Tutorenprogramm - Strömungsmechanik I
Wintersemester 2013/14
Kontinuitäts- und Bernoulligleichung - Musterlösung

1. Aufgabe

1. HGG und Bernoulli von $\boxed{0}$ nach $\boxed{k}$:

   außen: $p_{a_0} = p_{a_k} + \rho_L g(h_k - h_0) \quad \text{mit} \quad k = 1, 2$

   innen: $p_{a_0} = p_{a_k} + \frac{\rho_G}{2} v_{G_k}^2 + \rho_G g(h_k - h_0)$

   $$p_{a_0} + \rho_L g h_0 = p_{a_k} + \frac{\rho_G}{2} v_{G_k}^2 + \rho_G g(h_k - h_0) + \rho_L g h_0$$

   $$v_{G_k} = \sqrt{2g(h_k - h_0)(\frac{\rho_L}{\rho_G} - 1)}$$

   Konti: $v_{G_2} \frac{\pi d_2^2}{4} = v_{G_1} \frac{\pi d_1^2}{4}$

   $$\frac{v_{G_1}}{v_{G_2}} = \frac{d_2^2}{d_1^2} \quad \Rightarrow \quad \frac{d_1}{d_2} = \sqrt{\frac{v_{G_2}}{v_{G_1}}}$$

   $$\frac{d_1}{d_2} = \sqrt[4]{\frac{h_2 - h_0}{h_1 - h_0}}$$

2. Bernoulli von $\boxed{0}$ nach $\boxed{2}$:

   außen: $p_{a_0} = p_{a_2} + \rho_L g(h_2 - h_0)$

   innen: $p_{a_0} = p_{a_2} + (1 + \zeta_{Dr}) \frac{\rho_G}{2} v_{G_2}^2 + \rho_G g(h_2 - h_0)$

   $$d_1 = d_2 \quad , \quad \dot{V}_1 = \dot{V}_2 \quad \Rightarrow \quad v_{G_2} = v_{G_1}$$

   $$\zeta_{Dr} = \frac{h_2 - h_1}{h_1 - h_0}$$

Quelle: Herbst 2010

2. Aufgabe

1.
$$F = (p_a + \varrho_1 g H_1 - p_a - \varrho_2 g H_2)\frac{\pi d^2}{4} = \frac{\pi d^2 g}{4}(\varrho_1 H_1 - \varrho_2 H_2)$$

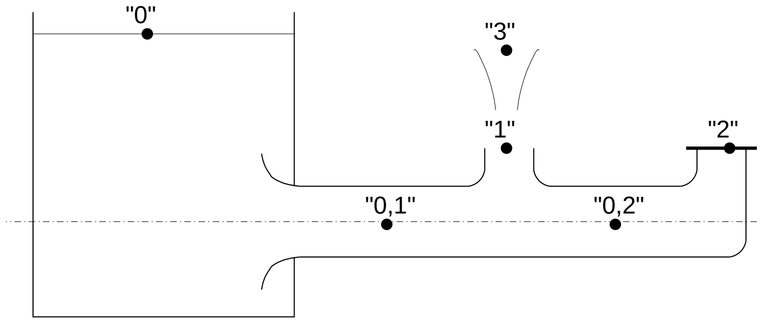
2. Druck in 2 ist gleich:
$$p_a + \varrho_1 g h_1 = p_a + \varrho_2 g h_2 + \varrho_1 g h_b$$
$$h_a + h_b = h_2 \quad \text{gleiches Volumen}: \varrho_1 A H_1 = \varrho_1 A h_1 + \varrho_1 A h_b$$
$$\Rightarrow h_b + H_1 - h_1 \quad \varrho_2 A H_2 = \varrho_2 A h_a \quad \Rightarrow h_a = H_2$$
$$\Rightarrow \varrho_1 g h_1 = \varrho_2 g H_2 + \varrho_1 g (H_1 - h_1) \Rightarrow h_1 = \frac{\varrho_2 H_2 + \varrho_1 H_1}{2\varrho_1}$$
$$h_2 = H_2 + H_1 - h_1$$

3. Bernoulli: $3 \to 4$
$$p_3 + \frac{\varrho_2}{2} v_3^2 = p_a + \frac{\varrho_2}{2} v_4^2 + \varrho_2 g h_a$$
Konti:
$$v_3 = v_4 = v_1 \quad h_a = H_2 \quad v_2 = v_1 \frac{4A}{\pi d^2} \quad v_1 = \frac{-dh_1}{dt}$$
Bernoulli: $1 \to 3$
$$p_a + \frac{\varrho_1}{2} v_1^2 + \varrho_1 g h_1 = p_3 + \frac{\varrho_1}{2} v_3^2 + \varrho_1 g h_b + \varrho_1 L \frac{dv_2}{dt}$$
$$\Rightarrow \varrho_1 g h_1 + \varrho_1 L \frac{4A}{\pi d^2} \frac{d^2 h_1}{dt^2} = \varrho_2 g H_2 + \varrho_1 g h_b$$
mit $h_b = H_1 - h_1$
$$\Rightarrow \underbrace{\varrho_1 L \frac{4A}{\pi d^2}}_{a} \frac{d^2 h_1}{dt^2} + \underbrace{2\varrho_1 g h_1}_{b} - \underbrace{(\varrho_2 g H_2 + \varrho_1 g H_1}_{c<0} = 0$$
DGL:
$$a \cdot \ddot{h_1} + b \cdot h_1 + c = 0$$

