Tutorenprogramm - Strömungsmechanik II
Wintersemester 2013/14
Grundgleichungen - Musterlösung

1. Aufgabe

1. $\dfrac{\partial \rho \vec{v}}{\partial t}$: lokale Impulsänderung

   $\nabla \cdot (\rho \vec{v}\vec{v})$: konvektive Impulsänderung

   $-\nabla p$: Druckkraft

   $\nabla \cdot \tau$: Reibungskraft

2. Vereinfachungen:
   inkompressibel: $\Rightarrow \varrho = konst$

   2-dimensionales Problem: $\Rightarrow w = 0, \quad \dfrac{\partial}{\partial z} = 0,$

   Konti: $\dfrac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0 \quad \Rightarrow \quad \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + v\dfrac{\partial \rho}{\partial y} + \rho\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) = 0$

   inkompressibel: $\quad \Rightarrow \quad \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

   Impuls: inkompressibel: $\rho\dfrac{\partial \vec{v}}{\partial t} + \rho\nabla \cdot (\vec{v}\vec{v}) = -\nabla p + \eta\nabla^2\vec{v}$

   2D: $\rho\nabla \cdot (\vec{v}\vec{v}) = \rho\nabla \cdot \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix} = \rho \begin{pmatrix} \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} \\ \frac{\partial vu}{\partial x} + \frac{\partial v^2}{\partial y} \end{pmatrix}$

   $$= \rho \begin{pmatrix} u\underbrace{\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right)}_{\equiv 0, \text{Konti}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \\ v\underbrace{\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right)}_{\equiv 0, \text{Konti}} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \end{pmatrix}$$

   $$\Rightarrow \quad x\text{-Impuls: } \rho\frac{\partial u}{\partial t} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

   $$\Rightarrow \quad y\text{-Impuls: } \rho\frac{\partial v}{\partial t} + \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

3. Vereinfachungen:
   eingeschwungen: $v = 0$, $p = p_a$

   Konti: mit $v = 0 \quad \Rightarrow \quad \dfrac{\partial v}{\partial x} = \dfrac{\partial v}{\partial y} = \dfrac{\partial v}{\partial t} = 0$ folgt $\dfrac{\partial u}{\partial x} = 0$

   Impuls: Mit $p = p_a \quad \Rightarrow \quad \dfrac{\partial p}{\partial x} = \dfrac{\partial p}{\partial y} = 0$

Mit $\dfrac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \dfrac{\partial^2 u}{\partial x^2} = 0$

$\Rightarrow$ x-Impuls: $\rho \dfrac{\partial u}{\partial t} = \eta \dfrac{\partial^2 u}{\partial y^2}$

$\Rightarrow$ y-Impuls: $0 = 0$

Nur noch lokale Impulsänderung und Reibungsterm $\rightarrow$ konvektive Terme sind verschwunden.

4. $\omega \uparrow \quad \rightarrow \quad \delta \downarrow \qquad \eta \uparrow \quad \rightarrow \quad \delta \uparrow$

Quelle: Frühjahr 2012

2. Aufgabe

1. Vereinfachungen:
   stationär: $\Rightarrow \frac{\partial}{\partial t} = 0, \qquad$ inkompressibel: $\Rightarrow \varrho = konst$

   2-dimensionales Problem: $\Rightarrow w = 0, \quad \frac{\partial}{\partial z} = 0, \qquad$ ausgebildet: $\Rightarrow \frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial x} = 0$

   Druck in x-Richtung konstant: $\Rightarrow \frac{\partial p}{\partial x} = 0$

   Konti: $\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{v}) = 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$
   inkompressibel, stationär: $\quad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

   Aus Konti folgt mit $\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial y} = 0 \quad \Rightarrow \quad v = \text{konst.} = 0$, da $v(y = 0) = 0$

   x-Impuls: $\rho\frac{\partial u}{\partial t} + \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}$
   y-Impuls: $\rho\frac{\partial v}{\partial t} + \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$

   Mit oben genannten Vereinfachungen, Konti und $v = 0$:

   x-Impuls: $\Rightarrow \quad 0 = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y}$
   y-Impuls: $\Rightarrow \quad 0 = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$

2. Geschwindigkeitsverteilung:
   Einsetzen von $\tau_{xx}, \tau_{xy}, \tau_{yy}$ ergibt $\Rightarrow \tau_{xx} = 0, \quad \tau_{yy} = 0, \quad \tau_{xy} = \eta\frac{\partial u}{\partial y}$
   $\Rightarrow$ mit $\frac{\partial u}{\partial x} = 0$: $\quad$ x-Impuls: $0 = \eta\frac{\partial^2 u}{\partial y^2}, \quad$ y-Impuls: $0 = \frac{\partial p}{\partial y}$

   $\Rightarrow u(y) = \frac{c_1}{\eta}y + c_2$

   Randbedingung:
   $u(y = 0) = 0 \quad \Rightarrow \quad c_2 = 0$
   $u(y = h) = u_W \quad \Rightarrow \quad c_1 = \frac{u_W \eta}{h}$
   $\Rightarrow u(y) = \frac{u_W}{h}y \quad$ "Couette Strömung"

Quelle: Herbst 2012