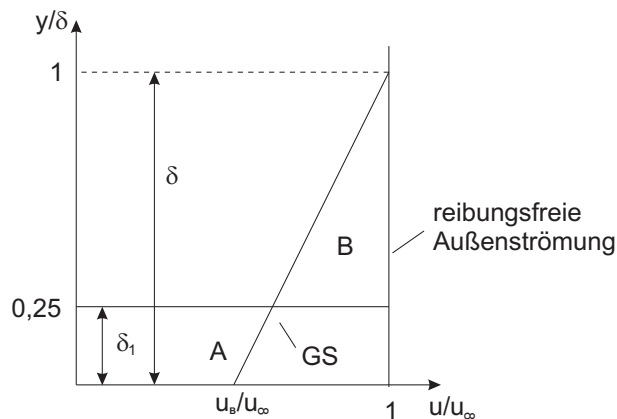


Tutorenprogramm - Strömungsmechanik II
Wintersemester 2012/2013
Laminare Grenzschichten - Musterlösung

1. Aufgabe

- a) $\frac{u_1(y)}{u_\infty} = a_0 + a_1\left(\frac{y}{\delta}\right)$
 1. RB.: $y = 0 : u = u_B \rightarrow a_0 = \frac{u_B}{u_\infty} = K$
 2. RB.: $y = \delta : u = u_\infty \rightarrow a_0 + a_1 = 1 \rightarrow a_1 = 1 - K$
 $\rightarrow \frac{u(y)}{u_\infty} = K + (1 - K)\frac{y}{\delta}$
- b) $K = 0.5$



Fläche A entspricht Fläche B

$$\frac{\delta_1}{\delta} = \int_0^1 \left(1 - \frac{u}{u_\infty}\right) d\left(\frac{y}{\delta}\right) = \frac{1}{2} \int_0^1 \left(1 - \frac{y}{\delta}\right) d\left(\frac{y}{\delta}\right) = 0.25$$

- c) Ebene Grenzschichtströmung: $\frac{\partial p}{\partial x} = 0$
 mit Euler-Gleichung $u_a \frac{du_a}{dx} = -\frac{1}{\rho} \frac{\partial p}{\partial x} \rightarrow \frac{du_a}{dx} = 0$
 $\rightarrow u_a = u_\infty = \text{const}$

v. Karman Int.-bez.: $\frac{d\delta_2}{dx} + \frac{1}{u_a} \frac{du_a}{dx} (2\delta_2 + \delta_1) - \frac{\tau_w}{\rho u_a^2} = 0$ (*)

$$\tau_w = -\tau(y=0) = \eta \frac{\partial u}{\partial y} \Big|_{y=0} = \eta \frac{u_\infty}{\delta} \frac{\partial \left(\frac{u}{u_\infty}\right)}{\partial \left(\frac{y}{\delta}\right)} \Big|_{\frac{y}{\delta}=0} = \eta \frac{u_\infty}{\delta} (1 - K)$$

$$\begin{aligned} \frac{\delta_2}{\delta} &= \int_0^1 \left[\frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty}\right)\right] d\left(\frac{y}{\delta}\right) \\ &= \int_0^1 \left[K - K^2 + (2K^2 - 3K + 1)\frac{y}{\delta} - (1 - K)^2 \left(\frac{y}{\delta}\right)^2\right] d\left(\frac{y}{\delta}\right) \\ &\rightarrow \frac{\delta_2}{\delta} = \frac{1}{6} + \frac{1}{6}K - \frac{1}{3}K^2 =: C \end{aligned}$$

einsetzen in (*): $C \frac{d\delta}{dx} - \eta \frac{u_\infty}{\delta} \frac{(1-K)}{\rho u_\infty^2} = 0$

$$\rightarrow \delta d\delta = \eta \frac{(1-K)}{C \rho u_\infty} dx$$

Integration: $\frac{1}{2} \delta^2(x) = \eta \frac{(1-K)}{C \rho u_\infty} x + C_1$ mit RB: $x \rightarrow 0: \delta \rightarrow 0 \rightarrow C_1 = 0$