4. Hinweis:
$$h(t) = \frac{-c}{b} + C_1 \sin\sqrt{\frac{b}{a}} t + C_2 \cos\sqrt{\frac{b}{a}} t$$
R.B.:
$$h_1(t=0) = H_1 \quad \sin 0 = 0 \quad \cos 0 = 1 \quad \Rightarrow C_2 = H_1 + \frac{c}{b}$$
$$v_1(t=0) = 0 \quad \Rightarrow C_1 = 0$$

Quelle: Frühjahr 2011

3. Aufgabe



1. Bernoulli $0 \to 1$: $\quad p_a + \varrho g H = p_a + \varrho g H_1 + \frac{\varrho}{2} v_1^2 + \frac{\varrho}{2} v_{0,1}^2 \lambda \frac{L_1}{D}$

   Konti: $\quad v_{0,1} = \frac{v_1}{4}$

   $\to v_1 = \sqrt{\frac{2g(H-H_1)}{1+\frac{\lambda}{16}\frac{L_1}{D}}}$

   Bernoulli $1 \to 3$: $\quad \frac{\varrho}{2} v_1^2 = \varrho g h_1 \quad \to h_1 = \frac{H-H_1}{1+\frac{\lambda}{16}\frac{L_1}{D}}$

2. siehe a): Bernoulli $0 \to 1$: $\quad p_a + \varrho g H = p_a + \varrho g H_1 + \frac{\varrho}{2} \bar{v}_{0,1}^2 \lambda \frac{L_1}{D} + \frac{\varrho}{2} v_{1,b}^2 \quad (*)$

   Bernoulli $0 \to 2$: $\quad p_a + \varrho g H = p_a + \varrho g H_1 + \frac{\varrho}{2} \bar{v}_{0,1}^2 \lambda \frac{L_1}{D} + \frac{\varrho}{2} v_2^2 + \frac{\varrho}{2} v_{0,2}^2 \lambda \frac{L_2}{D} \quad (**)$

   Mit Konti: $v_{0,2} = \frac{v_2}{4}$

   $(*) - (**)$: $\quad v_2 = v_{1,b} \sqrt{\frac{1}{1+\frac{\lambda}{16}\frac{L_2}{D}}}$

   Konti: $\quad \bar{v}_{0,1} = \frac{v_{1,b} + v_2}{4}$

   In $(*)$: $\quad \varrho g (H - H_1) = \frac{\varrho}{2} v_{1,b}^2 \left(1 + \frac{\lambda}{16} \frac{L_1}{D} \left(1 + \sqrt{\frac{1}{1+\frac{\lambda}{16}\frac{L_2}{D}}} \right)^2 \right)$

   $\to v_{1,b} = \sqrt{2g(H-H_1) / \left(1 + \frac{\lambda}{16} \frac{L_1}{D} \left(1 + \sqrt{\frac{1}{1+\frac{\lambda}{16}\frac{L_2}{D}}} \right)^2 \right)}$

3. instat. Bernoulli $0 \to 2$: $\quad p_a + \varrho g H = p_a + \varrho g H_1 + \frac{\varrho}{2} v_2^2 + \frac{\varrho}{2} v_{0,1}^2 \lambda \frac{L_1}{D} + \frac{\varrho}{2} v_{0,2}^2 \lambda \frac{L_2}{D} + \varrho \int_0^2 \frac{\partial v}{\partial t} ds$

   $\to p_a + \varrho g H = p_a + \varrho g H_1 + \frac{\varrho}{2} v_2^2 + \frac{\varrho}{2} v_{0,1}^2 \lambda \frac{L_1}{D} + \frac{\varrho}{2} v_{0,2}^2 \lambda \frac{L_2}{D} + \varrho L_1 \frac{dv_{0,1}}{dt} + \varrho L_2 \frac{dv_{0,2}}{dt}$

   instat. Bernoulli $0 \to 1$: $\quad p_a + \varrho g H = p_a + \varrho g H_1 + \frac{\varrho}{2} v_1^2 + \frac{\varrho}{2} v_{0,1}^2 \lambda \frac{L_1}{D} + \varrho L_1 \frac{dv_{0,1}}{dt}$

   DGL: $\quad \frac{dv_2}{dt} + \underbrace{\frac{2}{L_2}(1 + \lambda \frac{L_2}{16D})}_{=k_2} v_2^2 \underbrace{- \frac{2}{L_2}}_{=k_1} v_1^2 = 0$

