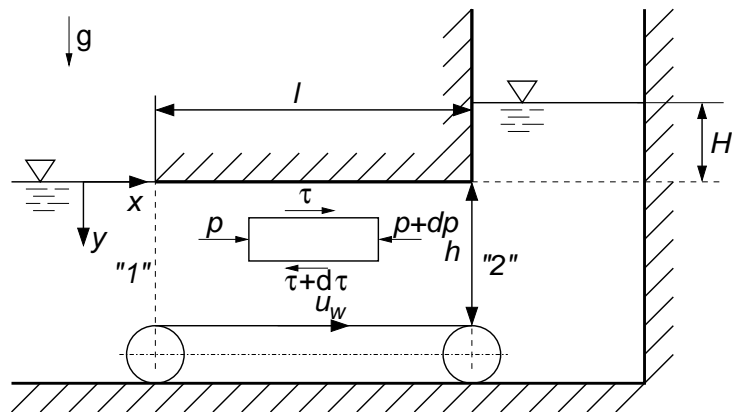


Tutorenprogramm - Strömungsmechanik I

Laminare Reibungsbehaftete Strömungen - Musterlösung

1. Aufgabe

1. Couette-Strömung (mit Druckgradient).
2. Kräftebilanz am Volumenelement in x-Richtung:



$$\left(p - \left(p + \frac{\partial p}{\partial x} dx \right) \right) dy + \left(\tau - \left(\tau + \frac{d\tau}{dy} dy \right) \right) dx = 0$$

$$\frac{d\tau}{dy} = -\frac{\partial p}{\partial x}, \quad \rightarrow \frac{\partial p}{\partial x} = \eta \frac{d^2 u}{dy^2}$$

Bernoulli/Hydrostatik :

$$p_2 = p_1 + \rho g H, \quad \frac{\partial p}{\partial x} = \frac{\rho g H}{l}$$

$$\rightarrow \frac{\rho g H}{l} = \eta \frac{d^2 u}{dy^2}$$

$$\rightarrow u(y) = \frac{\rho g H}{2\eta l} y^2 + c_1 y + c_2$$

Randbedingungen:

$$\left. \begin{array}{l} y = 0 : \quad u = 0 \\ y = h : \quad u = u_w \end{array} \right\} \rightarrow u(y) = \frac{\rho g H}{2\eta l} (y^2 - yh) + y \frac{u_w}{h}$$

Maximale Höhe $\rightarrow \dot{V} = 0$

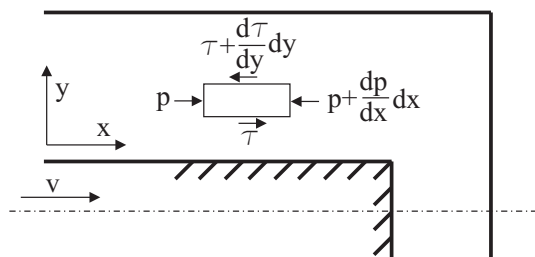
$$\rightarrow 0 = \int_0^h \left[\frac{\rho g H}{2\eta l} (y^2 - yh) + y \frac{u_w}{h} \right] dy$$

$$\rightarrow 0 = \frac{\rho g H}{2\eta l} \left(-\frac{h^3}{6} \right) + \frac{h u_w}{2}$$

$$\rightarrow H = \frac{6\eta l u_w}{\rho g h^2}$$

Quelle: Frühjahr 2009

2. Aufgabe



1. Bilanzierung an infinitesimal kleinem Element

$$\frac{dp}{dx} = -\frac{d\tau}{dy}$$

$$\tau = -\eta \frac{du}{dy} \implies \frac{d^2 u}{dy^2} = \frac{1}{\eta} \frac{dp}{dx}$$

1. Integration: $\frac{du}{dy} = \frac{1}{\eta} \frac{dp}{dx} y + C_1$ \wedge 2. Integration: $u = \frac{1}{2\eta} \frac{dp}{dx} y^2 + C_1 y + C_2$