$$\rightarrow \delta(x) = \sqrt{\frac{2\eta(1-K)}{C \rho u_\infty} x}$$

$$\begin{aligned}
\text{d) } P &= Fu_B = \int_0^L \tau_W B dx u_B \\
&\rightarrow \frac{P}{B} = u_B \int_0^L \eta \frac{u_\infty}{\delta(x)} (1-K) dx \\
&= \eta u_B u_\infty (1-K) \int_0^L \left(\frac{2\eta(1-K)}{C\rho u_\infty} x \right)^{-\frac{1}{2}} dx \\
&= \frac{\eta K u_\infty^2 (1-K)}{\sqrt{\frac{2\eta(1-K)}{C\rho u_\infty}}} \int_0^L x^{-\frac{1}{2}} dx \\
&= K u_\infty^2 \sqrt{\frac{\eta(1-K)C\rho u_\infty}{2}} 2\sqrt{L} \\
&= K u_\infty^2 \sqrt{2\eta(1-K)C\rho u_\infty L}
\end{aligned}$$

Quelle: Herbst 2008

2. Aufgabe

1.

$$\begin{aligned} \text{Haftbedingung:} \quad \frac{y}{\delta} = 0 &\rightarrow \frac{u}{u_a} = 0 &\Rightarrow & a_0 = 0 \\ \text{Grenzschichttrand:} \quad \frac{y}{\delta} = 1 &\rightarrow \frac{u}{u_a} = 1 &\Rightarrow & a_1 = 1 \\ &&\Rightarrow & \frac{u}{u_a} = \sin\left(\frac{\pi y}{2\delta}\right) \end{aligned}$$

2. $p(x=0) = p_0 \Rightarrow$ Bernoulli:

$$\begin{aligned} p(x) + \frac{\rho}{2} u_a^2 &= p_0 - C \frac{x^2}{2} + \frac{\rho}{2} u_a^2 = p_0 \\ u_a &= x \sqrt{\frac{C}{\rho}} \end{aligned}$$

$$\begin{aligned} 3. \quad \delta_1 &= \delta \int_0^1 \left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \delta \int_0^1 \left(1 - \sin\left(\frac{\pi y}{2\delta}\right)\right) d\left(\frac{y}{\delta}\right) = \delta \left[\frac{y}{\delta} + \frac{2}{\pi} \cos\left(\frac{\pi y}{2\delta}\right)\right]_0^1 \\ &\Rightarrow \delta_1 = \left(1 - \frac{2}{\pi}\right) \delta = k_1 \delta \end{aligned}$$

$$\begin{aligned} \delta_2 &= \delta \int_0^1 \frac{u}{u_a} \left(1 - \frac{u}{u_a}\right) d\left(\frac{y}{\delta}\right) = \delta \int_0^1 \left(\sin\left(\frac{\pi y}{2\delta}\right) - \sin^2\left(\frac{\pi y}{2\delta}\right)\right) d\left(\frac{y}{\delta}\right) \\ &= \delta \left[-\frac{2}{\pi} \cos\left(\frac{\pi y}{2\delta}\right) - \frac{1}{2} \frac{y}{\delta} + \frac{1}{2\pi} \sin\left(\frac{\pi y}{\delta}\right)\right]_0^1 \\ &\Rightarrow \delta_2 = \left(\frac{2}{\pi} - \frac{1}{2}\right) \delta = k_2 \delta \end{aligned}$$

$$\tau(y=0) = -\eta \frac{du}{dy} \Big|_{y=0} = -\frac{\eta u_a}{\delta} \frac{d\left(\frac{u}{u_a}\right)}{d\left(\frac{y}{\delta}\right)} \Big|_{\frac{y}{\delta}=0} = -\frac{\eta u_a \pi}{\delta} \frac{1}{2}$$

4. Einsetzen in die von Kármánsche Integralbeziehung:

$$k_2 \frac{d\delta}{dx} + \sqrt{\frac{\rho}{C}} \frac{1}{x} \sqrt{\frac{C}{\rho}} (2k_2 + k_1) \delta - \frac{\eta x \pi \rho}{2 \delta \rho x^2 C} \sqrt{\frac{C}{\rho}} = 0$$

Umformen:

$$\begin{aligned} \frac{d\delta}{dx} + \underbrace{\frac{2k_2 + k_1}{k_2}}_{\Gamma} \frac{\delta}{x} - \underbrace{\frac{\eta \pi}{2k_2 \sqrt{C} \rho}}_{\Omega} \frac{1}{\delta x} &= 0 \\ \frac{d\delta}{dx} + \Gamma \frac{\delta}{x} - \Omega \frac{1}{\delta x} &= 0 \end{aligned}$$

