{\delta\, x} = 0$$

$$\frac{d\delta}{dx} + \Gamma\frac{\delta}{x} - \Omega\frac{1}{\delta\, x} = 0$$

Differentialgleichung lösen:

$$x\frac{d\delta}{dx} = \Omega\frac{1}{\delta} - \Gamma\delta = \frac{\Omega - \Gamma\delta^2}{\delta} = \Gamma\left(\frac{\frac{\Omega}{\Gamma} - \delta^2}{\delta}\right)$$

$$\Rightarrow \quad \frac{1}{\Gamma}\left(\frac{\delta}{\frac{\Omega}{\Gamma} - \delta^2}\right) d\delta = \frac{1}{x}dx$$

$$\Rightarrow \quad -\frac{1}{2\Gamma}\ln\left(\frac{\Omega}{\Gamma} - \delta^2\right)\Bigg|_{\delta_0}^{\delta} = \ln x|_{x_0}^{x}$$

$$\Rightarrow \quad -\frac{1}{2\Gamma}\left[\ln\frac{\frac{\Omega}{\Gamma}-\delta^2}{\frac{\Omega}{\Gamma}-\delta_0^2}\right]=\ln\left(\frac{x}{x_0}\right)$$

Auflösen nach $\delta(x)$:

$$\left(\frac{\frac{\Omega}{\Gamma}-\delta^2}{\frac{\Omega}{\Gamma}-\delta_0^2}\right)^{-\frac{1}{2\Gamma}}=\frac{x}{x_0}$$

$$\Rightarrow \frac{\Omega}{\Gamma}-\delta^2=\left(\frac{x}{x_0}\right)^{-2\Gamma}\left(\frac{\Omega}{\Gamma}-\delta_0^2\right)$$

$$\Rightarrow \delta=\sqrt{\frac{\Omega}{\Gamma}-\left(\frac{x}{x_0}\right)^{-2\Gamma}\left(\frac{\Omega}{\Gamma}-\delta_0^2\right)}$$

mit

$$\Omega=\frac{\eta\,\pi}{2\,k_2\sqrt{C\,\rho}}=\frac{\eta\,\pi}{2\left(\frac{2}{\pi}-\frac{1}{2}\right)\sqrt{C\,\rho}},\qquad \Gamma=\frac{2k_2+k_1}{k_2}=\frac{2\left(\frac{2}{\pi}-\frac{1}{2}\right)+\left(1-\frac{2}{\pi}\right)}{\left(\frac{2}{\pi}-\frac{1}{2}\right)}=\frac{4}{4-\pi}$$

Quelle: Frühjahr 2010

3. Aufgabe

1. Randbedingungen:

   (i) $y = 0 \quad \Rightarrow \quad u = 0$ (Haftbedingung)

   (ii) $y = \delta \quad \Rightarrow \quad u = u_a$ (Grenzschichtrand)

   (iii) $y = 0$: Wandbindung aus x-Impulsgleichung für $y = 0$:

   $$v(y=0)\left.\frac{\partial u}{\partial y}\right|_{y=0} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\eta}{\rho}\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}$$

   mit $v(y=0) = -\frac{\dot{V}}{L\,B}$ und $\frac{\partial p}{\partial x} = 0$ (ebene Platte ohne Druckgradient) folgt:

   $$-\frac{\dot{V}}{L\,B}\left.\frac{\partial u}{\partial y}\right|_{y=0} = \frac{\eta}{\rho}\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}$$

2. Koeffizienten:

   Aus (i) folgt: $a_0 = 0$

   Aus (ii) folgt: $a_1 + a_2 = 1$

   Aus (iii) folgt mit $\left.\frac{\partial u}{\partial y}\right|_{y=0} = u_a\frac{a_1}{\delta}$ und $\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = \frac{2a_2u_a}{\delta^2}$:

   $$-\frac{\dot{V}}{L\,B}\frac{a_1u_a}{\delta} = \frac{2\eta}{\rho}\frac{a_2u_a}{\delta^2} \quad \Rightarrow \quad a_2 = -\frac{\dot{V}\,\rho\,\delta}{2\,L\,B\,\eta}a_1$$

   $$\Leftrightarrow \quad 1 - a_1 = -\frac{\dot{V}\,\rho\,\delta}{2\,L\,B\,\eta}a_1 \quad \Rightarrow \quad a_1 = \frac{1}{1-\frac{\dot{V}\,\rho\,\delta}{2\,L\,B\,\eta}}, \quad a_2 = 1 - a_1 = 1 - \frac{1}{1-\frac{\dot{V}\,\rho\,\delta}{2\,L\,B\,\eta}}$$

   $$\left[\Rightarrow \frac{u}{u_a} = \frac{1}{1-\frac{\dot{V}\,\rho\,\delta}{2\,L\,B\,\eta}}\left(\frac{y}{\delta}\right) + \left(1 - \frac{1}{1-\frac{\dot{V}\,\rho\,\delta}{2\,L\,B\,\eta}}\right)\left(\frac{y}{\delta}\right)^2\right]$$