R.B.:

$$\left. \begin{array}{l} 1) \quad y = 0 : \quad u = v \\ 2) \quad y = \frac{b_2 - b_1}{2} \equiv H : \quad u = 0 \end{array} \right\} \text{Haftbedingung}$$

aus 1) $C_2 = v$

aus 2) $0 = \frac{1}{2\eta} \frac{dp}{dx} H^2 + C_1 H + v = 0 \iff C_1 = -\frac{1}{2\eta} \frac{dp}{dx} H - \frac{v}{H}$

$$\implies u(y) = \frac{1}{2\eta} \frac{dp}{dx} H^2 \cdot \left(\frac{y^2}{H^2} - \frac{y}{H} \right) + v \cdot \left(1 - \frac{y}{H} \right)$$

$$\implies u(y) = \frac{1}{4\eta} \frac{dp}{dx} y \cdot (b_2 - b_1) \left(\frac{y}{b_2 - b_1} - 1 \right) + v \cdot \left(1 - \frac{2y}{b_2 - b_1} \right)$$

2. Volumenstrombilanz: $\dot{V} = 2 \cdot \int_0^H u(y) dy = -b_1 \cdot v$

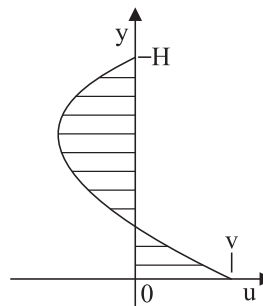
$$\int_0^H u(y) dy = \frac{1}{2\eta} \frac{dp}{dx} H^2 \left(\frac{1}{3} \frac{y^3}{H^2} - \frac{1}{2} \frac{y^2}{H} \right) + v \cdot \left(y - \frac{1}{2} \frac{y^2}{H} \right) \Big|_0^H$$

$$= -\frac{1}{12\eta} \frac{dp}{dx} H^3 + v \frac{H}{2} = -b_1 \frac{v}{2} \iff v \cdot \left(\frac{b_1}{2} + \frac{H}{2} \right) = \frac{1}{12\eta} \frac{dp}{dx} H^3$$

$$\iff \frac{1}{2\eta} \frac{dp}{dx} H^2 = 3 \frac{v}{H} (H + b_1) \iff \frac{dp}{dx} = \frac{6\eta v}{H^2} \cdot \left(1 + \frac{b_1}{H} \right)$$

$$\implies u(y) = 3v \cdot \left(1 + \frac{b_1}{H} \right) \left(\frac{y^2}{H^2} - \frac{y}{H} \right) + v \cdot \left(1 - \frac{y}{H} \right)$$

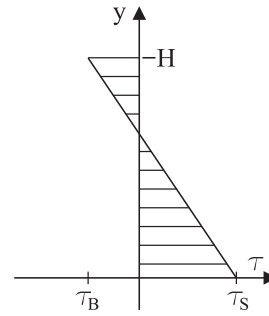
$$\implies u(y) = 6v \cdot \left(1 + \frac{2b_1}{b_2 - b_1} \right) \left(\frac{2y^2}{(b_2 - b_1)^2} - \frac{y}{b_2 - b_1} \right) + v \cdot \left(1 - \frac{2y}{b_2 - b_1} \right)$$



$$3. \quad \tau = -\eta \frac{du}{dy} = -\eta \left(3v \left(1 + \frac{b_1}{H} \right) \left(\frac{2y}{H^2} - \frac{1}{H} \right) - \frac{v}{H} \right)$$

$$\text{Stempel } y = 0: \quad \tau_S = 3\eta \frac{v}{H} \left(1 + \frac{b_1}{H} \right) + \eta \frac{v}{H}$$

$$\text{B-Wand } y = H: \quad \tau_B = -3\eta \frac{v}{H} \left(1 + \frac{b_1}{H} \right) + \eta \frac{v}{H}$$