Differentialgleichung lösen:

$$\begin{aligned} x \frac{d\delta}{dx} &= \Omega \frac{1}{\delta} - \Gamma \delta = \frac{\Omega - \Gamma \delta^2}{\delta} = \Gamma \left(\frac{\frac{\Omega}{\Gamma}}{\delta} - \delta^2\right) \\ &\Rightarrow \frac{1}{\Gamma} \left(\frac{\delta}{\frac{\Omega}{\Gamma} - \delta^2}\right) d\delta = \frac{1}{x} dx \\ &\Rightarrow -\frac{1}{2\Gamma} \ln\left(\frac{\Omega}{\Gamma} - \delta^2\right) \Big|_{\delta_0}^{\delta} = \ln x \Big|_{x_0}^x \end{aligned}$$

$$\Rightarrow -\frac{1}{2\Gamma} \left[\ln \frac{\frac{\Omega}{\Gamma} - \delta^2}{\frac{\Omega}{\Gamma} - \delta_0^2} \right] = \ln \left(\frac{x}{x_0} \right)$$

Auflösen nach $\delta(x)$:

$$\left(\frac{\frac{\Omega}{\Gamma} - \delta^2}{\frac{\Omega}{\Gamma} - \delta_0^2} \right)^{-\frac{1}{2\Gamma}} = \frac{x}{x_0}$$

$$\Rightarrow \frac{\Omega}{\Gamma} - \delta^2 = \left(\frac{x}{x_0} \right)^{-2\Gamma} \left(\frac{\Omega}{\Gamma} - \delta_0^2 \right)$$

$$\Rightarrow \delta = \sqrt{\frac{\Omega}{\Gamma} - \left(\frac{x}{x_0} \right)^{-2\Gamma} \left(\frac{\Omega}{\Gamma} - \delta_0^2 \right)}$$

mit

$$\Omega = \frac{\eta \pi}{2 k_2 \sqrt{C \rho}} = \frac{\eta \pi}{2 \left(\frac{2}{\pi} - \frac{1}{2} \right) \sqrt{C \rho}}, \quad \Gamma = \frac{2k_2 + k_1}{k_2} = \frac{2 \left(\frac{2}{\pi} - \frac{1}{2} \right) + \left(1 - \frac{2}{\pi} \right)}{\left(\frac{2}{\pi} - \frac{1}{2} \right)} = \frac{4}{4 - \pi}$$

Quelle: Frühjahr 2010

3. Aufgabe

1. Randbedingungen:

$$(i) \quad y = 0 \quad \Rightarrow \quad u = 0 \quad (\text{Haftbedingung})$$

$$(ii) \quad y = \delta \quad \Rightarrow \quad u = u_a \quad (\text{Grenzschichttrand})$$

(iii) $y = 0$: Wandbindung aus x -Impulsgleichung für $y = 0$:

$$v(y=0) \left. \frac{\partial u}{\partial y} \right|_{y=0} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{\eta}{\rho} \left. \frac{\partial^2 u}{\partial y^2} \right|_{y=0}$$

mit $v(y=0) = -\frac{\dot{V}}{LB}$ und $\frac{\partial p}{\partial x} = 0$ (ebene Platte ohne Druckgradient) folgt:

$$-\frac{\dot{V}}{LB} \left. \frac{\partial u}{\partial y} \right|_{y=0} = \frac{\eta}{\rho} \left. \frac{\partial^2 u}{\partial y^2} \right|_{y=0}$$

2. Koeffizienten:

Aus (i) folgt: $a_0 = 0$

Aus (ii) folgt: $a_1 + a_2 = 1$

Aus (iii) folgt mit $\left. \frac{\partial u}{\partial y} \right|_{y=0} = u_a \frac{a_1}{\delta}$ und $\left. \frac{\partial^2 u}{\partial y^2} \right|_{y=0} = \frac{2a_2 u_a}{\delta^2}$:

$$-\frac{\dot{V}}{LB} \frac{a_1 u_a}{\delta} = \frac{2\eta}{\rho} \frac{a_2 u_a}{\delta^2} \quad \Rightarrow \quad a_2 = -\frac{\dot{V} \rho \delta}{2LB\eta} a_1$$