Quelle: Herbst 2009

4. Aufgabe

a) Kontinuitätsgleichung:

$$\dot{V} = v_3 \cdot \frac{\pi d^2}{4} = v_1 \cdot \frac{\pi D^2}{4}$$

$$\Rightarrow v_3 = \frac{4\dot{V}}{\pi d^2}, \qquad v_1 = \frac{4\dot{V}}{\pi D^2}$$

Bernoulli $\boxed{0} \rightarrow \boxed{3}$:

$$p_i + \rho g H = p_a + \rho g h + \frac{\rho}{2} v_3^2 \left(1 + \lambda \frac{L}{d}\right) + \frac{\rho}{2} v_1^2 \left(\zeta_S + \zeta_V + \lambda \frac{2L}{D}\right)$$

$$\Rightarrow p_i + \rho g H = p_a + \rho g h + \frac{\rho}{2} v_3^2 \left(1 + \lambda \frac{L}{d} + \left[\frac{d}{D}\right]^4 \cdot \left[\zeta_S + \zeta_V + \lambda \frac{2L}{D}\right]\right)$$

$$\Leftrightarrow p_i = p_a - \rho g (H - h) + \frac{\rho}{2} v_3^2 \left(1 + \lambda \frac{2L}{d} + \left[\frac{d}{D}\right]^4 \cdot \left[\zeta_S + \zeta_V + \lambda \frac{2L}{D}\right]\right)$$

$$\Leftrightarrow p_i = p_a - \rho g (H - h) + \frac{\rho}{2} \left(\frac{16\dot{V}^2}{\pi^2 d^4}\right) \cdot \left(1 + \lambda \frac{2L}{d} + \left[\frac{d}{D}\right]^4 \cdot \left[\zeta_S + \zeta_V + \lambda \frac{L}{D}\right]\right) \qquad \left[\frac{N}{m^2}\right]$$

b) Torricelli:

$$v_{4,end} = \sqrt{2\frac{p_i - p_a}{\rho} + 2g(H + L)}$$

instationärer Bernoulli $\boxed{0} \rightarrow \boxed{4}$:

$$p_i + \rho g (H + L) = p_a + \frac{\rho}{2} v_4^2 + \rho \int_0^{2L} \frac{\partial v_1}{\partial t} ds + \rho \int_0^{L} \frac{\partial v_4}{\partial t} ds$$

Mit Hinweis $L \gg D$: $\int_0^{2L} \frac{\partial v_1}{\partial t} ds = \int_0^{2L} \frac{dv_1}{dt} ds \qquad \& \qquad \int_0^{L} \frac{\partial v_4}{\partial t} ds = \int_0^{4L)} \frac{dv_4}{dt} ds$

Kontinuitätsgleichung:

$$v_1 \frac{\pi D^2}{4} = v_4 \frac{\pi d^2}{4} \Rightarrow v_1 = v_4 \frac{d^2}{D^2} \qquad ; \qquad \frac{dv_1}{dt} = \frac{dv_4}{dt} \frac{d^2}{D^2}$$

$$\Rightarrow p_i + \rho g (H + L) = p_a + \frac{\rho}{2} v_4^2 + \rho \int_0^{2L} \frac{dv_4}{dt} \frac{d^2}{D^2} ds + \rho \int_0^{L} \frac{dv_4}{dt} ds$$

$$\Rightarrow p_i + \rho g (H + L) = p_a + \frac{\rho}{2} v_4^2 + \rho \frac{dv_4}{dt} L \left(2\frac{d^2}{D^2} + 1\right)$$

$$\frac{2\frac{p_i - p_a}{\rho} + 2g(H + L) - v_4^2}{2L\left(2\frac{d^2}{D^2} + 1\right)} = \frac{dv_4}{dt}$$

$$dt = \frac{2L\left(2\frac{d^2}{D^2} + 1\right) dv_4}{2\frac{p_i - p_a}{\rho} + 2g(H + L) - v_4^2}$$

Mit Hinweis: $a^2 = 2\frac{p_i - p_a}{\rho} + 2g(H + L)$

$a < |v_4|$, da $v_{4,end} = a$

$$\Rightarrow \Delta T = \frac{2L\left(2\frac{d^2}{D^2} + 1\right)}{2\sqrt{2\frac{p_i - p_a}{\rho} + 2g(H + L)}} Ln \left(\frac{v_{4,end} + v_4}{v_{4,end} - v_4}\right)$$

Einsetzen der Bedingung $v_4 = 0,9 \cdot v_{4,end}$:

$$\Rightarrow \Delta T = \frac{2L\left(2\frac{d^2}{D^2}+1\right)}{2\sqrt{2\frac{p_i - p_a}{\rho} + 2g(H+L)}} Ln(19) \qquad \left[\frac{m}{\sqrt{\frac{kgm^4}{m^2 s^2 kg} + \frac{m^2}{s^2}}}\right] = [s]$$

Quelle: Herbst 2013