Quelle: Frühjahr 2010

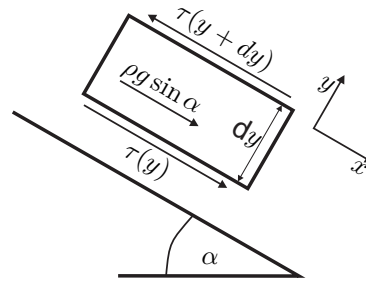
3. Aufgabe

a) Kräftebilanz am Volumenelement in x-Richtung:

$$\tau(y)dx - \tau(y + dy)dx + \rho g \sin \alpha dx dy = 0$$

$$\tau dx - \left(\tau + \frac{d\tau}{dy} dy \right) dx + \rho g \sin \alpha dx dy = 0$$

$$\frac{d\tau}{dy} = \rho g \sin \alpha \quad \left[\frac{kg}{m^2 s} \right]$$



- b)
1. Die Schubspannung ist am Übergang zur Luft null: $\tau(y = 2\delta) = 0$
 2. Die Schubspannung am Übergang der Fluide ist gleich: $\tau_1(y = \delta) = \tau_2(y = \delta)$
 3. Die Geschwindigkeit des Fluids an der Wand entspricht der Wandgeschwindigkeit (Haftbedingung): $u_1(y = 0) = -u_W$
 4. Die Geschwindigkeit am Übergang der Fluide ist gleich: $u_1(y = \delta) = u_2(y = \delta)$

c) Schubspannungsverteilung in den beiden Fluiden:

$$\tau_1 = \rho g \sin \alpha y + c_1$$

$$\tau_2 = \rho g \sin \alpha y + c_2$$

Mit Randbedingung 1:

$$\tau_2(2\delta) = 0 \Rightarrow c_2 = -2\rho g \sin \alpha \delta$$

$$\tau_2 = \rho g \sin \alpha y - 2\rho g \sin \alpha \delta$$

$$\tau_2 = \rho g \sin \alpha (y - 2\delta)$$

Einsetzen von Randbedingung 2:

$$\tau_1(y = \delta) = \tau_2(y = \delta) \Rightarrow \rho g \sin \alpha \delta + c_1 = \rho g \sin \alpha \delta - 2\rho g \sin \alpha \delta$$

$$c_1 = c_2 = -2\rho g \sin \alpha \delta$$

$$\tau_1 = \rho g \sin \alpha (y - 2\delta)$$

$$\text{Newtonsches Fluid: } \tau_n = -\eta_n \frac{du}{dy}$$

$$\frac{du_1}{dy} = \frac{\rho g \sin \alpha}{\eta_1} (-y + 2\delta)$$

$$\frac{du_2}{dy} = \frac{\rho g \sin \alpha}{\eta_2} (-y + 2\delta)$$

Unbestimmte Integration:

$$u_1 = \frac{\rho g \sin \alpha}{\eta_1} \left(-\frac{y^2}{2} + 2\delta y \right) + c_3$$

$$u_2 = \frac{\rho g \sin \alpha}{\eta_2} \left(-\frac{y^2}{2} + 2\delta y \right) + c_4$$

$$\text{Einsetzen von Randbedingung 3: } u_1(y = 0) = -u_W \Rightarrow c_3 = -u_W$$

$$\Rightarrow u_1 = \frac{\rho g \sin \alpha}{\eta_1} \left(-\frac{y^2}{2} + 2\delta y \right) - u_W$$

$$\text{Anwenden von Randbedingung 4: } u_1(y = \delta) = u_2(y = \delta)$$

$$\Rightarrow \frac{\rho g \sin \alpha}{\eta_1} \left(-\frac{\delta^2}{2} + 2\delta^2 \right) - u_W = \frac{\rho g \sin \alpha}{\eta_2} \left(-\frac{\delta^2}{2} + 2\delta^2 \right) + c_4$$

$$\Leftrightarrow \frac{\rho g \sin \alpha}{\eta_1} \frac{3\delta^2}{2} - u_W = \frac{\rho g \sin \alpha}{\eta_2} \frac{3\delta^2}{2} + c_4$$