$$\Leftrightarrow \quad 1 - a_1 = -\frac{\dot{V} \rho \delta}{2LB\eta} a_1 \quad \Rightarrow \quad a_1 = \frac{1}{1 - \frac{\dot{V} \rho \delta}{2LB\eta}}, \quad a_2 = 1 - a_1 = 1 - \frac{1}{1 - \frac{\dot{V} \rho \delta}{2LB\eta}}$$

$$\left[\Rightarrow \frac{u}{u_a} = \frac{1}{1 - \frac{\dot{V} \rho \delta}{2LB\eta}} \left(\frac{y}{\delta} \right) + \left(1 - \frac{1}{1 - \frac{\dot{V} \rho \delta}{2LB\eta}} \right) \left(\frac{y}{\delta} \right)^2 \right]$$

3. Ablöseprofil an Ablösestelle x_{ab} für gegebenes Profil:

Randbedingungen (i) und (ii) immer noch gültig $\Rightarrow a_0 = 0, \quad a_1 + a_2 = 1$

Randbedingung (iii) jetzt Ablösebedingung: $\tau(y=0) = 0 \Leftrightarrow \left. \frac{\partial u}{\partial y} \right|_{y=0} = 0$

$\Rightarrow a_1 = 0$ und damit: $a_2 = 1 - a_1 = 1$

Geschwindigkeitsprofil: $u = u_a(x_{ab}) \left(\frac{y}{\delta(x_{ab})} \right)^2$

Quelle: Frühjahr 2012

4. Aufgabe

a) Konti: $u_0 H_0 = u_a(x) H(x) \Rightarrow u_a(x) = \frac{u_0 H_0}{H(x)} = \frac{H_0 u_0}{H_0 - c \frac{x}{L}}$

$$\frac{du_a}{dx} = \frac{H_0 u_0}{(H_0 - c \frac{x}{L})^2 \frac{L}{c}}$$

Bernoulli: $\frac{dp}{dx} = -\rho u_a \frac{du_a}{dx}$

$$\Rightarrow \frac{dp}{dx} = -\frac{\rho u_0^2 H_0^2}{c (H_0 - c \frac{x}{L})^3}$$

b) $\frac{\partial(\frac{u}{u_a})}{\partial(\frac{y}{\delta})} = a_1 + a_2 \left(\frac{y}{\delta}\right)$

$$\frac{\partial^2(\frac{u}{u_a})}{\partial(\frac{y}{\delta})^2} = 2a_2$$

1.RB: $\frac{y}{\delta} = 0 \Rightarrow u = 0 \Rightarrow u_0(x) = 0$

2.RB: $\frac{y}{\delta} = 1 \Rightarrow u = u_a \Rightarrow a_1 + a_2 = 1$

3.RB: $\frac{y}{\delta} = 0 \Rightarrow \eta \frac{u_a}{\delta^2} \frac{\partial^2(\frac{u}{u_a})}{\partial(\frac{y}{\delta})^2} = \frac{dp}{dx}$

$$\Rightarrow \eta \frac{u_a}{\delta^2} 2a_2 = -\frac{\rho u_0^2 H_0^2}{c (H_0 - c \frac{x}{L})^3} \Rightarrow a_2(x) = -\frac{\rho u_0 H_0 \delta^2}{2\eta \frac{L}{c} (H_0 - c \frac{x}{L})^2}$$

$$\Rightarrow a_1(x) = 1 + \frac{\rho u_0 H_0 \delta^2}{2\eta \frac{L}{c} (H_0 - c \frac{x}{L})^2}$$

c) $\frac{\delta_1(y)}{\delta(x)} = \int_0^1 \left(1 - \frac{u(x,y)}{u_a(x)}\right) d\frac{y}{\delta} = \int_0^1 \left[1 - a_1 \frac{y}{\delta} - a_2 \left(\frac{y}{\delta}\right)^2\right] d\frac{y}{\delta} = \left[\frac{y}{\delta} - \frac{a_1}{2} \left(\frac{y}{\delta}\right)^2 - \frac{a_2}{3} \left(\frac{y}{\delta}\right)^3\right] \Big|_0^1 = 1 - \frac{a_1}{2} - \frac{a_2}{3}$

c) Nein, da negativer Druckgradient.

d) vergrößert, da Gradient der Geschwindigkeit $\frac{\partial u}{\partial y}$ an der Wand bei turbulenten Strömungen größer ist.

Quelle: Herbst 2008