3. Ablöseprofil an Ablösestelle x_{ab} für gegebenes Profil:

   Randbedingungen (i) und (ii) immer noch gültig $\quad \Rightarrow \quad a_0 = 0, \quad a_1 + a_2 = 1$

   Randbedingung (iii) jetzt Ablösebedingung: $\tau(y=0) = 0 \quad \Leftrightarrow \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$

   $\Rightarrow \quad a_1 = 0 \quad$ und damit: $a_2 = 1 - a_1 = 1$

   Geschwindigkeitsprofil: $\quad u = u_a(x_{ab})\left(\frac{y}{\delta(x_{ab})}\right)^2$

Quelle: Frühjahr 2012

4. Aufgabe

a) Konti: $u_0 H_0 = u_a(x) H(x) \quad \Rightarrow \quad u_a(x) = \dfrac{u_0 H_0}{H(x)} = \dfrac{H_0 u_0}{H_0 - c\frac{x}{L}}$

$$\Rightarrow \quad \frac{du_a}{dx} = \frac{H_0 u_0}{\left(H_0 - c\frac{x}{L}\right)^2 \frac{L}{c}}$$

Euler: $\dfrac{dp}{dx} = -\varrho u_a \dfrac{du_a}{dx}$

$$\Rightarrow \quad \frac{dp}{dx} = -\frac{\varrho {u_0}^2 {H_0}^2}{\frac{L}{c}\left(H_0 - c\frac{x}{L}\right)^3}$$

b) 1.RB: $u(y=0) = 0 \quad \Rightarrow \quad a_0(x) = 0$

2.RB: $u(y=\delta) = u_a \quad \Rightarrow \quad a_1(x) + a_2(x) = 1$

3.RB: $\eta \left.\dfrac{\partial^2 u}{\partial y^2}\right|_{y=0} = \dfrac{dp}{dx} \quad \Rightarrow \quad \eta \dfrac{u_a}{\delta^2} 2a_2(x) = -\dfrac{\varrho {u_0}^2 {H_0}^2}{\frac{L}{c}\left(H_0 - c\frac{x}{L}\right)^3}$

$$\Rightarrow \quad a_2(x) = -\frac{\varrho u_0 H_0 \delta^2}{2\eta\frac{L}{c}\left(H_0 - c\frac{x}{L}\right)^2}$$

$$\Rightarrow \quad a_1(x) = 1 + \frac{\varrho u_0 H_0 \delta^2}{2\eta\frac{L}{c}\left(H_0 - c\frac{x}{L}\right)^2}$$

c) $$\frac{\delta_1(x)}{\delta(x)} = \int_0^1 \left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \int_0^1 \left(1 - a_1\frac{y}{\delta} - a_2\left(\frac{y}{\delta}\right)^2\right) d\left(\frac{y}{\delta}\right)$$

$$= \left[\frac{y}{\delta} - \frac{a_1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{a_2}{3}\left(\frac{y}{\delta}\right)^3\right]_0^1 = 1 - \frac{a_1}{2} - \frac{a_2}{3}$$

$$\Rightarrow \quad \delta_1(x) = \delta(x)\left(\frac{1}{2} - \frac{\varrho u_0 H_0 \delta^2}{4\eta\frac{L}{c}\left(H_0 - c\frac{x}{L}\right)^2} + \frac{\varrho u_0 H_0 \delta^2}{6\eta\frac{L}{c}\left(H_0 - c\frac{x}{L}\right)^2}\right)$$

d) Nein, da konvergenter Kanal mit $\dfrac{dp}{dx} < 0$.

e) Der Reibungswiderstand wird vergrößert, da der Gradient der Geschwindigkeit an der Wand $\left.\dfrac{\partial u}{\partial y}\right|_{y=0}$ bei turbulenten Strömungen größer ist.

Quelle: Herbst 2008