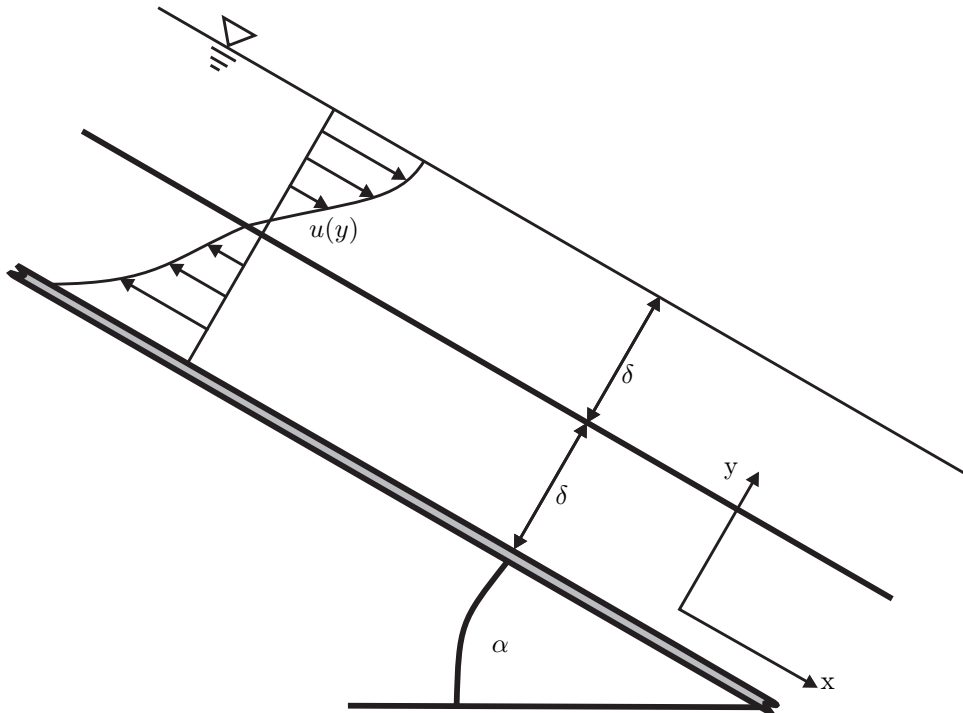
$$\Rightarrow c_4 = \frac{3\rho g \sin \alpha}{2} \delta^2 \left(\frac{1}{\eta_1} - \frac{1}{\eta_2} \right) - u_W$$

Geschwindigkeitsverteilung in beiden Fluiden:

$$0 \leq y \leq \delta : \quad u_1(y) = \frac{\rho g \sin \alpha}{\eta_1} \left(-\frac{y^2}{2} + 2\delta y \right) - u_W \quad \left[\frac{m}{s} \right]$$

$$\delta \leq y \leq 2\delta : \quad u_2(y) = \frac{\rho g \sin \alpha}{\eta_2} \left(-\frac{y^2}{2} + 2\delta y \right) + \frac{3\rho g \sin \alpha}{2} \delta^2 \left(\frac{1}{\eta_1} - \frac{1}{\eta_2} \right) - u_W \quad \left[\frac{m}{s} \right]$$

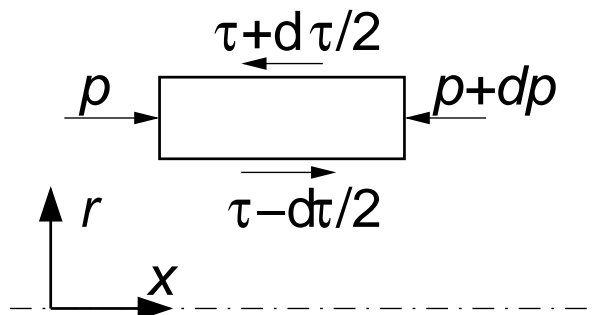
d) Qualitative Geschwindigkeitsverteilung im Fluid:



Quelle: Herbst 2013

4. Aufgabe

1. Der Geschwindigkeitsgradient besitzt ein negatives Vorzeichen.
2. Kräftebilanz am Volumenelement in x-Richtung:



$$\left(p - \left(p + \frac{dp}{dx} dx \right) \right) 2\pi r dr + \left(2\pi \left(r - \frac{dr}{2} \right) dx \right) \left(\tau - \frac{d\tau}{dr} \frac{dr}{2} \right) -$$

$$\left(2\pi \left(r + \frac{dr}{2} \right) dx \right) \left(\tau + \frac{d\tau}{dr} \frac{dr}{2} \right) = 0$$

$$-\frac{dp}{dx} - \frac{d\tau}{dr} - \frac{\tau}{r} = 0$$

$$-\frac{dp}{dx} - \frac{1}{r} \frac{d(\tau r)}{dr} = 0$$

$$\rightarrow \tau r = - \int_0^r \frac{dp}{dx} r dr$$

$$\rightarrow \tau = - \frac{dp}{dx} \frac{r}{2} + \frac{k_1}{r}$$

Symmetrie: $\tau(r=0) = 0, \rightarrow k_1 = 0, \rightarrow \tau = - \frac{dp}{dx} \frac{r}{2}$

Randbedingung für die Geschwindigkeit: $u(r=R) = 0$

Einsetzen der Beziehung für die Schubspannung:

$$c \sqrt{\left| \frac{du}{dr} \right|} = \frac{dp}{dx} \frac{r}{2}$$

$$\left| \frac{du}{dr} \right| = \left(\frac{dp}{dx} \right)^2 \frac{r^2}{4c^2}$$

Für $\frac{dp}{dx} < 0$ ist $\frac{du}{dr} < 0$.

$$\rightarrow \frac{du}{dr} = - \left(\frac{dp}{dx} \right)^2 \frac{r^2}{4c^2}$$

$$\rightarrow u = - \left(\frac{dp}{dx} \right)^2 \frac{r^3}{12c^2} + k_2$$

Mit der Randbedingung $u(r=R) = 0: \rightarrow u = \left(\frac{dp}{dx} \right)^2 \frac{1}{12c^2} (R^3 - r^3)$

3. Volumenstrom:

$$\dot{V} = \int_0^R 2\pi r u(r) dr = \frac{2\pi}{12c^2} \left(\frac{dp}{dx}\right)^2 \int_0^R R^3 r - r^4 dr = \frac{\pi}{20c^2} \left(\frac{dp}{dx}\right)^2 R^5$$

Mittlere Geschwindigkeit:

$$\bar{u} = \frac{\dot{V}}{\pi R^2} = \frac{1}{20c^2} \left(\frac{dp}{dx}\right)^2 R^3$$

Maximale Geschwindigkeit:

$$u_{max} = u(r=0) = \frac{1}{12c^2} \left(\frac{dp}{dx}\right)^2 R^3$$

$$\frac{u_{max}}{\bar{u}} = \frac{5}{3}$$

4. Mittlere Viskosität:

$$\bar{\eta} = \frac{1}{\pi R^2} \int_0^R 2\pi r \eta(r) dr$$

$$\frac{du}{dr} = -\frac{1}{4c^2} \left(\frac{dp}{dx}\right)^2 r^2$$

$$\eta = -\frac{2c^2}{\frac{dp}{dx} r}$$

$$\bar{\eta} = \frac{-1}{\pi R^2} \int_0^R \frac{4\pi c^2}{\frac{dp}{dx}} dr = \frac{-4c^2}{R} \frac{1}{\frac{dp}{dx}}$$

Quelle: Herbst 2009